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On the generation of stream rating curves

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ABSTRACT

Traditional methods for the calculation of rating curves from measurements of water level and discharge are criticised as being limited and complicated to implement, such that manual methods are still often used. Two methods for automatic computation are developed using least-squares approximation, one based on polynomials and the other on piecewise-continuous splines. Computational problems are investigated and procedures recommended to overcome them. Both methods are found to work well and once the parameters for a gauging station have been determined, rating data can be processed automatically. For some streams, ephemeral changes of resistance may be important, evidenced by scattered or loopy data. For such cases, the approximation methods can be used to generate a rating envelope as well, allowing the routine calculation also of maximum and minimum expected flows. Criticism is made of current shift curve practices. Finally, the approximation methods allow the specification of weights for the data points, enabling the filtering of data, especially decreasing the importance of points with age and allowing the computation of a rating curve for any time in the past or present.

1. Introduction

A rating curve is a relationship between the discharge Q of a stream and h, the stage or surface elevation, so that when routine measurements of stage at a gauging station are made, the flow can be estimated. The curve is calculated from a number of (h, Q) rating data points from that station, using relatively infrequent measurements of the velocity distribution, cross-section, and stage of the stream.

The problem of the automatic calculation of rating curves has received relatively little research attention. The main problem seems to be the perceived success and almost universal use of the power function

$$Q = C\,(h - h_0)^{\mu}, \tag{1}$$

where C, h_0 and μ are constants, and which is a straight line on $(\log Q, \log(h - h_0))$ axes. The reasons for it being a problem include:

- On one hand it is too simple, with only three parameters, and is limited in its accuracy and generality.
- On the other hand, it is too complicated, such that the three parameters occur nonlinearly and solving for them is difficult such that manual methods are often used.

The power function, and its representation as a straight line on logarithmic axes appears ubiquitously in books, standards, and lecture notes. Whereas it is sometimes a convenient approximation to the relationship $Q(h)$ over the whole range of data, in general it is not. It is an over-simplification of the real hydraulics at many gauging stations. Such a formula is valid for an infinitely-wide weir in infinitely-deep water or for uniform flow in an infinitely-wide rectangular channel. There is no reason for a real rating curve to follow such a function closely. Insufficient knowledge of hydraulics has led to a too-great belief in the power function, on one hand by practitioners, and on the other by theoreticians in related disciplines. This has led to complicated procedures in some organisations where sequences of power functions are used, and a great deal of trouble goes into the laborious manual fitting of piecewise-continuous straight lines on logarithmic axes by adjusting the offsets h_0 for each on different vertical $\log(h - h_0)$ axes.

The more general representation of Q by a polynomial of higher degree M has been in the background for some time:

$$Q = a_0 + a_1 h + a_2 h^2 + \ldots + a_M h^M = \sum_{m=0}^{M} a_m h^m, \tag{2}$$

where $a_0,\ a_1,\ \ldots, a_M$ are coefficients. It was presented by Herschy in the first edition of his book in 1985, most recently in Herschy (2009, p195), in International Standard 7066-2 (1988), and in Morgenschweis (2010, p384). Standard linear least-squares methods can be used to determine the coefficients. Mirza (2003) used it successfully with just $M = 3$, and in that scholarly work gave considerable attention to statistical matters.

Reading those sources and water industry websites, but also reading between the lines, it seems that the approximation by polynomials,

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despite its promise, has not often been adopted, and usually only implemented to low degree. Herschy wrote in the first edition of his book in 1985, and 24 years later again in the third edition, (Herschy, 2009, p195): “however some user experience is still required with this method before it is accepted as an alternative to the existing methods”, implying that its use has been languishing.

Fenton and Keller (2001, Section 6.3.2), suggested writing the polynomial for Q raised to the power ν, specified *a priori*:

$$Q^{\nu} = a_0 + a_1 h + a_2 h^2 + \ldots + a_M h^M = \sum_{m=0}^{M} a_m h^m, \tag{3}$$

which is actually a simple generalisation of the power function, Eq. (1), written in the form $Q = (a_0 + a_1 h)^{1/\nu}$, to $Q = (a_0 + a_1 h + a_2 h^2 + \ldots)^{1/\nu}$. They recommended a value of $\nu = \frac{1}{2}$, on the basis of that being the mean value in the hydraulic discharge formulae for a sequence of weir and channel cross-sections that modelled local and channel control (Fenton, 2001). The use of such a fractional value of ν has two effects:

1. For small flows, h and Q small, the data usually is such that
$$Q^{\nu} = a_0 + a_1 h, \tag{4}$$
with $\nu = \frac{1}{2}$, is a surprisingly good approximation when compared with power function approximations in which ν is a free parameter, as shown in Fenton (2015b, Section 3.4). In this small flow limit the more general polynomial approximation just has to simulate nearly-linear variation, which it can easily do.
2. For larger flows, when the higher degree terms in Eq. (3) become important, the use of Q^{ν} means that the magnitude of the dependent variable to be approximated is numerically much smaller, so that, instead of a range of say, $Q \approx 1$ to $10^4\ \mathrm{m^3\,s^{-1}}$, for $\nu = \frac{1}{2}$ a numerical range $Q^{1/2} \approx 1$ to 10^2 has to be approximated. This is a simple version of a *power transformation* used in more formal data analysis applications to stabilise variance and to make the data more normal distribution-like.

In recent years there have been a number of papers with a quite different way of looking at the problem, using Bayesian statistics. Le Coz et al. (2014) provided an excellent survey both of that field and the rating curve problem generally. However, all the papers they referred to used either a single power function or two or more of them, each in the belief that they were following hydraulic principles. It is the assertion here that there is little fundamental about the power function or the application of hydraulic theory, and here a rather different path will be followed, treating the problem as one of data approximation.

In that spirit, Coxon et al. (2015) used LOWESS (LOcally WeightEd Scatterplot Smoothing) to obtain rating curves for a huge number of sites. The method considered each stage-discharge measurement as the central point in a subset of the data points. The estimate of the discharge for the data point and its variance was generated by fitting a weighted linear regression to the selected data. Weights were dependent upon the differences in stage and gave most weight to data closest to the central measurement. To account for outlier points, two passes were made, then a weight function was used to weight each data point according to how far the point was from the first fitted line, reducing the impact of those furthest from it. The procedure could be used to satisfy the goal in this work, of developing methods for practical automatic computation. It seems good in principle, but there are a number of adjustable parameters and the reduction of importance of outlying points might deny the importance of some causative processes and trends at work. It functioned well for the demanding application that Coxon et al. required of it, where the main thrust was the quantification of uncertainty rather than the generation of approximations.

Fenton (2015b), hereafter referred to as Report I, considered several aspects of the problem of the automatic generation of rating curves. The present paper is based on that report, which contains more detail. Here first, the application of polynomial approximation methods is treated at length. Several mathematical reasons for problems associated with them are given, with physical explanations and methods for overcoming them. It is considered imperative to use series of Chebyshev polynomials rather than the simple polynomials shown above which are series of monomials h^m. Also it is desirable to approximate, not values of discharge Q, but Q^{ν}, where ν is a fractional exponent, as in Eq. (3). It is usually able to be taken to be $\frac{1}{2}$, but in extreme cases can be calculated by a method that is presented. Other than ν, the degree M of the polynomial series is the only free parameter. It is possible to use large values of M but if the data has gaps there will usually be one degree beyond which large fluctuations occur in between accurate approximation of the data points. To overcome that problem, an alternative approximation method is developed using piecewise-continuous splines, in which case the parameters of the problem are the number and stage values of knot points between which simple quadratic or cubic spline functions are used. A simple automatic method is suggested for the placing of those knots, just requiring there to be the same number of data points in each interval. This usually works well. Otherwise, in difficult cases the values of stage for the knot points can be specified. Results for both the polynomial method and the approximating spline method are presented. They are both found to perform well and have the potential to be standard procedures for rating curve generation. Then possible reasons for scatter of rating points are discussed. For such data, the methods can be modified to calculate additionally a rating envelope, giving likely maximum and minimum flow rating curves. For discrepant points it is suggested that current use of shift curves should be re-examined. Finally, the approximation methods are simply modified to allow the importance of data points to decrease with age. This allows the generation of a rating curve on any date in the past also, thereby determining any relatively slow long term changes in the stream.

2. Polynomial approximation

Eq. (3) is now generalised by considering the approximating function to be made up, not of a series of monomials h^m, but of more general functions $f_m(h)$:

$$Q^{\nu} = \sum_{m=0}^{M} a_m f_m(h) = a_0 f_0(h) + a_1 f_1(h) + \ldots + a_M f_M(h). \tag{5}$$

In application, the functions are specified *a priori*, and the unknown coefficients a_m found by least-squares fitting to data points (h_n, Q_n^{ν}) for $n = 1, \ldots, N$. Each of the functions applies over the whole data range of h, and so methods using them are *global* ones, as distinct from those in Section 3 below where piecewise-continuous *local* functions are used. We will consider the functions $f_m(h)$ each to be a polynomial of degree m so that the sum of such polynomials in Eq. (5), including the last one at $m = M$, is also a polynomial of degree M, and we can refer to methods using them as *polynomial approximation*. It will be found that Chebyshev polynomials for the $f_m(h)$ are particularly useful.

There are three problems here with the approximation: the rapid variation of data at the low flow end, the large range of discharge Q, and the ability of the approximating functions to describe arbitrary variations. These problems will be overcome, as is now described.

2.1. Exponent ν

2.1.1. Usual adequacy of $\nu = 1/2$

Traditionally, the power function has often been required to model all the data. By writing it in the form of Eq. (4), $Q^{\nu} = a_0 + a_1 h$, while it incorporates the usual rapid variation and large curvature on (Q, h) axes at low-flows, it is obvious that it is a limited approximation to the whole rating curve problem. Concerning the actual value of ν to use, Report I (Fig. 2) showed that for each of seven different stations,

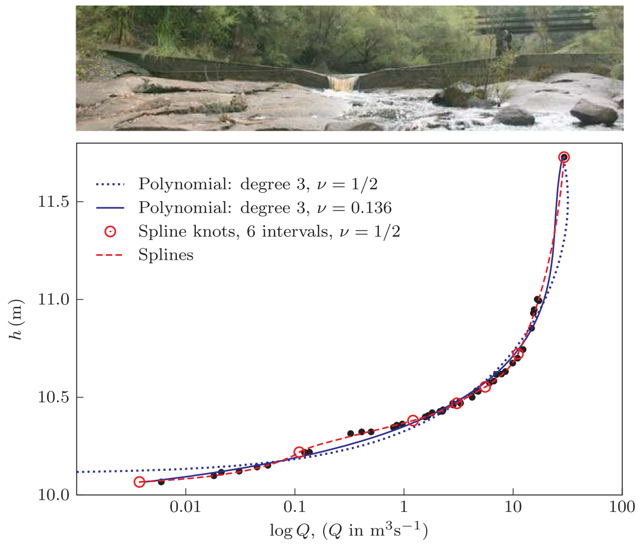

Fig. 1. Shannon River at Dog Pool, Australian Station 060185, 1964–1971. Data from the Hydrologic Reference Stations Data Set, Bureau of Meteorology, Australia.

selected randomly, adopting $\nu = \frac{1}{2}$ gave agreement with scattered data at the low flow end that was just as satisfactory as results from fitting a value of ν. McMahon and Peel (submitted for publication) used the polynomial approximation methods of Report I and this work to obtain 622 rating curves from 171 Australian Bureau of Meteorology Hydrologic Reference Stations. They found that the methods worked well - with the exception of about 0.5% of the stations, where there was difficulty approximating the low-flow data, as exemplified by the dotted line in Fig. 1 here. In such cases it is better to determine the actual value of ν from the low-flow data and use that, as is now described.

2.1.2. Calculating ν when necessary

To do this one can use any of several methods outlined in Report I (Section 3.2). The recommended method is to use the function $Q^\nu = a_0 + a_1 h$ (Eq. 4) and solving for a_0, a_1, and ν by nonlinear least-squares approximation of the points at the low-flow end, minimising

$$\varepsilon_2 = \sum_n (a_0 + a_1 h_n - Q_n^\nu)^2. \tag{6}$$

As two of the unknowns, a_0 and a_1 occur linearly in Eq. (4) and hence in the error contribution here, this form is computationally simpler than solving for the three unknowns in the highly nonlinear and embedded form of Eq. (1), $Q = C(h - h_0)^\mu$, when problems with complex values can occur if the term in brackets becomes negative as part of the iterative solution procedure.

It is not recommended to use all points to determine ν. Its value is best determined from the low-flow data when the simple linear approximation used in Eqs. (4) and (6) is adequate in that limited range. To determine how many of the points to use in determining ν, the author has experimented and as a rough guide found using about one third of all the data points usually satisfactory, but this can be varied to suit the circumstances, determined by plotting results.

Always ν has a value rather less than unity. Its precise value is not so important in its other useful function, that of providing a power transformation for the larger flow data, to be explained in Section 2.1.3 below.

Fig. 1 shows the results for a case where it was necessary to calculate ν, the most extreme one found. The physical nature of the purely local control is shown in the photograph; it is a compound V-notch weir. On the rating curve plot it can be seen that fitting a polynomial in the form of Eq. (3) with $M = 3$ did not work well with $\nu = \frac{1}{2}$. Using the method described immediately above to determine ν from low-flow data gave $\nu = 0.136$, significantly different. With that value, the subsequent polynomial approximation of Eq. (3) worked rather better as shown. However the data shows a certain oscillation due to the irregular nature of the structure. Better results were obtained using spline approximation, an alternative method to be described below, which still worked well simply with $\nu = \frac{1}{2}$. Taking a higher degree polynomial with $M = 4$ gave very poor results with large oscillations.

2.1.3. The role of ν as a power transformation

The use of a value of ν which is less than unity solves another problem with the approximation of rating data, and that is the large variation in the magnitude of Q. It can easily vary by a factor of 10^4 between lower and upper ends, even in a small problem such as shown in Fig. 1. The polynomial has to approximate variation of that magnitude with a similar relative accuracy overall. One way of overcoming this which was tried was to approximate values of log Q and use a polynomial for that quantity. This expanded the low flow region and contracted the high flow region as required, but sometimes too much so, and is not recommended. Instead, using a fractional value of ν with Q^ν, which also gives a much smaller numerical range to approximate, for example $Q^{1/2}$ varying by a factor 10^2, has been found to be satisfactory. Whether a small calculated value such as $\nu = 0.136$ in Fig. 1 or the commonly-assumed value $\nu = \frac{1}{2}$ in this work, it was found to eliminate problems that otherwise occurred if $\nu = 1$ were simply used.

2.2. Global basis functions in the approximating series

Whereas the previous two difficulties, of describing both low flows and overall flows varying by several orders of magnitude, are ones of accuracy, and are obvious, the worst problems associated with polynomial approximation are not obvious, and are to do with the approximating properties of the functions $f_m(h)$ in Eq. (5).

We present a hierarchy of approaches in increasing levels of power and desirability, starting from the conventional and immediately obvious one. They are illustrated in Fig. 2.

(a) $f_m = h^m$, as suggested in books and standards, shown in Eqs. (2) and (3), and represented in Fig. 2(a): this is a very fragile form if the numerical values of h are large, for example using elevation above sea level, when the range of h might be something like $h = 100$ m to $h = 110$ m, as illustrated for example, or if the stage is specified in centimetres, as is the practice in some countries. Over that relatively small range of stages, each monomial term h^m looks rather like all the others, with little apparent curvature. To describe any general variation with finite curvature, the individual contributions in the series would have to struggle, with large coefficients a_m.

(b) $f_m = (h - h_{\min})^m$, where $h_{\min}$, the minimum of all stages measured has been subtracted: the effects are shown in Fig. 2(b), and now the individual functions show more diverse behaviour. Common practice is to use a stage datum for rating curves just below the minimum, so this is effectively what might be used in practice.
However, in Section 6.3 of Report I an example was presented that showed using the monomial functions in the form of both (a) and (b), with and without subtraction of $h_{\min}$, individual contributions in the series of approximating terms had remarkable values of ± 10, 000 times that of the final sum! Although plotting and use of results was possible, it would be extremely fragile if passing coefficient values between programs and computers, as they would have to be specified to very high levels of precision.

(c) $f_m = y^m$, where y is a scaled variable in the interval $[-1, +1]$:

$$y = -1 + 2\frac{h - h_{\min}}{h_{\max} - h_{\min}}, \tag{7}$$

where $h_{\max}$ is the maximum of all the stage measurements: Fig. 2(c) shows how the first 4 or 5 such monomials are quite different, providing good grounds for approximation, however for larger

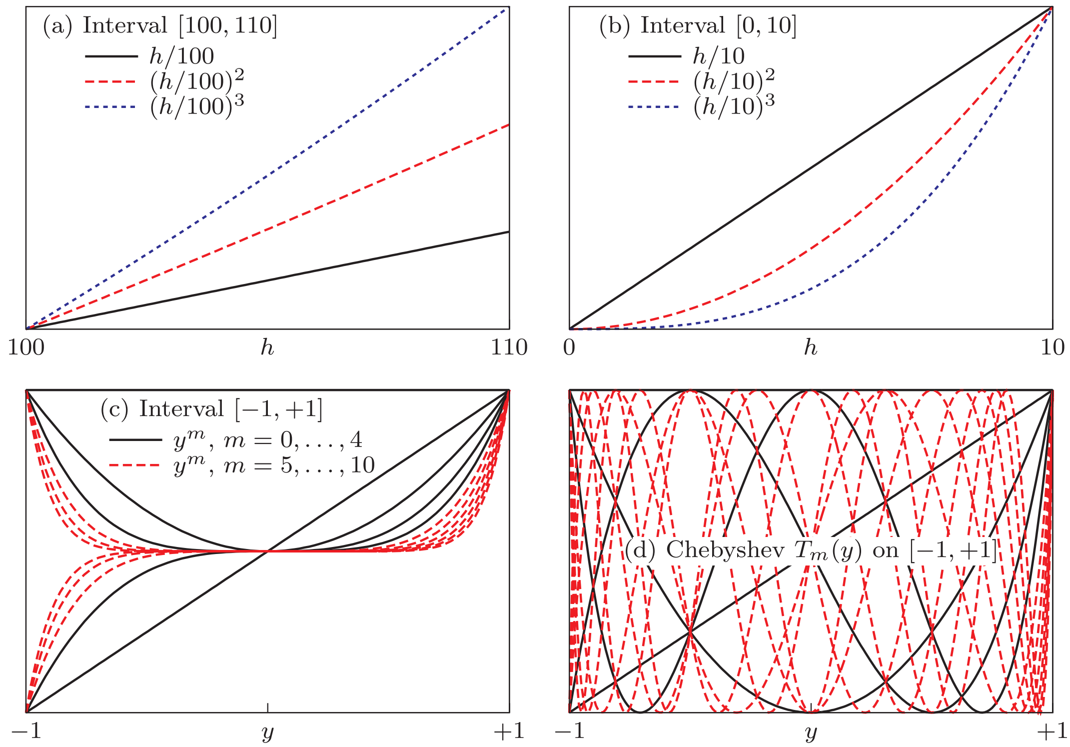

Fig. 2. Comparison of the ability of different basis functions $f_m(h)$ to represent varying data. Shown are three monomials on the intervals [100, 110] and [0, 10], and on [−1, +1] the first eleven monomials and Chebyshev polynomials.

values of m they all start to look similar to each other, making higher degree approximation more difficult. In Report I using y was found to give rather better results than the first two possibilities of (a) and (b), however not as good as the following fourth alternative.

(d) $f_m = T_m(y)$, still using y, but as the argument of a Chebyshev polynomial of degree m, which can be simply evaluated by $T_m(y) = \cos(m \arccos y)$, or recursively from $T_0(y) = 1$, $T_1(y) = y$, and for all $m \geqslant 2$, $T_m(y) = 2yT_{m-1}(y) - T_{m-2}(y)$: Fig. 2(d) shows that they all behave quite differently from each other, and hence can describe arbitrarily varying quantities without the problems noted above. They are simply implemented using the formulae given here.

2.3. Recommended formulation and solution for the coefficients

From the above, it is recommended to use the formulation

$$Q^\nu = \sum_{m=0}^{M} a_m T_m(y). \tag{8}$$

It is highly desirable to use a fractional value of ν for which a value of $\frac{1}{2}$ can usually be assumed. In exceptional cases ν can be determined using the procedure described in Section 2.1.2.

In view of the evidence cited in Section 2.2(b), it is considered imperative to use polynomials which have an orthogonal nature, capable of efficient description of arbitrarily-varying data. In any further reference to polynomial approximation in this work it is always Chebyshev polynomials that have been used. Each T_m is a polynomial of degree m so that the sum of the different Chebyshev polynomials to $m = M$ is itself a polynomial of degree M. While it can be called a Chebyshev series, it cannot be called a Chebyshev polynomial itself. The number of terms in the series and the degree M of the polynomial might typically be in the range from 3 to 10.

The coefficients a_m can be obtained by least-squares methods, minimising the sum of the weighted squares of the errors of the approximation over N data points,

$$\varepsilon_2 = \sum_{n=1}^{N} w_n \left(\sum_{m=0}^{M} a_m T_m(y_n) - Q_n^\nu \right)^2, \tag{9}$$

where the y_n are obtained from the h_n from Eq. (7). The w_n are the weights for each rating point, giving the freedom to weight some points more if one wanted the rating curve to approximate them more closely, or they could be set to be a decaying function of the age of the data point, so that the effects of changes with time could be examined. Or, a less-trusted data point could be given a smaller weight. Often, however, all the w_n will be 1.

Two ways of calculating the a_m are considered here:

(a) Normal equations Following the standard least squares procedure, the total error in Eq. (9) is differentiated with respect to each of the unknown a_m, $m = 0, \ldots, M$ and set to zero so that error is at a minimum, thus giving a system of $M + 1$ equations in the $M + 1$ unknowns, the so-called *normal equations* for the a_m. Interpreted in a matrix equation sense, the equations can be written $[A_{jm}][a_m] = [b_j]$, where to evaluate the elements in those matrices here, abandoning the usual convention that matrix row and column numbering starts at 1,

For j from 0 to M

$$b_j = \sum_{n=1}^{N} w_n Q_n^\nu T_j(y_n)$$

For m from 0 to M

$$A_{jm} = \sum_{n=1}^{N} w_n T_j(y_n) T_m(y_n)$$

The equations and the matrix are famously poorly-conditioned unless care is taken to use functions which have some form of orthogonal nature, as has been done here. While the Chebyshev polynomials are orthogonal under integration and summation with certain special weights, in the present case summing over an arbitrary sequence of y_n, they are not strictly orthogonal, but show sufficient diversity of behaviour that the matrix is not particularly poorly-conditioned. Here no problems were found using this method to obtain solutions.

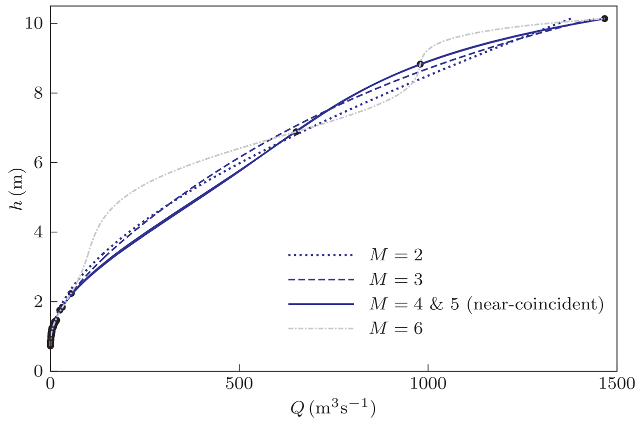

Fig. 3. Typical results using polynomial approximation for data with gaps, showing variation with degree *M* (Station: Pallamallawa on the Gwydir River, Australia, see p22).

(b) Optimisation methods: These minimise ε_2 in Eq. (9), using common software such as the Solver module in spreadsheets, or other software packages. They are simple to implement without a great deal of knowledge of the program details. The methods used internally might be gradient methods, such as a multi-dimensional Newton's method. Numerical solution does not seem to be difficult in the present work because the a_m occur linearly in the approximating function (8), and also because of the efficient nature of the approximating basis Chebyshev polynomials.

2.4. Degree of approximation M

With polynomial approximation there are few freedoms of choice. The use of Chebyshev polynomials is considered mandatory here, to guard against series with very poor convergence properties. Then, having decided to approximate in terms of a certain value of ν, probably $\frac{1}{2}$, the only computational parameter remaining is the degree of approximation M.

Fig. 3 shows a typical result from polynomial approximation for a data set with gaps. As M increases to from 2 to 3 to 4 and 5, better agreement is obtained, here coincidentally the latter two agreeing. For higher degree $M = 6$, however, the polynomials have begun to oscillate between the data points, and the results are unacceptable. For data with no large gaps, one can use higher degrees, but sooner or later the degree of approximation becomes too high and unacceptable oscillations appear.

In practice, one could determine the appropriate value of M for a particular station visually, based on evidence such as the figure here. It need only be done once.

2.5. Quantification of goodness-of-fit

It would be better to have a more automatic procedure to measure how well a particular function approximated the data, such as the statistical *coefficient of determination*, R^2. The author investigated this problem at some length, considering two error criteria. The first was the sum of the squares of the errors at each data point, the numerical value of ε_2 defined in Eq. (9), which could be used to obtain a value of R^2. This did not work well, as it did not detect over-fitting, when a higher degree of approximation fitted the data points better but allowed severe fluctuations between them, as shown in Fig. 3.

To try to identify that problem, a second error criterion was investigated, which was the integral of the absolute magnitude of the curvature of the approximating function over the whole curve. Unfortunately, both error criteria behaved differently for different stations, and no recommendation can be made here.

Criteria such as R^2 also do not show where omitted-variable bias exists. In the rating curve case there are several such variables, including those that are discussed in this work: unsteadiness, and channel changes, both short- and long-term.

In view of these problems, no goodness-of-fit results will be presented here, either for the polynomial representation or for the approximating spline method about to be described.

2.6. Extrapolation

The methods in this work approximate data over a finite range. They must not be used for extrapolation beyond either end of the data.

3. Approximating splines

A different approach was developed in Report I (Section 8), using piecewise-continuous approximation in the form of spline functions, where, rather than interpolating as in their usual application, they were required to perform least-squares approximation. The method is in a sense both local and partly-global, in that a sequence of local polynomials is used, each of which approximates just part of the range of data, but which is required to merge smoothly with its two neighbours, so that they and their approximated data points also have some influence. It seems to be relatively simple, to have a readily-understandable physical significance and a good level of continuity. Other than the degree of the splines (second or third, which seem to make little difference) the only adjustable quantities are the values of stage used as knot points, marking the boundaries between successive splines.

Consider a number of data pairs (h_n, Q_n^ν) for $n = 1, 2, \ldots, N$. Let the range of stages $[h_{\min}, h_{\max}]$ be subdivided into J intervals by $J + 1$ knot points, at each of which the stage is H_j, $j = 1, \ldots J + 1$ and $H_0 = h_{\min}$ and $H_J = h_{\max}$. They must be separate and ordered, such that $H_{j+1} > H_j$ for all the j. The points are to be approximated over each interval by a polynomial of degree $M = 2$ or 3. So, for $j = 1, \ldots, J-1$ and $H_j \leqslant h \leqslant H_{j+1}$:

$$P_j(h-H_j) = \sum_{m=0}^{M} c_{j,m}(h-H_j)^m, \tag{10}$$

where $P_j(h-H_j)$ is the polynomial of degree M which holds between stage knot points H_j and H_{j+1}, expressed as a function of $h-H_j$, the height of stage h above the preceding knot point.

At each knot there are continuity conditions between the polynomials on either side, agreeing in value and first derivative for quadratic splines, and additionally the second derivative for cubic splines. The H_j may be set as data or automatically allocated such that there are the same number of data points in each interval, for example, described below. The program calculates all the coefficients so that the sum of the squares of the differences between data points and functions are minimised. Only those data points falling within a certain interval contribute to the total sum using the function for that interval. It is not necessary to scale the h as recommended for polynomial approximation in Section 2, as it only appears as the local shifted value $h-H_j$ and the degree of the polynomial is low anyway.

The details of those operations are as follows. The sum of the squares of the errors is minimised using the polynomials as given in Eq. (10), as

$$\varepsilon_2 = \sum_{j=1}^{J} \sum_{n \in I_j} w_n \left(\sum_{m=0}^{M} c_{j,m}(h_n-H_j)^m - Q_n^\nu \right)^2, \tag{11}$$

where I_j is the set of data point reference numbers in interval j; the summations are a mathematical statement of the obvious "over all the intervals, consider every data point in that interval and calculate its contribution to the total error using the polynomial for that interval".

The spline nature of the approximation requires the satisfaction across each interior knot of the continuity of function value plus all derivatives up to $M-1$ at the knot points, giving extra conditions on all the coefficients $c_{j,m}$. From Eqs. (10) at left and right of each interior knot point, and using $\delta_j = H_{j+1}-H_j$ for the interval length, the conditions become, for $j = 2...J$:

$$c_{j+1,0} = \sum_{m=0}^{M} c_{j,m}\delta_j^m, \quad c_{j+1,1} = \sum_{m=1}^{M} mc_{j,m}\delta_j^{m-1}, \quad \text{and}$$
$$c_{j+1,2} = c_{j,2} + 3c_{j,3}\delta_j, \text{ if } \quad M = 3. \tag{12}$$

The solution procedure to determine the unknown $c_{j,m}$ by minimising ε_2 in Eq. (11) is first to use the continuity Eqs. (12) to eliminate those variables on the left sides, then to solve for the remaining $c_{j,m}$, as for polynomial approximation. Differentiation leading to the Normal Equations could be used, but it is simpler to use optimisation software, which was the case in this work. Because of the high degree of continuity enforced by equations (12), the net number of unknowns is not large, just $J + M$. In the present work, quadratic splines, $M = 2$, were found to be satisfactory, and J might have a maximum value of, say, 10.

To allocate the knot points H_j, they could simply be equally spaced between minimum and maximum values. Another automatic method, that which was used here, was simply programmed and which often worked well, was where knot points were calculated such that there were approximately the same number of data points in all intervals. However, the knots can also simply be assigned as data. Such hand adjustment of the knot positions was sometimes useful to describe regions of high curvature. It was usually found necessary to place a knot point near isolated data points for extreme floods. It should be emphasised that only the values of the stage knot points H_j were specified, there was no attempt to pin the approximation by specifying the discharge there.

The approximating spline method worked well in all the examples considered, with occasional improvement by hand allocation of knot points. It never failed badly, unlike the polynomial method if too high a degree M were used as in Fig. 3.

4. Plotting of rating curves

Horizontal discharge axis In this work as described above, using $Q^{1/2} = \sqrt{Q}$ proved very helpful for computations. However, using it as a plotting co-ordinate caused the low flow region to be too condensed for display purposes. Attempts were made instead to use log Q for the computations, but this caused the high flow region to be too condensed, with difficulties of approximation. The hybrid solution was adopted, of obtaining solutions using $Q^{1/2}$ but for plotting the traditional horizontal scale of log Q has been mostly used here, with its ability to represent variation of several orders of magnitude. The exception is the use of Q in Fig. 3.

Vertical stage axis Use of logarithmic scales for stage has often made representation and understanding difficult and caused errors. Report I (Section 4.3) reported on an elementary mathematical mistake in current software using logarithmic axes. Unless one were using the traditional power function of Eq. (1) with $h-h_0$ there is little reason to use a logarithmic scale for stage, as it typically varies from a value close to zero, opening out unreasonably on the logarithmic scale, to at most about 10 m.

Tradition is that the stage has an arbitrary local datum somewhere below the lowest possible water level, so that stages plotted are typically in the range of greater than 0 m to something like 10 m or rather less, but where they have no external physical significance. Use of the transformed variable y (Eq. 7), or the spline method, mean that there is no advantage to specifying and using these traditional local values of stage. It would be possible to use the actual elevation above mean sea level, which would have certain advantages, as that is often important in water engineering. There would no longer be such a need to maintain an arbitrary local benchmark and refer measurements to that, especially if satellite navigation systems could be directly used for water level measurements.

5. Results

To examine the performance of the methods described above, data from seven sites were considered, three from Australia, two from Bangladesh, and two from the United States of America. Wherever necessary results were converted to *SI* units. Results have already been shown for one in Fig. 1, and another in Fig. 3 using polynomial approximation only. Results for that and another five are shown in Fig. 4. In each case two rating curves are presented, one from polynomial approximation, the other from approximating splines. All have been obtained using $\nu = \frac{1}{2}$, thus approximating values of $Q^{1/2}$. In all cases the polynomial degree M shown on each figure was chosen as the smallest value that was compatible with accuracy. Often increasing M by a single degree beyond that led to unacceptable fluctuations, as has been seen in Fig. 3, which is for the same station as Fig. 4(e). For the spline method, all results are for quadratic splines, $M = 2$. No advantage was found using cubic splines, $M = 3$. In the first four cases in Fig. 4, automatic allocation of knot points was used, with about the same number of data points in each interval; in the two remaining examples and in Fig. 1, especially at large flows where there were few data points, the knot points H_j were allocated manually by specifying in a data file. The method used is shown on each figure.

(a) Avon River at Stratford, Vic., Australia, site number 225201A, 2012-06-05 to 2015-04-24 There is a large gap in this data. The polynomial approximation of low degree, $M = 3$, and the quadratic splines with only three intervals, have worked well, both giving a similar plausible bridging of the gap. There are actually two points close together for the highest flows. Using higher degree polynomials showed over-fitting, with the result that both points were almost interpolated, with large oscillations between data points.
(b) Brahmaputra River, Bangladesh 1992 The data was taken from Mirza (2003) by digitising a fairly small figure, so the accuracy here might be questionable. Both methods handle this fairly simple problem quite well. It might be thought that neither method has treated the two points at $h \approx 19$ m, $Q \approx 40,000\ \mathrm{m^3\,s^{-1}}$ well, but it should be remembered that the approximations are for discharge, plotted horizontally, and so they actually perform quite well in passing between those two points and the three or four points on the other side of the curves with roughly the same stage.
(c) Choctawhatchee River near Bellwood, AL, USA, USGS Station 02361500, 2000-12-07 to 2015-05-22 The only real problem for approximation seems to be for the lowest flows, where there is a quite characteristic scatter of points. Both methods seem to perform satisfactorily.
(d) Ganges River, Bangladesh 1992 The data was also digitised from a small figure in Mirza (2003). This is a demanding problem, if one accepts that the fine structure ("lumps") in the data are real. It has been chosen here to do that as a test of the model, although it might be over-fitting it. If one knew the reliability of the data, it might have been possible to use a smoother approximation, with a polynomial of lesser degree and/or splines with fewer intervals. Both polynomials and splines, with high levels of approximation, have performed well in describing the complicated variation. The polynomial can do this because the data points are uniformly distributed. The splines have performed well, even if the placement of the knots has yet again been performed automatically, and there are actually few data points in each interval.
(e) Gwydir River at Pallamallawa, NSW, Australia, Site number 418001, 1991-01-05 to 1998-07-29 Again, this shows data with some large gaps. Using splines, automatic allocation of equally-populated intervals did not give such good results for high flows. So, for the first time in this figure, knots were allocated by hand, placing one at 8 m, which caused the splines to pass very close to the high

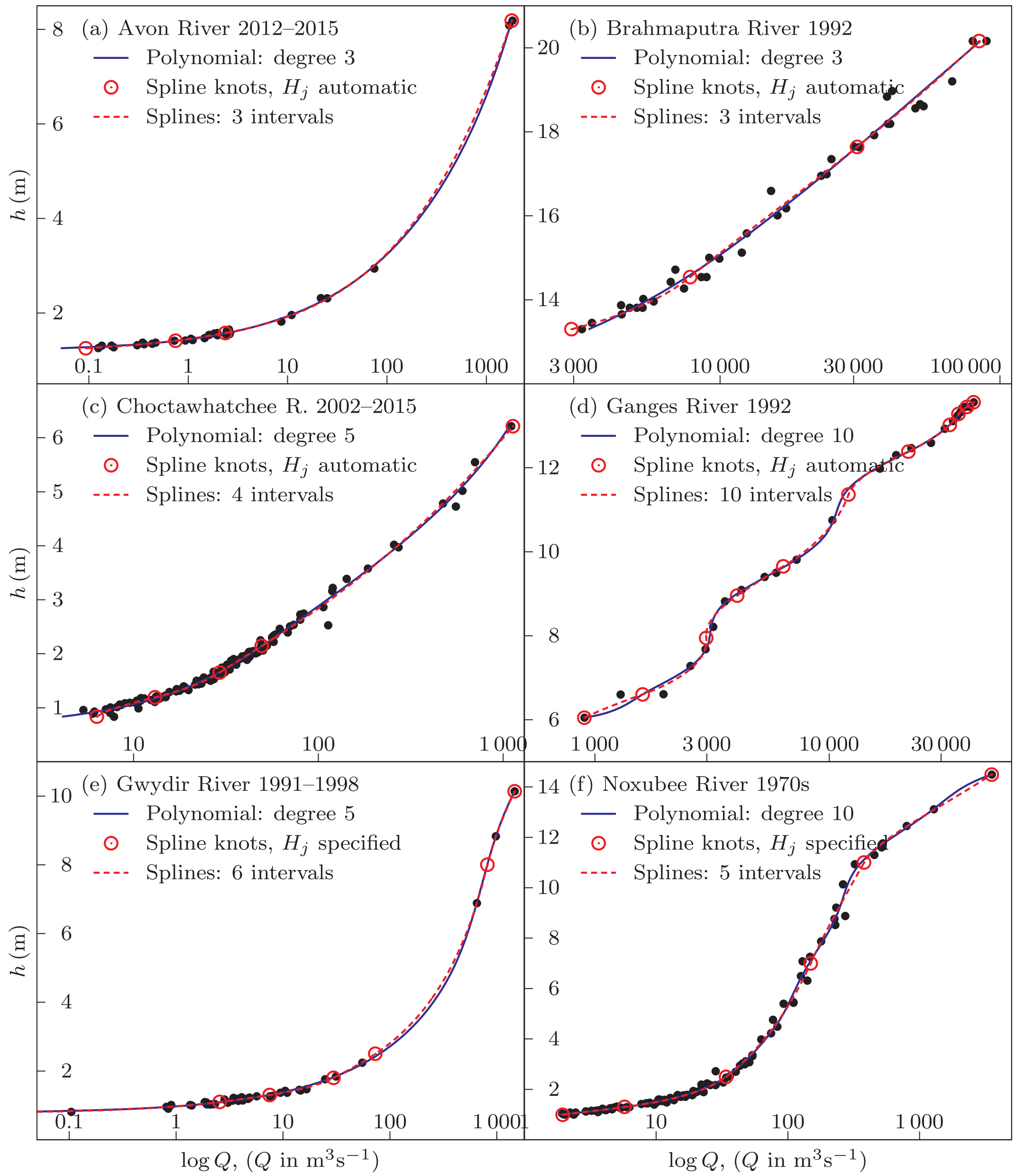

Fig. 4. Typical rating curves obtained from polynomial and piecewise-continuous spline approximations.

flow points as shown. Some of those data points were obtained partly from a boat floating above flooded agricultural land — possibly a rougher approximation would have been more justified. In any case with splines one has quite a lot of power to experiment with point placement.

(f) Noxubee River near Geiger, AL, USA, USGS Station 02448500,1970s This data set shows three regions, of local, channel and overbank control. The polynomial method has worked well here, but with a high degree of approximation necessary to pick up the points of high curvature, which has led to some possible overfitting. The spline approximations have worked well also, with only five hand-allocated knots. This is a site where the greater freedom to allocate spline knot points has been of benefit.

6. Scattered and looped data

The problem of the scatter of data points is now considered. Two main causes will be identified, and two different ways of representing the data. Firstly, short-term changes in the stream will be considered. It is suggested that this scatter can be incorporated and quantified by the computation of a rating envelope, so that maximum and minimum expected flows can also be calculated and published. Secondly, long-term stream changes can be identified and described by a procedure that enables the calculation of a rating curve also for any date in the past.

6.1. A model of a rating curve determined by channel resistance

A model of a stream is now developed to try to understand how channel changes affect rating curves. This will apply to both channel and overbank control, where the boundary resistance of the stream determines the rating. It is assumed that in the vicinity of the gauging station the long wave equations hold, a pair of partial differential equations, one expressing volume conservation, the other momentum conservation. Fenton (2015a), Sections 1.4 & 1.5 showed for waves of a sufficiently long period, that the time derivative in the momentum conservation equation can be neglected, obviously enough, but not so obviously, also that fluid inertia terms can be neglected in that limit,

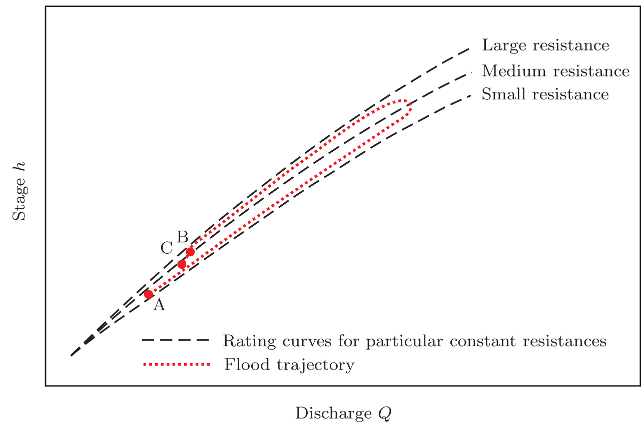

Fig. 5. Scattered data and erratic later measurements: shown are idealised rating curves for different constant resistances and a trajectory of a possible flood event during which resistance changes with the flow. A rating might be taken at any point on the trajectory, as might a routine stage measurement.

leaving the only terms in the equation those due to resistance and gravity, surprisingly even for high Froude number flows. Modelling the resistance here by the Gauckler-Manning equation then gives the momentum equation in the form in which the only approximation is that wave motions are long, typical of flood waves, and of course also valid for steady flow:

$$Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_\eta}, \tag{13}$$

where n is Manning's resistance coefficient, A is stream cross-sectional area, P is its wetted perimeter, and $S_\eta = -\partial\eta/\partial x$ is the magnitude of the slope of the free surface, where η is the surface elevation, which might also be a function of time t. For the special case of a steady uniform flow, $S_\eta = S_0$, the bed slope, and the expression becomes the usual statement of the Gauckler-Manning equation. It is surprising that the expression here is such a simple and general extension of that familiar form.

We consider the different terms on the right of Eq. (13), to see how different channel quantities affect the rating:

Resistance coefficient n: this might be able to change relatively quickly, and its value can depend on the recent flow history, due to the following effects:

- Small changes in bed grain arrangement and armouring can have a finite effect and could occur quite quickly, if, for example, overlying particles with high resistance are swept away by an increase in flow, or start to deposit in interstices.
- Bed forms – ripples, dunes, anti-dunes *etc.*. If conditions are right for the development or changing of bed forms, the effects on resistance could be important. This would take longer than for bed grain re-arrangement, as more material has to be transported. This has been treated by Simons and Richardson (1962).
- Sediment transport – if the bed grains are moving, then the stress required to move them appears additionally, whether they are moving along the bed, rolling, jumping, or carried suspended in the flow. It could be quite variable, depending on the different thresholds of movement.
- Vegetation – grasses, reeds, trees standing in the water *etc.*. This would have a more slowly varying effect, partly continuous and possibly seasonal.

The relative change in rated discharge due to a change Δn is obtained by the differential of Eq. (13) with respect to n, giving $\Delta Q/Q = -\Delta n/n$, which might not be small. Fenton (2015a), Section 1.6.2 showed the wide scatter of channel resistance coefficients in a large field study and hypothesised that some of that was due simply to different arrangements of bed particles. Fig. 6 below shows a situation where a range of variation Q in a large alluvial river over three years was some 25%-50%. That may also be partly due to fill and scour, however the rapidity of change in the data suggested it was mainly due to resistance changes.

Geometric term $A^{5/3}/P^{2/3}$ – effects of bed fill or scour: as a model we consider the channel to be rectangular, of width B, such that $A = B(h-Z)$, where Z is the bed elevation, and to be wide such that $P \approx B$ so that the term becomes $A^{5/3}/P^{2/3} \approx B(h-Z)^{5/3}$. The relative change in rated discharge due to a change ΔZ in the bed level due to scour or fill is then $\Delta Q/Q = -5/3 \times \Delta Z/(h-Z)$, in which $h-Z$ is the water depth at the gauging station. It is asserted here, that that relative amount of scour or fill, ΔZ/Depth, is usually small. In this case, because finite amounts of material have to be moved to effect the change in bed elevation, such changes to the rating will be relatively slow. They may be of a consistent long-term nature, leading to gradual change in the rating curve. A method to identify and calculate such change will be presented in Section 6.4.

Unsteady surface slope changes: The last term in Eq. (13) is the surface slope, which for a change ΔS_η gives the differential $\Delta Q/Q = \frac{1}{2}\Delta S_\eta/S_\eta$. The slope might change because of long-term downstream channel changes. However there can be rapid changes as a flood wave passes, which could have a finite effect on the rating, but only at times immediately before and after a flood peak when the surface slope is sufficiently different from the mean bed slope. Some important data are obtained during floods when there may be unsteady effects, which it would be good to correct, and of course subsequent routine computation of discharge should ideally be corrected for unsteadiness around flood peaks. The problem was considered at some length in Fenton and Keller (2001), Section 4, which included an extra correction to the Jones method, which obtains an estimate of S_η by assuming that the flood is propagating as a wave without change. However, effects of changing resistance described above are also important but unable to be quantified, so that unsteadiness will not be considered further here.

6.2. Scattered data and results due to a moveable bed

Let us consider the mechanism by which changes in resistance cause the data to be scattered. Fig. 5 shows rating curve axes, stage versus discharge. Three dashed lines represent hypothetical rating curves for a stream with different constant resistance values. The dotted line represents a hypothetical flood event, showing the actual relationship between stage and discharge at each time. We now consider such an event in detail.

The initial point A is for a low flow, over a bed with relatively small-resistance after a period of steady flow during which the bed has been steadily armoured and smoothed. The flood event then begins. The flow increases quickly in time, but initially not enough to change the nature of the bed, and the flood trajectory follows a curve corresponding to the initial resistance. After some distance on the figure, the bed is no longer stable, grains move, are exposed and bed forms might develop. Accordingly, the resistance is greater and the trajectory crosses contours of increasing resistance. There may also be unsteady effects due to the front of the flood being steeper and the instantaneous flow being greater, which mimic the effects of changing resistance. After the flood peak, with maximum discharge, and high transport, resistance continues to increase. A little time later the stage is a maximum. Subsequently the flow continues to decrease but the transport rate is still enough for resistance to increase. Gradually however that stops, and the flow gradually decreases. At point B, with relatively low flow, with the disturbed bed still in place, the resistance is still high. Now gradually over time the sediment transport acts to reduce the bed-forms and pack the bed grains so that resistance is less. As time passes, the bed will gradually be worked back to a low-resistance state as initially, but where the more-or-less steady flow as at point C might be different from the initial one.

The path has followed a loop, with two discharges for each stage, and *vice versa*, before and after the flow maximum. This is often described as a *looped rating curve*, however it is really just a *trajectory*, a path in time on rating axes that may be more or less looped. A rating point obtained during such an event could be anywhere on it. If we consider a long period of time, with many such flow events, the space between the upper and lower constant-resistance curves will contain many such flow trajectories, and many individual rating points obtained at moments on those trajectories, leading to a scatter of points between the bounds. The more stable the bed, the less the scatter. Usually, rating measurements are not made frequently enough for any trajectory to be identified, and the rating points seem randomly scattered. By approximating all data points as described above, we obtain what we call the rating curve.

If a new rating point were found to be discrepant from established results, it might be possible to correct for any effects of unsteadiness as described above, but as suggested there, effects of changing resistance due to changing bed composition and bed forms might also be important. We do not know those conditions, and do not know how much to correct. The conventional shift curve approach is to arbitrarily distort the rating curve locally by a straight line on logarithmic axes to pass through the new point (Sauer, 2002, Section 8; WMO, 2010, Section 1.12). The view here is that even a short time after the discrepant measurement, when resistance in the stream is still the same, the flow might be quite different, and be far from the shifted interval, so that the shift is not relevant. The next time that the interval of the shifted curve is visited by a flow event, conditions in the stream might be quite different, and the previous discrepant point and shift no longer have a special relevance. Accordingly there is little need to ascribe great importance to a discrepant data point not agreeing with the rating curve. The view here is that it is an approximating curve passing through a more-or-less scattered cloud of points, where at least some of that scatter is due to fluctuations in the preceding flows and instantaneous state of the bed when each point was determined.

Now considering the subsequent regular routine measurements of stage to estimate flows, each measurement might be made at any point on the continuing trajectory, but should fall within the band of resistance values as modelled in the figure. This ephemeral nature of the resistance in mobile bed streams means that it is not possible to predict accurately the flow at any later time. It is not known what flows and bed changes will occur in the future up until the moment a routine stage measurement is made and the rating curve required to give a corresponding flow. This existence of a certain level of uncertainty for some rating sites suggests the concept and different treatment of the next section, using the approximation methods developed here to calculate curves for not just the likely mean discharge, but also for minimum and maximum possible values.

Of course, the data for many sites shows little scatter. In the small sample presented in this paper, of the six run-of-river results shown in Fig. 4 about three show evidence of significant scatter. A single rating curve might still be considered satisfactory but augmentation by maximum and minimum flow curves might be useful.

6.3. A generalisation for scattered data - the rating envelope, maxima and minima

The existence of a finite band of results leads to an extension of the idea of a single rating curve: the calculation of a *rating envelope*, inside of which all or most of the individual rating points fall, and to provide expressions for curves approximating both upper and lower bounds, as well as the conventional rating curve approximating all points. The effects of long-term changes would have to be subtracted, however, possibly using time-decaying weights, as described in the following section.

For gauging stations where data is scattered, rather than being a problem, this can be viewed as providing extra information as to the range of discharges that can be expected. It would make some sense also to publish routinely the minimum and maximum flows expected for each stage reading.

In keeping with the approach in this work, methods of data approximation can be used to calculate the upper and lower envelopes. The method suggested here is, first to calculate the approximation to all the points, the rating curve, and then to delete those points which lie below it. Then to approximate the remaining points, again deleting those points which lie below it and repeat the process of approximation and deletion as many times as necessary, to give the upper envelope, a fit to the uppermost points. Then this procedure would be repeated for the lower envelope, successively deleting all points above each curve. As approximately half the data points are lost with each pass, the number of passes is limited. The two data points with the smallest and largest values of independent variable h might always be retained to ensure that the final envelope would extend from the smallest to largest values of stage. In practice what one would be doing is approximating the 1/8 or 1/16, say, of all data points, those which lie furthest from the approximation to all the points. In the spirit of approximation, it can be called an envelope, even if some points might still lie outside it.

Fig. 6 shows an example for three years of gaugings from the Red River, Viet Nam. The flood hydrographs for those years, shown in Report I (Section 11.3), were quite characteristic, with the main events being more seasonal rather than with individual events, the flows coming off the high mountains in southern China. The flow trajectory has several large loops, barely identifiable in the figure here. The procedure described above for computing envelopes was applied, using polynomial approximation with degree $M = 6$. Four passes of the halving procedure for each of the upper and lower envelopes were applied, so starting with 217 data points, at the end there were about $217/2^4 \approx 15$ for each envelope. It can be seen that the method worked well, giving mean, maximum, and minimum rating curves.

6.4. Identification of long-term changes and calculation of a rating curve for any day in the past or present

A feature of both approximation methods, using polynomials and piecewise-continuous splines, is that the importance of each data point can be weighted. For example, less weight might be given to a point whose accuracy was doubtful. Or, points can be weighted according to their age, so that the oldest points have the smallest contributions, and the most recent gaugings can be rationally incorporated to give the most recent rating curve. In fact, the rating curve can be constructed for any day, now or in the past. For that day, t_0 say, one could assume that points more recent than that would have zero weight, and those older have a weight which is a function of their age. Thus if t_n is the time when point n was established, then in the sums of squares, Eqs. (9) & (11), the weights are

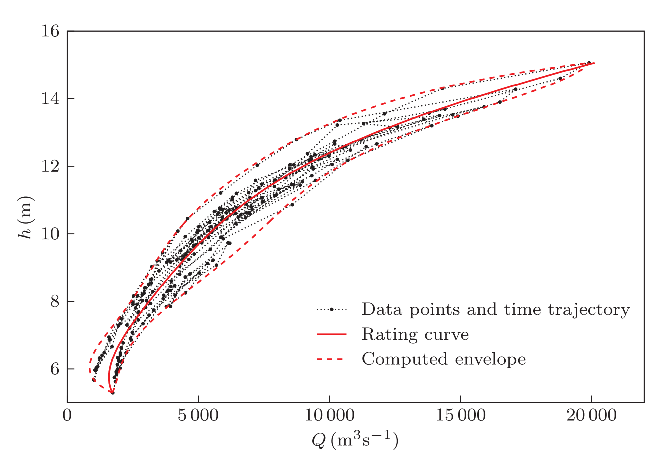

Fig. 6. Rating curve and upper and lower envelopes to the data, Station 41 on the Red River, Viet Nam, 1995–1997.

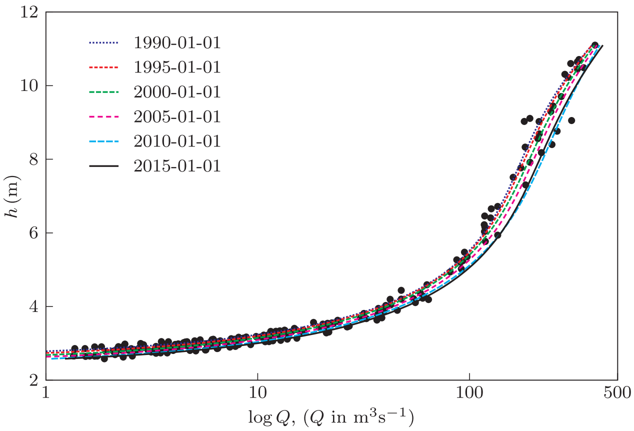

Fig. 7. Calculation of rating curves on specific days using weights that are a function of measurement age, here using Eq. (15) with a data half-life of 2 years — Noxubee River near Geiger, AL, USA, USGS Station 02448500, from 1984-10-02 to 2015-05-11.

given by

$$w_n = \begin{cases} F(t_0 - t_n), & \text{if } t_0 \geqslant t_n; \\ 0, & \text{otherwise} \end{cases}, \tag{14}$$

where $F(t_0-t_n)$ is a function of the age of the data t_0-t_n at the date of the rating curve to be calculated. Here, all rating points later than t_0 have been assigned a weight of zero. It might not be too outrageous, in fact, if data were scarce, also to include later rating points in some manner, as providing some information at least. This will not be pursued here.

The simplest method of diminishing the importance of older points would be to ignore all points with an age greater than a certain amount. The author tried this, but for high flows, where there were not enough points to define the curve properly, poor results were obtained. A better method seems to be to use a smoother function, decaying into the past, to keep all the points to some extent in determining the shape of the curve. A good example is the exponential weight factor $F(t_0-t_n) = \exp(-\alpha(t_0-t_n))$, where α is a decay constant. Writing $\tau_{1/2}$ for the "half-life", the age at which the weight decays by a factor of $\frac{1}{2}$, then the expression can be written

$$F(t_0-t_n) = \left(\frac{1}{2}\right)^{(t_0-t_n)/\tau_{1/2}}. \tag{15}$$

This was applied to 31 years of data from USGS Station 02448500 on the Noxubee River near Geiger, AL, USA, with results shown in Fig. 7. The polynomial method with a degree $M = 3$ was used, with a $Q^{1/2}$ fit and with a "half-life" of $\tau_{1/2} = 2$ years, found to work well. The results show how the rating curve, and presumably the bed, has generally moved steadily down over 31 years.

7. Conclusions

It has been asserted that the problem of rating curve generation benefits little from simple hydraulic formulae, and is essentially one of data approximation, for which least-squares methods can be used. They too have their problems if used without knowledge and overcoming of intrinsic computational difficulties. Two methods have been developed and applied. Both use a power transformation of discharges. One uses polynomial approximation, in which it is necessary to scale the stage and then to use a polynomial which is a series of Chebyshev polynomials. These have much better numerical properties than simple polynomials and are easily implemented. The only important parameter is the degree of polynomial, however monitoring of results for different degrees is necessary, as there is usually one degree beyond which the results show unacceptable oscillations. The other method developed here uses piecewise-continuous splines together with least-squares approximation. This requires possibly just the input of the number of computational knot points. Depending on the results, some specification and manipulation of knot point stage values might be necessary. The spline approach is more robust and flexible, but is slightly more complicated to program. As the natures of the polynomial and spline approximations are different, but programming details of the solution methods are similar, the use of both would be convenient and would provide a test and check on results.

The nature of scattered rating data at some stations has been discussed, and it has been shown that the scatter of data points in mobile bed streams might be predominantly due to changes in resistance in the stream. Those changes can be ephemeral, depending on the arrangement of bed grains or of bed forms, so that we never really know what the immediate resistance and hence the rating is. A method has been developed to compute upper and lower bounds to the rating data, giving an envelope to scattered data, so that, for routine stage measurements, not only the most likely mean discharge, but also the possible maximum and minimum values could be published.

It is suggested that if a discrepant rating point is determined, there is no point in locally distorting the rating curve by using a shift curve, as the next time that the interval of validity of the shift is visited by a flow event, conditions in the stream might be quite different.

Both approximation methods developed here allow the specification of a weight for each data point which can be specified as diminishing with age of the data, such that the methods can be used to generate the present rating curve or that on any particular day in the past.

An incidental practical benefit of the approaches here is the possibility of no longer using an arbitrary local datum for rating data and curves, instead using the actual elevation above mean sea level, thereby merging more with other hydraulic applications that might use the results.

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