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# Rating Curves: Part 2 – Representation and Approximation

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**Abstract:** A rating curve represents a supposedly-unique relationship between the stage (surface elevation) of water at a gauging station and the flow past that station. A significant problem is that the flow can vary from none to that of large floods, and this whole range has to be represented, including everyday small flows. A traditional way of doing this is to use log-log axes, but these have a number of problems. This paper considers the simple hydraulics of typical natural geometries of controls and river cross-sections and shows that in many situations the stage at the gauging station will vary roughly like the square root of the discharge, but with different relationships at different flow ranges. This suggests routinely plotting stage against the square root of discharge as representing and approximating rating curves.

**Keywords:**. discharge, flood, flow, gauging, hydrograph, plotting, rating curve, river.

## 1 INTRODUCTION

In the introduction to the companion paper to this one (Fenton, 2001) the nature of rating curves was described, including the comment by Brown, quoted by Chester (1986), who called for the need to examine and "rationalise the whole question of discharge rating curves" as it "has received relatively little attention in technical literature" and "is covered in a general fashion in the stream gauging manuals". In the companion paper it was remarked that since 1983 there has been little progress along those suggested directions.

The companion paper provided a summary of one part of the report by Fenton and Keller (2001), this paper deals with another aspect – how they might be plotted and approximated. This uses some elementary hydraulics to guide the path.

The idea of plotting a fractional power of discharge was first put forward by Chester (1986), who advocated plotting discharge to the power 0.4, which has been implemented in some practical software. In this paper it is suggested that 0.5 is to be preferred, as it is more representative of the hydraulics and natural features of rivers and controls. Thus, it is advocated that one actually plots on $(\sqrt{Q},\eta)$ axes.

## 2 LOGARITHMIC SCALES

A problem with rating curves is that they have to represent a relationship between stage, which might vary by 10 metres or so, and discharge which can vary by several orders of magnitude, from 0 to hundreds of thousands of discharge units. A traditional solution is to plot the discharge using a logarithmic scale, effectively expanding the region for small flows and contracting that for large. In many books and standards (for example, Herschy, 1995, and Australian Standard AS 3778.2.3, 1990) it is shown how it is convenient to use a logarithmic scale for the stage $\eta$ as well, in that often it is found that that by subtracting of some arbitrary value $\eta_0$ such that if one plots the logarithm of the discharge against the logarithm of $\eta-\eta_0$, points on the rating curve approximately fall on a straight line. The implication of this is that the discharge obeys a law of the form

$$Q = C(\eta-\eta_0)^n, \tag{1}$$

where $C$ and $n$ are constants. Such a relationship might be valid for low flow, and there might even be a similar but different relationship for high flows. To handle this more complicated situation, the curve could be broken up into a small number of segments, each of which is a straight line on the log-log plot, such as performed by

Herschy, (1995, #4.5). This does seem a rather arbitrary procedure, however. Rather better would be to approximate it with a larger number of such segments, when the use of straight-line approximation would be sound. This is widely done in practice.

Generally, however, the use of log-log plots is fraught with difficulties, and the continuing and enthusiastic advocacy of their use in texts and standards is surprising. Some of the difficulties include

- The cease-to-flow point, actually just $(0,\eta_0)$, cannot be shown on a log-log plot, as on those axes it is at a horizontal position of $(-\infty,-\infty)$. This is not necessarily an important disadvantage. The important problem is that the cease-to-flow point needs to be found before plotting the figure, by the following method.
- The cease-to-flow point can be found by a nonlinear analytical procedure involving the use of three data points at which values for $\eta$ and $Q$ are known, and substituting into a formula for $\eta_0$ (see, for example #4.4 of Herschy, 1995). While this is a relatively simple method, it is not clear that it is safe to advise it as a technique for routine practice. The method depends on choosing three points deemed to satisfy (1) exactly. There is no room for a recognition that it would be better to incorporate more data points in a least-squares sense. The formula given is badly conditioned if $n \approx 1$, and breaks down completely if $n = 1$.
- Subsequently least-squares methods can be used to find values of *C* and *n*. The log-log plot allows for this in a linear sense, for taking the logarithm of both sides of (1) gives $\log Q = \log C + n \log(\eta - \eta_0)$, and by using $\log Q$ and $\log(\eta - \eta_0)$ the solution follows by standard methods.
- For small flows and heads the data points are artificially separated, and small differences physically become large differences with a tendency to attach more importance to these points in the least squares procedure than is really the case.
- For large flows, the reverse holds, and the points are compressed, large differences are apparently compressed, and the points are rendered less important.
- One disadvantage is that the use of log-log scales is open to misinterpretation and abuse. In one hydrographic office an elementary misunderstanding has lead to a plotting mistake being made which rendered all points incorrectly plotted, although this was only of any consequence only at the low-flow end.
- In a similar spirit to this is the observation of the quite ubiquitous use of a plot of the logarithm of the stage as simply $\log \eta$, where the stage is relative to an arbitrary datum, rather than the plotting the logarithm of the stage relative to the cease-to-flow stage, $\log(\eta - \eta_0)$. There is no theoretical justification for plotting $\log \eta$; even if the stage discharge relationship were exactly as given by (1) it would not plot as a straight line on $\log Q$ - $\log \eta$ axes. Figure 4.12 of Herschy shows how this is the case.

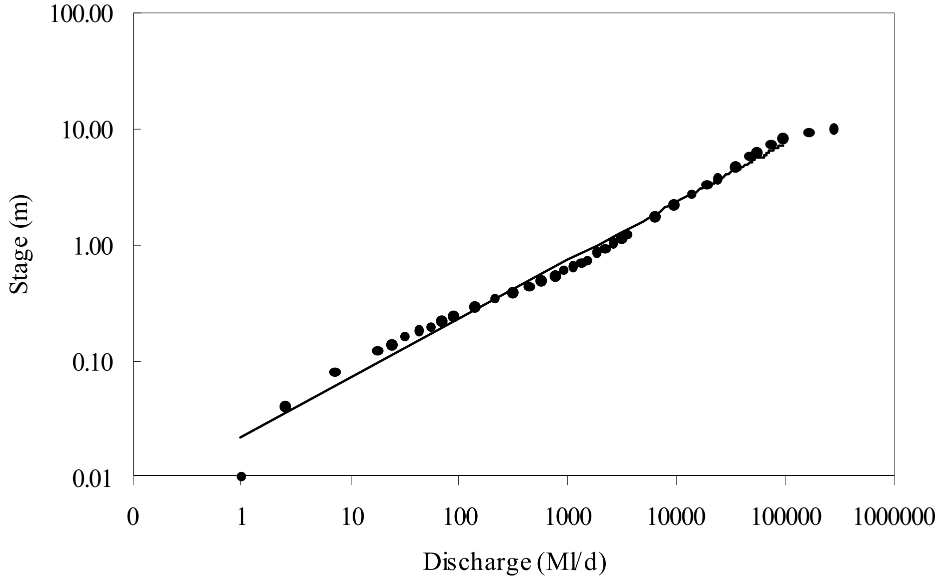


Figure 1. Stage-discharge Relationship Using Log-log Axes for Pallamallawa on the Gwydir River, with a Line of Best Fit on These Axes to All but the Last Two Points

Some problems of the naïve use of log-log scales are demonstrated by an example here, from Pallamallawa on the Gwydir River. Data sets for this and other stations on the same river are given in Fenton and Keller (2001). It is not suggested here that the simple-minded procedure about to be followed here is implemented in practice at that site. In Figure 1 a set of points from the rating curve are plotted and without any attempt to determine the cease-to-flow point, a global straight-line approximation fitted to all but the two points of highest discharge, where, corresponding to overbank flow occurring, a significant discontinuity occurred. On the axes shown the straight line does not seem unreasonable. To obtain the linear fit, however, a least-squares procedure has been used in log-log space that means that as this opens out the points very much at the low-flow end, that they contribute more than their real importance. This is illustrated by plotting on linear axes, as shown in Figure 2, using both the same data and line of best fit, now curved, it becomes obvious how the wide-spacing of data points at the low-flow end on the logarithmic plot has distorted the result considerably, and in reality, the plausibly-satisfactory results on log scales are not acceptable. Even if a single straight line were not fitted, the shrinking of the scale at the upper end is such as to render apparently small changes or errors innocuous, whereas in reality they are important, as revealed by Figure 2.

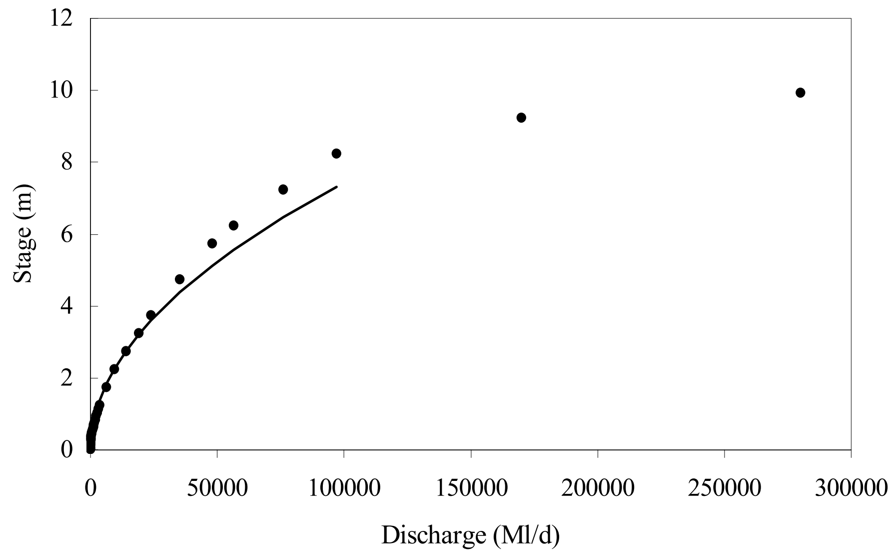


Figure 2. The Same Data and Approximation as Figure 1 but Using Natural Axes.

Of course, a higher degree polynomial fit in terms of the logarithms of the variables could be implemented, in the spirit of what is done later in this paper, and better fits could be obtained. However that would still not overcome some of the problems of using the logarithmic scales. This seems not to have been done elsewhere, and Standards and other sources seem often just to recommend a simple linear plot such as has been shown here, for which there is little justification.

## 3 THE POSSIBILITY OF USING $(Q^\nu, \eta)$ SCALES

Plotting rating curves on simple linear axes could be used to overcome some of the problems of logarithmic scales, however the requirement to include large flows means that the region of small flows becomes graphically insignificant, as shown in Figure 2. That figure strongly resembles a plot of the square root function. More generally it could be a plot of the function $Q^\nu$, where $\nu$ is any number somewhat greater than 0 and somewhat less than 1). This suggests using such an alternative scale, as by Chester (1986), who advocated that $Q^{2/5}$ should be used. The choice for the power 2/5 was made by assuming the discharge formula for a V-shaped section control. The use of a 2/5 power scale for discharge has been implemented as an option in the widely used HYDSYS package. Although the justification for using 2/5 has not been particularly convincing, the assumption of a power law scale is interesting and potentially useful.

Here three different types of controls are considered which might determine the behaviour of a rating curve over part of its range at least. Simple hydraulic theory is used in each case to develop a theoretical model, showing how stage can approximately be linearly related to some fractional power of the discharge, $Q^\nu$, for some model cross-sections. It is concluded that $\nu = 1/2$ such that $Q^{\nu} = \sqrt{Q}$ is a choice which probably more generally models the hydraulics of gauging stations.

To provide a model, a family of cross-sections are considered, of both sharp-crested and broad-crested weirs, and of the waterway itself. Monomial variation of the breadth *b* as a function of elevation above the lowest point *z* is considered, such that $b(z) = \beta z^m$, where $\beta$ is a constant for a particular section. This assumption incorporates and generalises some simple cross-sections as shown in Figure 3. For $m = 0$, $b(z) = \beta$, which is a constant so that this describes a rectangular section. As *m* increases, a sequence of flat bottomed U-shaped sections is obtained, modelling wide mature streams and even an approximation to trapezoidal sections. For $m = 1/2$ the cross-section is a parabola, which could be used as an approximate model for many natural cross-sections. As *m* increases the bottom becomes increasingly sharp, until at $m = 1$ a sharp V-shaped section is obtained. If one were to continue increasing *m* this would correspond to sections with a deep valley (and in fact, infinite surface gradient), as shown by the dashed line for $m = 2$ in the figure. In some sense these could model some mountain streams, as well as the opening out of the river as it reaches bank level. However for more general natural topographies it seems not a particularly important case, and values of *m* only between 0 and 1 will be considered.

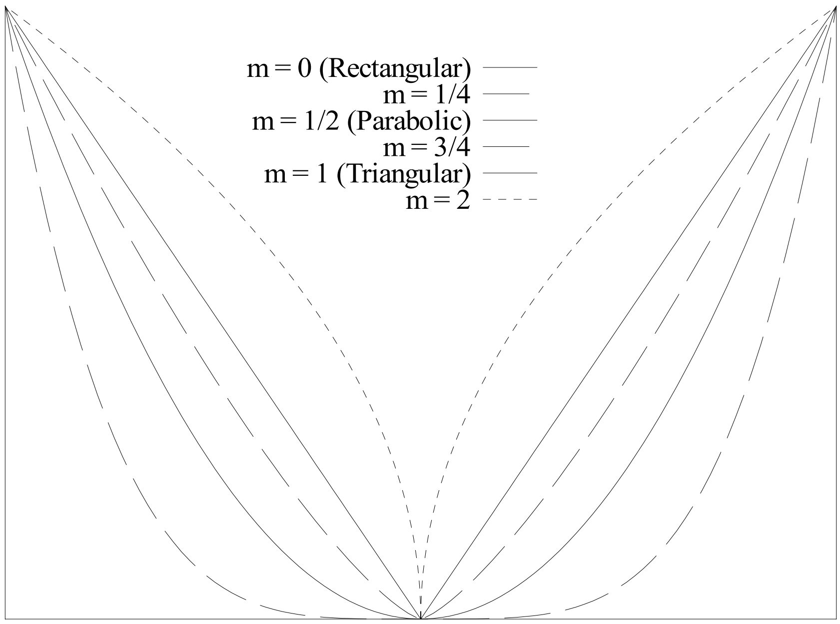


Figure 3. Cross-Sections Belonging to the General Family Given by the Monomial $b(z) = \beta z^m$.

## 3.1 Low Flows

For low flows it is assumed that there is a local downstream control, such as a weir constructed so as to provide that control, or a naturally-occurring feature such as a rock ledge or number of rocks in the stream. In Fenton and Keller (2001) it was shown theoretically that in the limit of low flow, even if there is a finite length of river between control and gauge, that the system acts like a reservoir with a horizontal surface, and the rating curve behaviour is precisely that of the downstream control. Two types of control structures are considered: in Appendix A the downstream control is modelled by a family of sharp-crested weirs, and in Appendix B by a similar family of broad-crested weirs, such as might be installed, or to approximate a natural feature. The surprising result was obtained that in both cases the behaviour is the same, even if the formula is different, and for both types of weir it is shown that $Q \sim H^{3/2+m}$, namely that the discharge varies like the total depth there to the power $3/2+m$ (of course the 3/2 law for the rectangular weir - $m = 0$ - is well known). More importantly, this means that the depth at the weir $H \sim Q^{1/(3/2+m)}$, and as it is necessary just to add on the difference between the datum for the weir and gauging station it is concluded that recorded stage at the gauging station should also vary like $Q^{1/(3/2+m)}$. Of course, it is not usually known what the value of *m* is – this problem will be addressed below.

### 3.2 Intermediate Flows

For larger flows, as the depth increases, the local control will drown out and will cease to control the flow at the gauging station. Possibly further downstream other local controls might occur, but it is just as likely that the control becomes a channel control, where the frictional nature of the waterway controls the flow. In Appendix C this is modelled by a uniform flow which is governed by Manning's law. Once again a family of sections described by the same monomials is considered, but in this case where the dimensions are those of the waterway as a whole, and not just a weir in it. The result found, after assuming that the waterway is rather wider than it is deep, so that some integrals can be evaluated, is that the depth of flow, and the inferred recorded stage at the gauging station should vary like $Q$ raised to the power $1/(m+5/3)$. Chézy's law is also considered, and an equivalent power of $1/(m+3/2)$ obtained - which happens to be the same value obtained for both sharp and broad-crested weirs. It seems remarkable that three different theories, based on three different assumptions and processes, should throw up such consistent results.

### 3.3 Large Flows

For flows which approach bank-full and over-bank flows, it is possible that a rather different value of $m$ could be used, possibly larger than 1, so that the section contains a deep central section and a convex-up shape on both sides away from that, corresponding to the widening of the river at the top of its banks, as shown in Figure 3. As this seems a very variable and uncertain situation, it will not be considered here.

### 3.4 Results

The results are summarised in Table 1, which considers five representative sections, ranging from rectangular through to triangular, and for each shape, what the exponent $\nu$ in $Q^{\nu}$ would be such that this quantity would vary linearly with stage according to four different determinants: sharp and broad-crested weirs and uniform channel flow according to Manning and Chézy friction laws. As already noted, three of these columns have the same values.

| Nature of weir cross-section or stream cross-section | Exponent $m$ | Exponent of $Q$ giving a linear relationship with stage | | | |
|---|---|---|---|---|---|
| | | Weir flow | | Uniform channel flow | |
| | | Sharp-crested weir $1/(m+3/2)$ | Broad-crested weir $1/(m+3/2)$ | Manning $1/(m+5/3)$ | Chézy $1/(m+3/2)$ |
| Rectangular | 0.00 | 0.67 | 0.67 | 0.60 | 0.67 |
| Shallower U-shaped | 0.25 | 0.57 | 0.57 | 0.52 | 0.57 |
| Parabola | 0.50 | 0.50 | 0.50 | 0.46 | 0.50 |
| Sharper U-shaped | 0.75 | 0.44 | 0.44 | 0.41 | 0.44 |
| V-shaped | 1.00 | 0.40 | 0.40 | 0.38 | 0.40 |

Table 1. Family of Cross-Sections Between a Rectangular and a V-shaped Stream and the Corresponding Exponent of $Q$ such that a Rating Curve based on Uniform Flows Would be a Straight Line

The important question then arises as to what value of the exponent should be used? It can be seen that over the whole table, numbers range from 0.38 to 0.67, and for a value of 0.5 to correspond to sections which have a finite, intermediate curvature. As in Fenton and Keller (2001), who did not produce as detailed a study as this, a value of about 0.5 should be an average representative value, if a single value is to be chosen. A value of 0.4 might be too extreme, as in many local controls by natural features, a well-developed V-notch section to be relatively rare, and features like rock ledges, providing the impoundment at low levels, to be much more common, while at higher flows a flat-bottomed stream is still much more likely. In general, both on the scale of local controls and stream cross-sections, U-sections are rather more likely than V-sections.

The value of 0.5 implies that the stage-discharge relationship will tend to show stage varying approximately like $\eta \sim Q^{1/2} = \sqrt{Q}$, which has a certain charm to it as the square root function is more familiar and standard than any other fractional power. Also, its inverse, the square function is slightly easier to handle and make interpretations from using mental arithmetic than the inverse function of the 0.4 scale raising quantities to the power 2.5. If special plotting paper were drawn up, with tick marks at major values as with a log scale, this would not matter, but in practice with non-specialist standard software as used to produce the figures, with equally-spaced tick marks, the square root scale is simpler.

For these reasons, as yet based on few examples, the idea of plotting stage-discharge relationships on $\left(\sqrt{Q},\eta\right)$ axes is pursued. At low flows the relationship should be roughly linear, such that it should be relatively easy and accurate to establish the cease-to-flow point. At larger flows the relationship should also be roughly linear, but not necessarily the same relationship as the low-flow one at all. For higher flows some deviation from linearity is expected. Using these scales should contract large flows and expand small flows on the plots. The tendency to be linear, at least at the bottom end and without marked curvature at the upper end, means that it may be possible to use global means of approximation, as the curves should show rather less irregular behaviour. It is not expected that any curve will approximate a single straight line.

# 4 USE OF $\left(\sqrt{Q},\eta\right)$ SCALES FOR REPRESENTING RATING CURVES

The data from the previous example are shown plotted on $(\sqrt{Q},\eta)$ axes in Figure 4. It can be seen that the low-flow points still collapse into a relatively small region, but they do locally form a straight line of finite gradient. The low flow region is shown, the behaviour is clear, but unlike the log-log plot it does not dominate the plot. One can still extract low-flow information, but in that region the results resume their real importance, and plotting stage to within 1cm vertically would be satisfactory without moving a point an apparently large distance as on the log-log plot. It is clear that a piecewise-linear representation of the rating curve would be adequate. The two data points for high overbank flow, unsurprisingly, do not seem to follow the trend of the previous data (they were obtained partly from readings taken from a boat over flooded farmland!). In general, for such a discontinuity it might be better to use something like piecewise linear approximation.

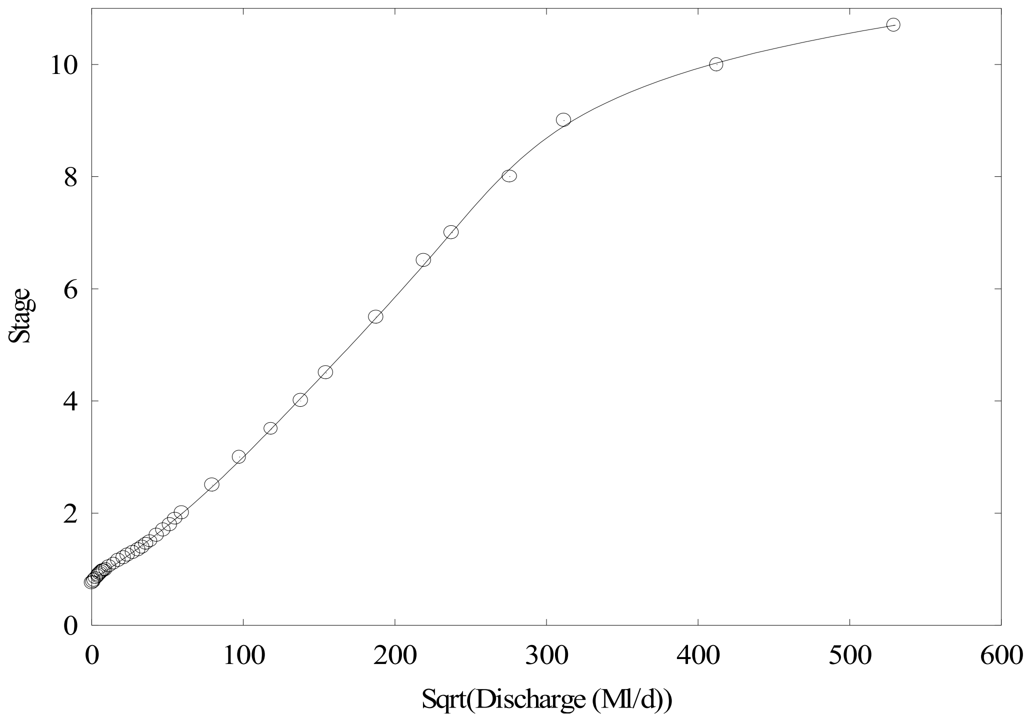


Figure 4. Rating Curve for Pallamallawa, Showing the Data and a 6th degree Polynomial Fit.

However, here the possible power of a global approximation is shown in *approximating* the actual rating data so as to automatically generate data for the rating curve at Pallamallawa. A trial is made of a global approximation, expressing $\sqrt{Q}$ as a sixth-degree polynomial in stage, using the methods described in the Appendices of Fenton and Keller (2001). The results shown in Figure 4, are encouraging. At the cease-to-flow end of the data, the polynomial was able to describe the region dominated by the local control. Possibly even more usefully, it seems able to make a plausible continuous relationship that incorporates the two high flow points.

An advantage that the power plots have, for both 2/5 and 1/2 exponents, is that the treatment of the cease-to-flow point is more satisfactory than with a log-log scale. If the cease-to-flow point is known, it can be plotted without any special treatment. If it is not known, it seems that the tendency of the points to lie on straight or very-nearly straight lines provides a reasonable way of extrapolating the low-flow data to determine the cease-to-flow stage. On the other hand, using a log-log scale, if the cease-to-flow point is not known it has to be found by using an approximation such as equation (1) and finding $\eta_0$ such that the expression plots as a straight line, all requiring non-trivial operations.

## 5 CONCLUSIONS AND RECOMMENDATIONS

The hydraulics of three different possible determinants of parts of the range of rating curves have been considered, namely both sharp and broad-crested weirs and uniform flows. By considering a family of cross-sections, from rectangular through U-shaped to triangular cross-sections, it was found that in most cases there was a tendency for the variation of stage to be like the square root of the discharge, so that if the stage and the square root of discharge were used for plotting rating curves, important parts of them would appear as straight lines, or close to straight lines. This would make the determination of the cease-to-flow point rather easier, and make approximation and description of the curves rather better.

This work has concentrated on exploring the hydraulics, especially in the Appendices, and has provided relatively few examples of real rating curves and their representation on axes of stage and the square root of discharge. A number of extra examples have been provided in Fenton and Keller (2001), which support the conclusions here.

## 6 ACKNOWLEDGEMENTS

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## APPENDIX A. DISCHARGE OVER SHARP-CRESTED WEIRS

Consider the downstream control to be modelled by a sharp-crested weir of arbitrary width $b(z)$, where $z$ is the surface elevation above the cease-to-flow point. Applying the conventional theory of sharp-crested weirs the general expression is obtained:

$$Q = C \int_0^H b(z) \sqrt{2g(H-z)}\, dz , \tag{A.1}$$

where $Q$ is discharge, $C$ is a coefficient of discharge, $g$ is gravitational acceleration, and $H$ is the total depth above the cease-to-flow point. While the hydraulics of this conventional approach are very poor (pressure over the crest is not zero as assumed, horizontal velocity distribution is also not as assumed) it seems to work well in practice, as the coefficient $C$ shows relatively little variation with head for a particular shape.

Now (A.1) for monomial variation is evaluated, $b(z) = \beta z^m$, which incorporates and generalises some simple cross-sections as shown in Figure 3. Equation (A.1) now becomes

$$Q = C \int_0^H \beta z^m \sqrt{2g(H - z)}\, dz \,. \tag{A.2}$$

The variable $\theta = z / H$ is introduced, and (A.2) becomes

$$\begin{aligned} Q &= C\beta\sqrt{2g}\, H^{3/2+m} \int_0^1 \theta^m \sqrt{1-\theta}\, d\theta \\ &= C\beta\sqrt{2g}\, H^{3/2+m}\; B(m+1, 3/2) \end{aligned} \tag{A.3}$$

where $B(m+1, 3/2)$ is a Beta function (see Abramowitz and Stegun, 1965). For our purposes the numerical value of the result is not so important – what has been shown is that the dependence of discharge on head is like $H^{3/2+m}$.

Some special cases are:

Rectangular weir: $m = 0$, and $b(z) = \beta = B$, the constant width of this rectangular weir, the familiar expression is obtained:

$$Q = \tfrac{2}{3} C\beta\sqrt{2g}\, H^{3/2} \,.$$

The theory on which this is based has very little theoretical validity, but when the coefficient of discharge *C* is studied experimentally, it is found not to vary much with head *H*, and practically it can be deduced that for a downstream control which is a rectangular weir, possible if one such were constructed but admittedly rather unlikely if the downstream control were a natural feature, then the head over the weir varies like *Q* to the power $2/3 \approx 0.67$.

For a triangular or V-notch weir, $m = 1$, $\beta = 2 \times \tan(\theta / 2)$:

$$Q = \tfrac{4}{15} C\beta\sqrt{2g}\, H^{5/2} \,.$$

Also in this case when the coefficient of discharge *C* is studied experimentally, it is found not to vary very much with head *H*, and practically it can be deduced that for a downstream control which is a triangular weir, then the head over the weir varies like *Q* to the power $2/5 = 0.4$. It was this result that led Chester (1986) to his original suggestion.

In many cases in nature, however, it is unlikely that the downstream control will be a notch with a sharp V-shape, but may well be more like a U-shape, as shown by some of the intermediate cases in Figure 3. For a parabola $m = 1/2$, half-way between the above two cases, then $b(z) = \beta\sqrt{z}$, and

$$Q = \tfrac{\pi}{8} C\beta\sqrt{2g}\, H^2 \,,$$

with the deduction that head over the weir varies like *Q* to the power $1/2 = 0.5$. This might also be the value if the control were trapezoidal in cross-section, having a horizontal crest and side walls which are sloping straight lines, a combined rectangular and triangular weir, with the effective exponent being somewhere between the values of 0.67 and 0.4 of those two cases, possibly not far from the value here of 0.5.

## APPENDIX B. DISCHARGE OVER BROAD-CRESTED WEIRS

Now consider a family of broad-crested weirs, of the same cross-sections as shown in Figure 3. Some natural controls will be more of this nature for low flows, when the control might, for example, be a rock ledge. The basic theory used here is the rather conventional simple one, that flow upstream of the weir is subcritical, that the broad-crested weir allows transition to supercritical flow, and that tailwater does not interfere with this.

Surprisingly, in this case the simple theory is more applicable than is the case for thin-plate weirs described above. The expression for the specific energy relative to the channel bottom is

$$E = H + \alpha \frac{Q^2}{2gA^2}, \tag{B.1}$$

where $\alpha$ is a Coriolis energy coefficient, with a magnitude of roughly 1.1, which is included here for generality, but it does not affect our results. Assume that critical flow on the broad-crested weir will provide the control, the depth being such that energy is a minimum. Now, it is easily shown that if the breadth $b(z) = \beta z^m$, then the area

$$A = \frac{\beta H^{m+1}}{m+1}. \tag{B.2}$$

Substituting into (B.1), differentiating with respect to $H$ and setting to zero for the minimum, gives

$$H = \left( \frac{\alpha Q^2 (m+1)^3}{g\beta^2} \right)^{1/(3+2m)}. \tag{B.3}$$

However, the stage upstream at the gauging station, assuming that for these low flows that the Froude number there is small so that $E = \eta$, giving

$$\eta = \left( \frac{\alpha Q^2 (m+1)^3}{g\beta^2} \right)^{1/(3+2m)} + \alpha \frac{Q^2}{2gA^2}.$$

Substituting (B.2) and then (B.3), after some manipulations the surprisingly simple result for the stage relative to the crest of the weir is obtained

$$\eta = \left( \frac{\alpha Q^2 (m+1)^3}{g\beta^2} \right)^{1/(3+2m)} \left( 1 + \frac{1}{2(m+1)} \right).$$

This shows that the ratio of the depth at the gauging station to that over the broad-crested weir is $1 + 1/2(m+1)$. This is a generalisation of the well-known result for rectangular broad-crested weirs ($m = 0$) that the ratio is $3/2$. For our purposes this is not so important, however. What is important is the result that the stage varies like $Q$ to the power $2/(3+2m) = 1/(3/2+m)$, which is precisely the result that obtained above for sharp-crested weirs using a different theory!

## APPENDIX C

Now consider the case where the local control is drowned out and effectively it is the frictional nature of the channel which provides the control. To first approximation flow past the gauging station will be assumed to be uniform, given by Manning's and Chézy's laws.

Consider a channel cross-section given by the monomial $b(z) = \beta z^m$, where the same terminology as above is used, where this expression was used for both thin-plate and broad-crested weirs. The channel bottom is at $z = 0$ and the surface at $z = H$. It is convenient to consider a symmetrical channel, with the equation of the bank, relative to the centreline, to be

$$y(z) = b(z)/2 = \beta/2 \times z^m. \tag{C.1}$$

The cross-sectional area of the section is given by (B.2). The wetted perimeter of the section is given by

$$P = 2\int_0^H \sqrt{1 + \left(\frac{dy}{dz}\right)^2}\, dz.$$

In fact, on substituting (C.1) for $y(z)$ this expression cannot be evaluated in terms of simple functions. It is easier to re-write it as

$$P = 2\int_0^H \frac{dy}{dz}\sqrt{\left(\frac{dz}{dy}\right)^2 + 1}\, dz ,$$

where it is recognised that in most rivers the depth is much less than the width, so that the square of the slope $(dz/dy)^2$ is a small quantity and the binomial expansion of the square root can be used to give

$$P \approx 2\int_0^d \frac{dy}{dz}\left(1 + \frac{1}{2}\left(\frac{dz}{dy}\right)^2\right) dz .$$

Substituting (C.1) gives

$$P = \beta\, H^m \left(1 + \frac{H^{2-2m}}{m(1-m/2)\beta^2}\right), \tag{C.2}$$

showing how the wetted perimeter varies with total depth above the cease-to-flow point. A bit more insight into this can be gained if it is recognised that the surface width *B* is given by $B = \beta\, d^m$ , and the expression can be written in terms of total depth and surface width as

$$P = B\left(1 + \frac{1}{m(1-m/2)} \times \frac{H^2}{B^2}\right).$$

To first order the wetted perimeter is equal to the surface width, while there are corrections proportional to the square of the ratio of depth to width, which will be a small quantity in many circumstances.

Now using Manning's equation for steady uniform flow

$$Q = \frac{A}{n}\left(\frac{A}{P}\right)^{2/3}\sqrt{\bar{S}} ,$$

where *n* is the friction coefficient, and $\bar{S}$ is the mean slope. Substituting (B.2) for area *A* and (C.2) for perimeter *P* gives

$$Q = \frac{\beta\sqrt{\bar{S}}}{n(m+1)^{5/3}} \times H^{m+5/3} \times \left(1 - \frac{2}{3}\frac{H^{2-2m}}{m(1-m/2)\beta^2}\right).$$

This seems to be too complicated for simple deductions, so the second term in the brackets will be neglected, which has already been shown to be small. The result for is that in a waterway where the breadth varies like depth to the power *m*, the discharge varies like depth to the power $m+5/3$. Solving for *H* gives $H \sim Q^{1/(m+5/3)}$ , and as stage is given by the depth plus an arbitrary reference level, it is concluded that if the flow were uniform at all depths there should be a linear relationship between stage and $Q^{1/(m+5/3)}$ . If Chézy's friction law had been used, the fraction $5/3$ would be replaced by $3/2$ .