WATERCOMP '89 The First Australian Conference on Technical Computing in the Water Industry
Melbourne 30 May-1 June 1989

# Computing Floods and River Flows Using Quasi-Characteristics

C. ZOPPOU
Senior Engineer, Hydrology and Water Resources Unit, ACT Electricity and Water
J.D. FENTON
Professor of Fluid Mechanics, Department of Civil Engineering, University Auckland

SUMMARY This paper examines the quasi-characteristic method for the solution of the Saint Venant equations governing the motion of flows, waves and floods in rivers and channels. The apparently new approach combines the more desirable features of traditional numerical methods, but does not seem to suffer from their disadvantages. It is explicit, efficient, accurate and simple to implement. However, it is shown that the existing formulation has demanding stability criteria if friction, inflow and/or a non-prismoidal channel are considered. A simple modification of the method yields a scheme which has finite stability criteria but which are not particularly restrictive. The performance of the method is compared with existing numerical methods in the solution of a particular problem.

## 1. INTRODUCTION

Although the one-dimensional equations describing unsteady open channel flow were first derived by Saint Venant over a century ago, it has only been in the last decade or so that significant research and effort have been devoted to solving these equations using numerical methods.

The Saint Venant equations can be written in the form:

$$\frac{\partial h}{\partial t} = -u\frac{\partial h}{\partial x} - \frac{A}{B}\frac{\partial u}{\partial x} + \frac{q}{B} - u\frac{A_x}{B}, \qquad (1)$$

and

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} - u\frac{\partial u}{\partial x} + gS_o - gS_f + \frac{q}{A}(u_i - u), \qquad (2)$$

$h$ = depth of flow,
$u$ = mean velocity of flow,
$t$ = time,
$x$ = distance along channel,
$g$ = acceleration due to gravity,
$A$ = cross-sectional area of flow,
$B$ = top width of flow,
$S_o$ = bed slope,
$S_f$ = resistance slope, given by the Manning equation $= n^2u^2/R^{4/3}$, in which $R$ = hydraulic radius,
$n$ = resistance coefficient in the Manning equation,
$u_i$ = $x$-Component of lateral inflow velocity,
$q$ = lateral inflow per unit length, and
$A_x$ = $\partial A/\partial x\,|_h$ is a measure of the departure of the channel from a prismatic cross-section.

The numerical solution of the two quasi-linear partial differential equations, (1) and (2), is usually accomplished by the use of the method of characteristics or finite difference methods. Although numerous models based on these approaches have been proposed, they all suffer from several limitations. The recently-developed method of quasi-characteristics combines the most desirable features of the more conventional methods (Fenton, 1985). The new approach is simple and explicit. In its original presentation, however, much was made of the fact that the method is unconditionally stable. Subsequent inclusion of friction, inflow and non-prismoidal terms showed that this was not the case. In this paper the stability of the method is examined and *via* the stability analysis a modification is proposed which renders the method stable. Stability criteria are developed, and these are found to be relatively generous.

## 2. COMPUTATIONAL METHODS

A survey of the computational methods used to solve the St. Venant equations numerically has been given by Zoppou and O'Neill (1981), and rather longer discussions are in Liggett and Cunge (1973) and Zoppou (1979). The more common methods can be divided into three groups: those which use the method of characteristics, those which use finite difference methods, and those which use a combination of the two.

### 2.1 Characteristic-based methods

In the method of characteristics the original partial differential equations are converted into an equivalent, but simpler, system of ordinary differential equations. These equations, which are exact, are solved by integrating numerically over the region in $(x, t)$ space. The numerical technique used to perform the required integration determines the accuracy of the solution. A disadvantage of the method is that information is obtained at irregular values of $x$ and $t$. However, the method is robust and can produce greater detail in regions of rapid change. It is ideal for modelling the propagation of bores.

The method of Specified Interval Characteristics was developed to overcome the difficulties associated with the variable grid produced by the conventional characteristic method. Here a rectangular computational grid is employed, but the essential travelling wave nature of the characteristic method is retained.

Unfortunately, there are circumstances where the application of the method of characteristics will not provide a convenient formulation to a problem. For example, the trivial change of including off-stream storage produces a significant increase in the complexity of the characteristic equations (Liggett, 1968). Schemes based on the method of characteristics however are generally stable, accurate and efficient.

### 2.2 Finite-difference schemes

Because of their general applicability, finite difference schemes have gained a wide acceptance for the numerical solution of hydraulic problems. In these schemes the partial derivatives in the original equations are replaced by difference quotients, and the solution is obtained at discrete points on a rectangular lattice in the $(x, t)$ plane.

Finite difference schemes can be divided into two groups, either explicit or implicit. In explicit schemes the finite-difference equations are written for each computational point on the forward time level in terms of the known values at grid points on the previous time level. The explicit scheme produces a large number of simple linear equations which can be solved directly for the unknown dependent variables on the forward time level. Strict adherence to the Courant criterion limiting the time step is necessary (Roache, 1972). This represents a severe limitation on the computational time increment that can be used by explicit models. With the exception of rapidly varying transients, explicit models are seldom used for the simulation of floods in natural channels.

In implicit schemes the partial differential equations are written in finite difference form which relate unknowns on the forward time level to values on the previous time level. Usually a system of simultaneous equations must be solved. Although implicit finite difference schemes are stable under certain conditions, large time steps may be used at the sacrifice of accuracy. Unfortunately, the spatial and temporal approximations used in finite difference schemes introduce numerical diffusion.

With the exception of numerical schemes based on the method of characteristics, all the conventional numerical methods neither reveal nor utilise the propagating wave features of the solutions they are describing.

## 3. THE QUASI-CHARACTERISTIC METHOD

This method has been described in detail by Fenton (1983, 1985). It was claimed to be unconditionally stable, non-diffusive, and exact for equations with constant coefficients. It will be shown below that the first statement is true only for idealised problems which do not involve friction, inflow or which consider prismoidal channels.

Equations (1) and (2) can be shown, after some complicated mathematics, to yield the relatively simple expressions

$$h(x, t+\Delta t) = \frac{1}{2}(h_+ + h_-) + \frac{c(x,t)}{2g}(u_+ - u_-) + \Delta t\,(q/B - uA_x/B) + O(\Delta t^2), \qquad (3)$$

and

$$u(x, t+\Delta t) = \frac{1}{2}(u_+ + u_-) + \frac{g}{2c(x,t)}(h_+ - h_-) + \Delta t\,(g(S_o - S_f(u)) + q(u_i - u)/A) + O(\Delta t^2), \qquad (4)$$

where $h_\pm = h(x_\pm, t)$, and $u_\pm = u(x_\pm, t)$, (5)

and where $x_\pm = x - \Delta t\,(u(x,t) \pm c(x,t))$. (6)

Function values at $(x, t+\Delta t)$ are obtained from values at $(x_\pm, t)$ values which must be interpolated from known point values at time level $t$. As such, the equations provide an unusual method of solution. There is no attempt to approximate derivatives, they form an interpolation-only scheme for advancing the solution in time. The actual process of interpolation can be carried out by any means, and in this formulation there is total freedom so to do. It was found that spline approximation, whether cubic, exponential or taut could be used with high accuracy. The method requires no low-order approximation using finite differences.

The method is most closely related to grid-orientated characteristic schemes where straight line approximations to the characteristics are used. Indeed, such approximations would yield schemes such as this. The important difference is, however, that the present approach can be used for systems where no characteristic invariants exist. Such a case is the stage-discharge formulation of the St. Venant equations, for which equations equivalent to (3) and (4) were presented by Fenton (1985). The method is of first order accuracy, however schemes of higher accuracy can be developed (Fenton, 1983).

In the 1985 paper Fenton presented the results of several model computations for the motion of waves in canals for the case of prismatic channels with no friction. The method was found to perform very well. However, when one of us (C.Z.) attempted to implement the method for some practical calculations including the effects of friction, the method was found to have finite stability criteria, and indeed to become unstable for time steps of the order of those used by conventional finite difference schemes.

## 4. MODIFIED QUASI-CHARACTERISTIC SCHEME

### 4.1 Stability analysis

In this section a stability analysis of a family of quasi-characteristic schemes is presented. This is much more than of theoretical interest because precise limits can be given for the stability of the original method. More importantly, however, it enables a simple modification to the original scheme which has much better stability properties. Here the scheme given by equations (3) and (4) is written in modified form

$$h(x, t+\Delta t) = \frac{1}{2}(h_+ + h_-) + \frac{c(x,t)}{2g}(u_+ - u_-) + \Delta t\,(q/B - u_* A_x/B) + O(\Delta t^2), \qquad (7)$$

and

$$u(x, t+\Delta t) = \frac{1}{2}(u_+ + u_-) + \frac{g}{2c(x,t)}(h_+ - h_-) + \Delta t\,(g(S_o - \alpha u_*^2) + q(u_i - u_*)/A) + O(\Delta t^2), \qquad (8)$$

where $u$ in the inflow, friction and non-prismoidal terms has been replaced by $u_*$, an as-yet unknown velocity scale, and the friction slope $S_f$ has been replaced by $\alpha u_*^2$, where $\alpha$ is given by $n^2/R^{4/3}$. In common with most stability analyses it is necessary to linearise these equations about some undisturbed state. The equations

$$h = h_o + y\,,\ u = u_o + v\,,\ c = c_o + \cdots,$$

are substituted, where $c_o = \sqrt{gd_o}$, $d_o$ is the depth corresponding to a steady flow dominated by friction, $u_o$ is the corresponding velocity such that $S_o = \alpha u_o^2$, and $y$ and $v$ are small deviations from that steady flow. In this presentation the dependence of $n$ on $h$ is ignored.

Strictly speaking, one should consider the stability of small perturbations to the computational scheme, however it is notationally simpler to replace the scheme with its linearised equivalent and neglect the terms $\Delta t q/B$ in equation (7) and $\Delta t q u_i/A$ in equation (8) which play no role in the stability. The frictional term $\Delta t\, g(S_o - \alpha u_*^2)$ becomes $-2\,\Delta t\,\alpha g u_o v_*$ after linearising.

The linearised computational scheme becomes

$$y(x, t+\Delta t) = \frac{1}{2}(y_+ + y_-) + \frac{c_o}{2g}(v_+ - v_-) - \Delta t\,(v_* A_x/B), \quad (9)$$

and

$$v(x, t+\Delta t) = \frac{1}{2}(v_+ + v_-) + \frac{g}{2c_o}(y_+ - y_-) - \Delta t\,\alpha g u_o v_* - \Delta t\, q v_*/A, \quad (10)$$

in which $x_\pm = x - \Delta t\,(u_o \pm c_o)$.

It is assumed that $y$ and $v$ can be represented by a sine wave of complex amplitude $Y(t)$ and $V(t)$ respectively:

$$y(x,t) = Y(t)\,e^{ikx}, \quad \text{and} \quad v(x,t) = V(t)\,e^{ikx}, \quad (11)$$

where $k$ is the wavenumber of the disturbance. Substituting these into equations (9) and (10) and dividing through terms like $\exp ik(x - u_o\Delta t)$ gives two equations which can be represented by the matrix difference equation

$$\begin{bmatrix} Y(t+\Delta t)\,e^{iku_o\Delta t} \\ V(t+\Delta t)\,e^{iku_o\Delta t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} Y(t) \\ V(t) \end{bmatrix}, \quad (12)$$

where the matrix is given by

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \frac{c_o}{g}(-i\sin\theta - \gamma\phi) \\ \frac{-ig}{c_o}\sin\theta & \cos\theta - 2\beta\phi \end{bmatrix}, \quad (13)$$

in which $i = \sqrt{-1}$, $\gamma$ is the non-dimensional number corresponding to the non-prismatic nature of the channel:

$$\gamma = \frac{\Delta t A_x g}{c_o B}, \quad (14)$$

$\theta$ is the dimensionless number $\theta = k c_o \Delta t$, a measure of how far the computational solution proceeds in a single time step, $\beta$ is the dimensionless number dominated usually by friction in the channel but containing an inflow term:

$$\beta = \Delta t\,\alpha g u_o + \frac{1}{2}\frac{\Delta t\, q}{A}, \quad (15)$$

and where $\phi$ is a phase number which depends on the velocity $u_*$ used in the computational scheme. It will be seen that the stability of the method rests on the choice of this quantity.

For the scheme to be stable, the magnitudes of all the eigenvalues of the matrix must be less than unity (or equal to unity for neutral stability). The eigenvalues $\lambda$ are given by solutions of the quadratic

$$\lambda^2 + 2\lambda(\beta\phi - \cos\theta) + 1 - 2\beta\phi\cos\theta - i\gamma\phi\sin\theta = 0. \quad (16)$$

If the phase number $\phi$ is real then using some tedious and rather lengthy mathematics involving the Schur-Cohn criterion for the magnitudes of the eigenvalues gives the criteria to be satisfied if the scheme is to be stable:

$$\gamma^2 \le 4\beta^2, \quad (17)$$

$$\gamma^2\phi^2\sin^2\theta \le 4\cos^2\theta\,(1 - \beta^2\phi^2), \quad (18)$$

$$\gamma^2\phi^2\sin^2\theta \le 4\beta\phi\cos\theta\,(1 - \beta\phi\cos\theta). \quad (19)$$

## 4.2 Consideration of alternative schemes

Now, three computational alternatives will be considered.

*Scheme 1:* $u_* = u(x, t)$

This is the original scheme as proposed in Fenton (1985), which yields equations (3) and (4), and is intuitively the most obvious to use: the velocity at point $(x, t)$ to be used in non-prismatic, frictional and inflow terms is the velocity at that point and time. In this case it can be shown that $\phi = \exp iu_o\Delta t$, which is complex, and the criteria for stability are very difficult to obtain analytically. Simple numerical experimentation with equation (16), showed that provided $|\gamma| \le 2\beta$ it was conditionally stable, however demanding limitations were placed on the Courant number $C = c_o\Delta t/\Delta x$ ($\Delta x$ the space step) in the computations. It is recommended that this scheme not be used.

*Scheme 2:* $u_* = u(x - u\Delta t, t)$

This is the most obvious alternative, as it builds in the advective nature of the flow problem: the appropriate velocity at $(x, t)$ is that which was upstream at a distance such that it arrives at point $x$ at time $t + \Delta t$. In this case $\phi = 1$ and the criteria (17-19) become

$$\gamma^2 \le 4\beta^2, \quad (20)$$

$$\gamma^2 \le 4\frac{\cos^2\theta}{\sin^2\theta}(1 - \beta^2), \quad (21)$$

$$\gamma^2 \le 4\frac{\beta\cos\theta}{\sin^2\theta}(1 - \beta\cos\theta). \quad (22)$$

The presence of the cosine function in the numerator of the right hand sides means that if $\theta \to \pi/2$ that side goes to zero, and no stable computations can be performed for a finite value of $\gamma$. This can easily be shown to be a limit such that the Courant number $C \le 1/2$, slightly better than in the above Scheme 1.

*Scheme 3:* $u_* = \frac{1}{2}(u_+ + u_-)$

This scheme is an implicit recognition of the characteristic nature of the method: the velocity to use is the mean of that obtained by travelling along the quasi-characteristics, the same quantity used in the first term in equation (3). In this case, $\phi = \cos\theta$, and the occurrence of the cosine terms on both sides of the inequalities (17-19) means that they can be cancelled, giving the criteria

$$\gamma^2 \le 4\beta^2, \quad (23)$$

$$\gamma^2 \le \frac{4}{\sin^2\theta}(1 - \beta^2\cos^2\theta), \quad (24)$$

$$\gamma^2 \le \frac{4\beta}{\sin^2\theta}(1 - \beta\cos^2\theta). \quad (25)$$

To ensure that none of the right hand sides goes to zero it is necessary that $\beta \le 1$, and then a comparison of the hierarchy of inequalities shows that the only criteria for stability of the method is that

$$\beta \leq 1, \quad \text{and} \quad |\gamma| \leq 2\beta. \tag{26}$$

The scheme is thus conditionally stable, but with criteria dependent only on the friction and non-prismoidal terms. Unlike many other computational schemes there is no restriction on the Courant number, and this scheme might prove useful in practice. It is helpful that the criterion for $\gamma$ in equation (26) involves the *magnitude* of $\gamma$, so that it is the same for channels which increase or decrease in area.

An important special case is that where the channel is prismoidal, but where friction exists. This is not a problem, as the criterion $\beta \leq 1$ can still be satisfied. It is interesting though, that even this method would be unstable for a non-prismoidal channel with no friction.

In view of the above, it seems that this is the scheme to use in practical computations, and so it is recommended that in all the additional terms in equations (3) and (4) that the local velocity $u$ be replaced by $1/2\,(u_+ + u_-)$.

### 4.3 Interpretation of dimensionless numbers

Consider the dimensionless number $\beta$ as defined in equation (15):

$$\beta = \Delta t \alpha g u_o + \frac{1}{2}\frac{\Delta t\, q}{A}$$

$$= \frac{\Delta t\, g\, \alpha u_o^2}{u_o} + \frac{1}{2}\frac{\Delta t\, q}{A}$$

$$= \frac{\Delta t\, g\, S_o}{u_o} + \frac{1}{2}\frac{\Delta t\, q}{A}. \tag{27}$$

In this form $\beta$ is capable of some physical interpretation. Equation (2) shows that $gS_o$ is the contribution of bed slope to the fluid acceleration, hence the numerator $\Delta t\, g\, S_o$ is the change of velocity due to the bed slope alone over one computational time step. The term $\Delta t\, g\, S_o/u_o$ is then a dimensionless bed slope number, equal to the change of velocity of a free fluid particle in one time step divided by a typical actual fluid velocity. That is, it is the fractional change of fluid velocity due to the bed slope in a time step.

The second term in $\beta$ has a similar interpretation: $\Delta t\, q$ is the volume of inflow entering the channel per unit length in a computational step, that is, it is the area rate of inflow, and dividing by the actual area $A$, the whole term is the fractional change of flow area in a computational step due to inflow. This would be expected to be very small. It is not clear physically why the weighted sum of these two fractional contributions, one to fluid velocity, the other to flow area, should be less than unity (equation (26)), or any other definite amount.

The other limitation on the flow step in equation (26), $|\gamma| \leq 2\beta$ is also capable of some physical interpretation. Here the inflow contribution to $\beta$ is neglected. Writing the ratio of the terms gives

$$\frac{\gamma}{\beta} = \frac{\Delta t\, A_x\, g}{c_o B}\;\frac{u_o}{\Delta t\, g\, S_o} = \frac{u_o A_x/B}{c_o S_o}. \tag{28}$$

From equation (1) the numerator is the contribution to the rate of increase of flow depth of the non-prismoidal term. The denominator is the rate of change of elevation experienced by a disturbance as it propagates at the wave speed $c_o$ while travelling over a slope $S_o$. Thus an interpretation of the ratio $\gamma/\beta$ is that it is a dimensionless measure of the non-prismoidal term, measuring the relative contribution to the rate of change of water depth caused by the non-prismoidal term to the rate of change of the bottom elevation as experienced by a disturbance. Unfortunately it is not clear physically why this number should be limited to 2, as given by equation (26).

## 5. COMPARISON OF NUMERICAL METHODS

The results from the quasi-characteristic method for a model problem will be compared here with those from various schemes, including a nonlinear implicit finite difference model with Newton Raphson iteration, a linear implicit finite difference model with the double sweep algorithm, an explicit method, specified interval characteristics and a characteristic grid based model.

With the exception of the explicit model a similar computational time increment was adopted for all the models. The explicit model is based on the diffusion finite difference scheme with upwind and downwind differencing used at the boundaries (Roache, 1972). The specified interval and characteristic grid models used second order integration. The quasi-characteristic model is only of first order accuracy, however more accurate schemes may be implemented (Fenton, 1983).

A 10 kilometre long rectangular channel was chosen for the comparisons. It was 7.0 meters wide and had a bed slope of 0.001. The Manning resistance coefficient was 0.015. A steady uniform rating curve was used as the downstream boundary condition and a Pearson Type III distribution was used to define the upstream discharge hydrograph. The simulated discharge hydrographs, at the downstream extremity of the channel, obtained using these numerical models are shown in Figure 1. For this example the results indicate that all the models produce very similar results.

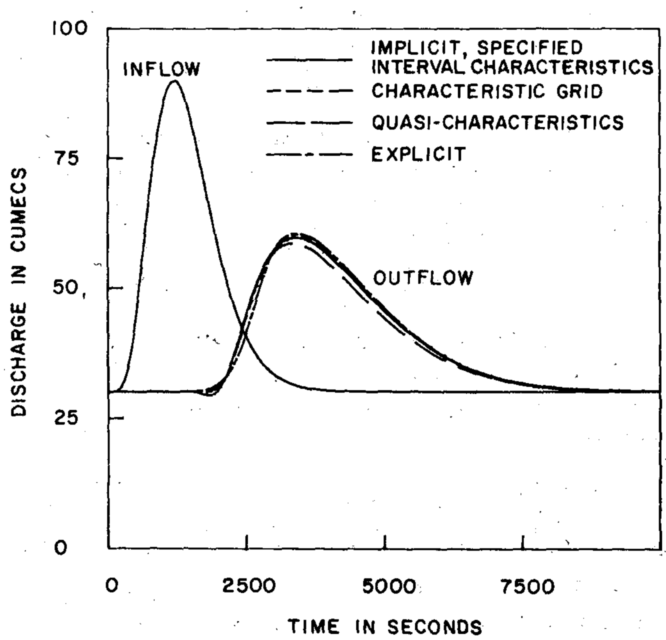


**Figure 1.** Comparison between traditional numerical models and the quasi-characteristic method

Reducing the computational time increment produced closer agreement between the model results. The computer resources required and the computational time increment used for each scheme are shown in Table 1. The real advantage of the quasi-characteristic scheme lies in its simplicity of coding, for it separates the operations of spatial approximation and time stepping, and the fundamental time-stepping can be carried out with the relatively simple expressions (7) and (8), where the velocity to be used in the non-wave-like terms, is $u_* = \frac{1}{2}(u_+ + u_-)$.

A computational time increment of 10 seconds was used for the explicit model because it was unstable for larger time increments. Although the explicit model is the most efficient the hypothetical example represents a rapid variation in comparison with floods in natural channels. For practical problems numerical models with an unrestricted computational time increment are the most flexible and efficient. With the exception of the explicit and the nonlinear implicit finite difference model all the numerical models require similar computer resources.

| MODEL | COMPUTATIONAL TIME INCREMENT (seconds) | EXECUTION TIME (seconds) |
|---|---|---|
| Linear implicit | 60 | 13.2 |
| Nonlinear implicit | 60 | 74.4 |
| Characteristic grid | NA | 13.5 |
| Specified interval Characteristics | 60 | 10.8 |
| Explicit | 10 | 3.5 |
| Quasi-Characteristics | | |
| (Linear interpolant) | 60 | 14.2 |
| (Cubic interpolant) | 60 | 14.7 |
| (Taut interpolant) | 60 | 15.0 |

**Table 1.** Computer resources required for the example.

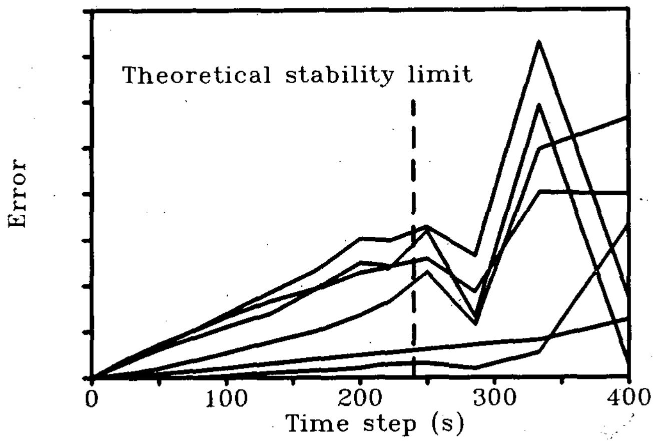


**Figure 2.** Demonstration of computational instability beyond the theoretical limit

A range of computational time increments was used with the quasi-characteristic model. Figure 2 shows the behaviour of the scheme as it depends on the computational time step used. The different curves correspond to the error at different points in the reach at the end of computations, having reached a given time. The theoretical stability limit $\beta \leq 1$ gave a limiting time step of 240s, much greater than that used in computations. It can be seen that up until this time step, the accumulation of error is proportional to the time step, as would be expected from a scheme which has a truncation error of order $O(\Delta t^2)$, but where the number of time steps required to reach a certain time level is proportional to $\Delta t^{-1}$. Beyond this limiting time step, where the method is theoretically unstable, the error accumulation starts to fluctuate wildly, showing that the theoretical limit does seem to be valid in practice.

## 6. CONCLUSIONS

The quasi-characteristic model is explicit, efficient, and simple to implement. The method produces results at predefined computational points and any method of spatial approximation can be used. Perhaps the most notable feature of the method is that it seems to be a general technique applicable to a wide range of hydraulic problems. Higher order schemes are simple to formulate.

## 7. REFERENCES


Fenton, J.D. (1983) A Taylor series method for numerical fluid mechanics, *Proc. 8th Australasian Fluid Mech.Conf. Newcastle.*, 1C13-1C16.

Fenton, J.D., (1985) A family of schemes for computational hydraulics, *21st I.A.H.R. Congress, Melbourne*, Subject 2a, 23-27.

Liggett, J.A. (1968) Mathematical flow determination in open channels, *J.Engnrng. Mech. Div., ASCE*, **94**, 947-963.

Liggett, J.A. and Cunge, J.A. (1973) Numerical methods of solution of the unsteady flow equations, in *Unsteady Flow in Open Channels*, Ed. K. Mahmood and V. Yevjevich, Water Resources Publications: Fort Collins, Colorado.

Roache, P.J. (1972) *Computational Fluid Dynamics*, Hermosa: Albuquerque.

Zoppou, C. (1979) *Numerical models and solution of unsteady flow equations for open channels*, Thesis (M.Eng.Sc.), Univ. of Melbourne.

Zoppou, C. and O'Neill, I.C. (1981) Numerical methods and boundary conditions for the solution of unsteady flow problems, *Proc. Conf. on Hydraulics in Civil Engnrng., Sydney, October*, I.E.Aust. National Conf. Pub. No. 81/12, 16-24.