

Discussion

A Boussinesq-type model for flow over trapezoidal profile weirs

By YEBEGAESHET T. ZERIHUN and JOHN D. FENTON, *Journal of Hydraulic Research*, IAHR, 2007, 45(4), 519–528.

Discussor:

OSCAR CASTRO-ORGAS, Research Engineer, *Department of Agronomy, University of Cordoba, Apdo. 3048, 14080, Cordoba, Spain. Tel.: +34-607457243, e-mail: oscar@tecagsl.com*

The Discussor wishes to congratulate the Authors for their novel and interesting approach to model curved streamline flows by taking into account both the curvilinear effects and wall friction. The novelty of their research lies not only in the theoretical approach originally due to Boussinesq (1877), but also in a successful result for flows over trapezoidal weirs. This type of weir was studied theoretically for low heads $H_o/L_w < 0.5$ by Montes (1998), whereas Fritz and Hager (1998) considered higher heads, for which streamline curvature plays a major role. This discussion aims to (1) add a theoretical background on the theory of Boussinesq-type models, and (2) a guidance for further developments clarifying some points of engineering interest.

The Authors have focused their analysis on the use of the momentum approach, closely following the path established by Boussinesq (1877). He assumed a linear variation of the velocity component normal to the channel bed to include the effects of a non-hydrostatic pressure distribution on the one-dimensional open channel flow momentum equation, rather than a linear variation of the streamline curvature, as stated by the Authors. It should be added that not only the momentum approach was proposed for this type of problem. Another contribution based on the energy principle was that of Fawer (1937), who also proposed a Boussinesq-type equation, by taking into account the free surface and the channel bed curvatures. It is commonly believed that the Fawer theory is only valid for straight channel bottoms, as a particular case investigated by Jaeger (1956), which is not true, however. Fawer (1937) presented a direct comparison between his energy approach and that of Boussinesq based on the momentum approach for curved streamline flow over a wavy bed. This was the first attempt to highlight the inherent differences between the energy and the momentum principles in hydraulics.

Boussinesq's approach contains the friction slope, while Fawer includes the slope of the energy grade line. Liggett (1993) and Montes (1998) have demonstrated that the latter is equal to the gradient of dissipated energy for steady flows. This gradient represents the transformation of mechanical into thermodynamic energy, implying that the "head-loss" is determined by the thermal energy state of the system. The friction factor of the Darcy-Weisbach equation originates from the momentum approach (Keulegan 1942, Montes 1998). It appears that

there exists not yet a theoretically-based thermodynamic equation quantifying the dissipation of energy inside a fluid mass. Therefore, the momentum approach should be used when frictional forces are important, such as for long rough channels. This discussion supports the momentum approach, as did the Authors, to include the friction effects on flow over trapezoidal-profile weirs.

However, the question of when the energy approach can be used remains. As shown by Fawer (1937), Hager and Hutter (1984), Hager (1985) and Matthew (1991), among others, the energy principle is an accurate approach when the energy loss can be neglected and, hence, the total energy line is practically horizontal. Then, the irrotational flow theory applies and the flow may be treated as a potential flow. For developing boundary layer flow, the energy of the free surface streamline remains constant and the friction effects can be included in the energy equation by means of the boundary layer displacement thickness, permitting the use of an energy approach (Montes 1998), without resorting to any estimation of the energy slope.

In their theoretical approach, the Authors used with Eq. (1) the implicit assumption of a centrifugal term at a vertical section. As they are dealing with the fluid flow equations in the Cartesian coordinate system, the inclusion of "centrifugal terms" does not seem to be appropriate. In the fluid flow equations the centrifugal acceleration appears when a system of reference different from the Cartesian is used. This is the case of the bed-fitted coordinates proposed by Dressler (1978) or the intrinsic coordinates centred in a streamline used by Hager and Hutter (1984). In Cartesian coordinates, no centrifugal forces are present and the deviations from the hydrostatic pressure distribution in vertical sections are explained simply in relation to the vertical acceleration, as shown by Montes (1998), who proposed

$$p = \rho g(\eta - z) + \rho \int_z^\eta u^2 \frac{\partial}{\partial x} \left(\frac{v}{u} \right) dz \quad (D1)$$

in which u and v are the velocity components in the x and z directions. The Authors' Eq. (2) could have been more correctly justified from a theoretical point of view by using Eq. (D1). The Authors' Eq. (2) is indeed an interesting approach for the pressure distribution in a vertical section. It contains an empirical

factor ω_o , representing the contributions of the bed curvature to the non-hydrostatic pressure component. Since ω_o defines in a vertical section the “shape” of the pressure curve, it would have been of interest to study its effect on some of the relevant flow features. Of particular interest would be the effect of ω_o on the vertical pressure distribution and on other flow features, such as the free surface profile and the discharge coefficient of the weir.

Attention should be paid to the definition of the flow depth in a given Boussinesq type model. In short, the analysis of the Euler equations in the bed-fitted system of Dressler (1978) yields the flow depth defined as the normal to the channel bed. The integration of the Euler equations performed by Hager and Hutter (1984) defined the flow depth as a vertical projection of an equi-potential line. The Authors used Cartesian coordinates and the flow depth was defined vertically, as was also done by Matthew (1991). This aspect has not been carefully considered, or discussed in the literature. The selection of the reference system itself to develop a Boussinesq type model is an important decision, which greatly depends on the type of flow to be simulated. For a rectangular channel section as considered by the Authors, the Cartesian system appears appropriate. However, to simulate other flows in a complex channel geometry, like a tunnel spillway with a curved bed or a complex channel transition in a spillway, the bed-fitted coordinates are more appropriate permitting a precise definition of the cross-sectional geometry normal to the channel bed. As an example, a typical sloping tunnel spillway of circular section and variable slope presents a non-circular section on a vertical plane, complicating the analysis if Cartesian coordinates are used.

The authors used equations like those of Gauckler-Manning-Strickler or Darcy-Weisbach (the latter not being explicitly stated in the paper) to compute the friction slope. It is well known that these equations are valid only for fully developed channel flows. It was shown by Kindsvater (1964) or Montes (1998), that the flow over broad-crested weirs ($H_o/L_w < 0.5$) may be explained by the boundary layer theory. It was further demonstrated that the flow is composed of a developing boundary layer of thickness δ (Fig. D1) with a nearly irrotational core flow. While further justification is necessary as to whether the flow over trapezoidal profile weirs is fully or partially developed, the use of these equations for fully developed flows seems to be a good approach, given the results obtained by the Authors. Note that the friction effects are of relevance for low heads ($H_o/L_w < 0.5$), indicating that a direct comparison with the boundary layer theory commonly used in engineering practice is of interest. Also, the assumption of a constant Boussinesq coefficient β made by the Authors does not seem necessary, as it could be easily determined with the

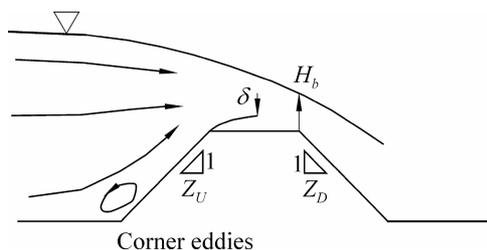


Figure D1 Flow features over trapezoidal profile weir

logarithmic velocity profile (Chow 1959), with a scant increase in the model complexity.

The Authors' computational approach to the flow equations using a two-point boundary value technique is noteworthy. However, in the Discussor's opinion, this method is by no means easily accessible to common hydraulic engineers. Therefore, it is believed that it would be of interest if the authors used their computer program not only to validate their model to determine the head-discharge relationship, but also to produce a simple design equation $C_d = C_d(H_o/L_w)$ as a salient result to be used by practicing civil engineers. As a complement, other relevant flow features such as the minimum bottom pressure and its position could then also be obtained, to complement the work developed for round-crested weirs (Hager 1991). It would be useful to obtain information regarding the effect of the downstream weir slope Z_D on the discharge characteristics of the weir using the Boussinesq-type momentum model. Any variation of Z_D would result in a change of the brink depth H_b (Weyermuller and Mostafa 1976) and, hence, in a modification of the discharge coefficient, as can be seen from the historical data of Bazin (Tracy 1957). A decrease in Z_D causes a reduction in H_b resulting in a stronger curved flow pattern over the crest domain, with the corresponding increase in the discharge coefficient. This flow phenomenon was also observed for spillway flows (Hager 1991) and in the transition from a mild to a steep slope (Montes 1994). The upstream weir slope Z_U also affects C_d as it influences the streamline pattern of the approach flow. The upstream slope further influences the development of a separation bubble at the grade break, which precludes a clear boundary layer development over the crest if no crest rounding is used. Of particular interest could be the discharge reduction caused by corner eddies, which is not yet studied to the Discussor's knowledge.

References

- Boussinesq, J. (1877). Essai sur la théorie des eaux courantes. *Mémoires présentés par divers savants à l'Académie des Sciences*, Paris, 23, 1–680.
- Chow, V.T. (1959). *Open channel hydraulics*. McGraw-Hill, New York.
- Dressler, R.F. (1978). New nonlinear shallow flow equations with curvature. *J. Hydr. Res.* 16(3), 205–220.
- Fawer, C. (1937). Etude de quelques écoulements permanents à filets courbes. Thesis, Université de Lausanne. La Concorde, Lausanne, Switzerland.
- Fritz, H.M., Hager, H.W. (1998). Hydraulics of embankment weirs. *J. Hydr. Engng.* 124(9), 963–971.
- Hager, W.H., Hutter, K. (1984). Approximate treatment of plane channel flow. *Acta Mech.* 51, 31–48.
- Hager, W.H. (1985). Equation of plane moderately curved open channel flows. *J. Hydr. Engng.* 111(3), 541–546.
- Hager, W.H. (1991). Experiments on standard spillway flow. *Proc. ICE* 91, 399–416.
- Jaeger, C. (1956). *Engineering fluid mechanics*. Blackie and Son, Edinburgh.

- Keulegan, G.H. (1942). Equation of motion for the steady mean flow of water in open channels. *Journal of Research, National Bureau of Standards* 29, 97–111.
- Kindsvater, C.E. (1964). Discharge characteristics of embankment shaped weirs. U.S. Dept. Interior, Geological survey water supply paper 1617-A, Washington DC.
- Liggett, J.A. (1993). Critical depth, velocity profile and averaging. *J. Irrig. Drain. Engng.* 119(2), 416–422.
- Matthew, G.D. (1991). Higher order one-dimensional equations of potential flow in open channels. *Proc. ICE* 91, 187–201.
- Montes, J.S. (1994). Potential flow solution to the 2D transition form mild to steep slope. *J. Hydr. Engng.* 120(5), 601–621.
- Montes, J.S. (1998). Hydraulics of open channel flow. ASCE Press, Reston, Va.
- Tracy, H.J. (1957). Discharge characteristics of broad-crested weirs. U.S. Dept. Interior, Geological survey circular 397, Washington DC.
- Weyermuller, R.G., Mostafa, M.G. (1976). Flow at grade-break from mild to steep slope flow. *J. Hydr. Div. ASCE* 102(HY10), 1439–1448.

Reply by the Authors

The authors acknowledge the discussion. Interesting issues are raised from a theoretical point of view for Boussinesq-type models and discharge characteristics of trapezoidal profile weirs. Only the main points of the discussion will be considered.

1 Governing equations

As the discussor stated, two approaches - the momentum and energy approach - are commonly used to derive the governing open channel flow equations. Strelkoff (1969) pointed out that equations based on the momentum principle are inherently different from those derived based on the energy approach. The energy approach incorporates a term to account for internal losses and is completely different from that included in the momentum equation for external resistance. For modeling free surface flows with significant streamline curvature, Boussinesq-type energy and momentum equations were developed in the past based on different simplifying assumptions. Zerihun (2004) pointed out the links between the different forms of such equations, and described thoroughly the impact of the simplifying assumptions on their application. Jaeger (1957) presented Fawer's equation for steady curved flow. Contrary to the discussor's opinion, this equation does not have any terms that account for bed curvature effects. It is believed that it would be of interest if the discussor presented the generalized form of the Fawer's equation.

The discussor raised issues regarding the choice of the reference system. Numerous attempts have been made to develop flow equations for rapidly varied flow situations using different coordinate systems. An orthogonal bed-fitted curvilinear coordinate system was used by Dressler (1978), Berger and Carey (1991) and Dewals *et al.* (2006). Hager and Hutter (1984) and Hager (1985) developed methods that employed projected depth

rather than true vertical depth of the flow which was employed by Matthew (1991). A Cartesian coordinate system in the vertical and longitudinal directions was employed by Steffler and Jin (1993) and Jin and Li (1996). The choice of this coordinate system for such depth-averaging method significantly reduces the computational complexity by reducing the flow equations into a one-dimensional. Other advantages of the Cartesian coordinate system that employs vertical flow depth for developing higher-order equations are given by Matthew (1991).

For an arbitrary channel section, the governing equation can be written as (Zerihun 2004)

$$\begin{aligned}
 & (1 + Z_b'^2) \frac{\partial Q}{\partial t} + 2\beta \left((1 + Z_b'^2) + \frac{\bar{d}}{2} \frac{\partial^2 H}{\partial x^2} + \omega_0 \bar{d} Z_b'' \right) \frac{Q}{A} \frac{\partial Q}{\partial x} \\
 & + \left(gA(1 + Z_b'^2) - \beta \frac{Q^2 B}{A^2} \left((1 + Z_b'^2) \right. \right. \\
 & \left. \left. + \omega_0 Z_b'' \left(2\bar{d} - \frac{A}{B} \right) \right) \right) \frac{\partial H}{\partial x} \\
 & + \beta \frac{Q^2 Z_b'}{2A} \frac{\partial^2 H}{\partial x^2} + \beta \frac{Q^2 \bar{d}}{2A} \frac{\partial^3 H}{\partial x^3} \\
 & + \omega_0 \beta \frac{Q^2}{A} (\bar{d} Z_b''' + Z_b'' Z_b') + gA(1 + Z_b'^2) (S_f + Z_b') \\
 & - \beta \frac{Q^2 B}{2A^2} \left(2\bar{d} - \frac{A}{B} \right) \left(\frac{\partial H}{\partial x} \frac{\partial^2 H}{\partial x^2} \right) \\
 & - \beta \frac{HQ^2}{A^2} \left((1 + Z_b'^2) + 2\bar{d} \left(\frac{1}{2} \frac{\partial^2 H}{\partial x^2} + \omega_0 Z_b'' \right) \right) \frac{\partial B}{\partial x} \\
 & = 0, \tag{R1}
 \end{aligned}$$

in which \bar{d} is the centroidal depth below the free surface, B channel width, and z the vertical coordinate. For steady, curved free surface flow in a rectangular channel, Eq. (R1) reduces to Eq. (1). From the practical point of view, Eq. (R1) can be used to solve real life flow problems rather than flow through an ideal structure.

The momentum correction coefficient β reflects the effect of non-uniform velocity distribution. In curved free surface flow, the streamline curvature strongly influences the velocity distribution over flow depth. In addition to channel geometry, channel roughness affects the magnitude of β (e.g. Li and Hager, 1991). This suggests that β varies from section to section for non-uniform open channel flows and its value must be determined based on a known distribution of flow velocity before applying Eq. (1). A numerical procedure was applied to estimate β using measured velocity data (Zerihun 2004). The influence of β on the accuracy of the model was examined and the result was presented in section 8.1 of the original paper. The discussor should carefully read this section rather than commenting the authors' approach.

The nature of the pressure equation (2) can be understood by examining the first step of the derivation procedure, rather than using unrelated Eq. (13), as suggested by the discussor. It is important to note that different methods employ different assumptions for developing flow equations. By integrating Euler's equation, the vertical pressure gradient can be written as

(Zerihun 2004, Fenton and Zerihun 2007)

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = -g - \beta \frac{Q^2}{A^2} \frac{\kappa}{\cos \theta}. \quad (\text{R2})$$

Equation (15) implies that the centrifugal term ($\kappa/\cos\theta$), which is a function of the streamline curvature parameters, strongly influences the magnitude of the second term. Based on different assumptions for the variation of this term at a vertical section, the authors' method yields equations that incorporate different degrees of dynamic pressure corrections (e.g. Zerihun and Fenton 2006).

2 Weighting parameters and bed pressure

The effect of the weighting factor ω_0 on the prediction of the proposed model depends on the bed profile. For flow over trapezoidal profile weirs or curved flow in a constant slope channel ($Z_b'' = Z_b''' = 0$), ω_0 does not affect the model results. For flow over curved beds, however, the value of ω_0 significantly affects the model results for pressure and free surface profiles especially in the regions where the boundary curvature is substantial. For such cases, the model results are in good agreement with the experiments if $\omega_0 \cong 0.95$ (e.g. Zerihun 2004, Zerihun and Fenton 2006). Zerihun (2004) has also studied the variation of the bed pressure for transcritical flow over trapezoidal profile weirs for different H_0/L_w values. The result for flow over short- and broad-crested weirs showed that only one absolute minimum bed pressure occurs near the downstream edge of the weir crest. In this region, the supercritical flow possesses a higher centrifugal acceleration due to the negative curvature of the flow surface.

The discussor highlighted the effect of the upstream and downstream side slopes of the trapezoidal profile weirs on the weir discharge characteristics. As mentioned in section 1 of the original paper, this issue is beyond the scope of this research.

References

Dewals, B.J., Erpicum, S., Archambeau, P., Detrembleur, S., Pirotton, M. (2006). Depth-integrated flow modeling taking into account bottom curvature. *Journal Hydr. Res.* 44(6), 785–795.

- Dressler, R.F. (1978). New nonlinear shallow flow equations with curvature. *Journal Hydr. Res.* 16(3), 205–220.
- Berger, R.C., Carey, G.F. (1991). A perturbation analysis and finite element approximate model for free surface flow over curved beds. *Intl. Journal Numerical Methods in Engng.* 31, 493–507.
- Fenton, J.D., Zerihun, Y.T. (2007). A Boussinesq approximation for open channel flow. *32nd IAHR Congress*, Venice, Italy, CD-ROM, 1–10.
- Jaeger, C. (1957). *Engineering fluid mechanics*. St. Martin's Press, New York, NY.
- Jin, Y., Li, B. (1996). The use of one-dimensional depth-averaged moment of momentum equation for the non-hydrostatic pressure condition. *Canadian Journal Civil Engng.* 23, 150–156.
- Hager, W.H., Hutter, K. (1984). Approximate treatment of plane channel flow. *Acta Mechanica* 51, 31–48.
- Hager, W.H. (1985). Equation of plane, moderately curved open channel flows. *Journal Hydr. Engng.* 111(3), 541–546.
- Li, D., Hager, W.H. (1991). Correction coefficients for uniform channel flow. *Canadian Journal Civil Engng.* 18(1), 156–158.
- Matthew, G.D. (1991). Higher order, one-dimensional equations of potential flow in open channels. *Proc. Instn. Civ. Engrs.* London 91, 187–201.
- Strelkoff, T. (1969). One-dimensional equations of open channel flow. *Journal Hydraulics Division ASCE* 95(3), 861–876.
- Steffler, P.M., Jin, Y. (1993). Depth averaged and moment equations for moderately shallow free surface flow. *Journal Hydr. Res.* 31(1), 5–17.
- Zerihun, Y.T. (2004). A one-dimensional Boussinesq-type momentum model for steady rapidly varied open channel flows. Ph.D. Thesis. Department of Civil and Environmental Engineering, The University of Melbourne, Melbourne Australia.
- Zerihun, Y.T., Fenton, J.D. (2006). One-dimensional simulation model for steady transcritical free surface flows at short length transitions. *Advances in Water Resources* 26 (11), 1598–1607.