ELSEVIER

PII: S0309-1708(96)00015-2

Advances in Water Resources, Vol. 20, No. 1, pp. 37–43, 1997

0309-1708/97/$17.00+0.00

Groundwater waves in aquifers of intermediate depths

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(Received 20 June 1995)

In order to model recent observations of groundwater dynamics in beaches, a system of equations is derived for the propagation of periodic watertable waves in unconfined aquifers of intermediate depths, i.e. for finite values of the dimensionless aquifer depth $n\omega d/K$ which is assumed small under the Dupuit–Forchheimer approach that leads to the Boussinesq equation. Detailed consideration is given to equations of second- and infinite-order in this parameter. In each case, small amplitude ($\eta/d \ll 1$) as well as finite amplitude versions are discussed. The small amplitude equations have solutions of the form $\eta(x,t) = \eta_0 e^{-kx} e^{i\omega t}$ in analogy with the linearized Boussinesq equation but the complex wave numbers k are different. These new wave numbers compare well with observations from a Hele–Shaw cell which were previously unexplained. The "exact" velocity potential for small amplitude watertable waves, the equivalent of Airy waves, is presented. These waves show a number of remarkable features. They become non-dispersive in the short-wave limit with a finite and quite slow decay rate affording an explanation for observed behaviour of wave-induced porewater pressure fluctuations in beaches. They also show an increasing amplitude of pressure fluctuations towards the base, in analogy with the evanescent modes of linear surface gravity waves. Copyright © 1996 Elsevier Science Ltd

INTRODUCTION

Groundwater dynamics near the coast are of interest for at least two reasons. Firstly, the intrusion of saltwater has serious consequences for agriculture. Secondly, the disposal of waste water from coastal developments can only be done properly if the ground water boundary conditions at the coast are understood.

The tidal oscillations of the watertable will vanish within the first 50 m or so from the high water mark in an unconfined aquifer. However, the steady effects in terms of an overheight above mean sea level and the intrusion of salt have far reaching consequences.

It has been known for a while that a sinusoidal tidal motion on a vertical beach will cause an inland watertable over height relative to mean sea level. This effect can be modelled partly through the non-linearity of the Boussinesq equation, see e.g., Philip[1] and Smiles and Stokes,[2] and the result thus derived for a vertical beach was shown by Knight[3] to be exact. Real beaches which are sloping show a further over height due to the slope and a simple quantification of this effect was obtained by Nielsen.[4] The latter solution is however restricted to fairly steep slopes and to shallow aquifers.

In the following we derive new equations for flow in aquifers of intermediate depth. We start with the

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shallow aquifer solutions and account for the effects of vertical flow and non-hydrostatic pressure in a stepwise fashion. Some of the results are similar to previous work by Dagan,[5] Parlange *et al.*[6] and Fenton,[7] but several results are new and increased clarity is obtained by clearly separating the effects of non-shallow depth from the effects of finite amplitude. While this paper has grown from a study of tidal watertable motions and focusses on periodic motions most of the results are applicable to unsteady problems in general.

2 THE SCHEME FOR DERIVING INTERMEDIATE DEPTH EQUATIONS

Consider the propagation of watertable waves in one horizontal direction in an unconfined aquifer over a horizontal impermeable base at mean depth d as shown in Fig. 1. The model is based on the Darcy flow assumption

$$\mathbf{u} = -K\nabla h^* \tag{1}$$

where $\mathbf{u} = (u, w)$ is the discharge per unit area, $h^* = h^*(x, z, t) = z + p/\rho g$ is the piezometric head and K is the saturated hydraulic conductivity. The vertical co-ordinate z is reckoned from the impermeable base, and the depth from the phreatic surface where $p = 0$ to the base is $h = h(x, t)$. Capillary fringe effects are neglected.

. Under the Dupuit–Forchheimer assumption of hydrostatic pressure the horizontal velocity u is uniform over the depth and the equation which combines (1) with depth-averaged continuity is the Boussinesq equation

$$\frac{\partial h}{\partial t} = \frac{K}{n}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \tag{2}$$

where n is the porosity. This equation is a finite amplitude equation for shallow aquifers, i.e., it is assumed that $n\omega d/K \ll 1$, where ω is the angular frequency.

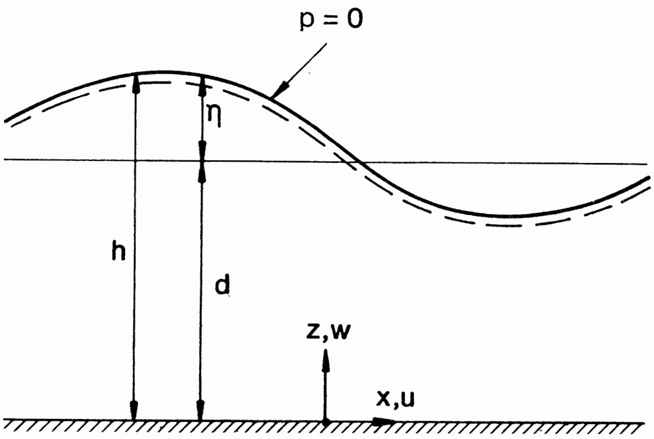

Fig. 1. Definition sketch.

We wish to derive new differential equations, which are valid when this shallow aquifer assumption is relaxed. The approach is a step-wise approach where the first approximation corresponds to the Dupuit–Forchheimer assumption and eqn (2) i.e., $h_1^* = h_1^*(x, t) = h$ and $u_1 = u_1(x) = -K(\partial h/\partial x)$. Subsequent approximations are found using the continuity equation and the Darcy equation as follows. If u_1 varies with x, continuity requires the existence of a vertical velocity given by

$$w_1 = -\int_0^z \frac{\partial u_1}{\partial x}dz = K\frac{\partial^2 h}{\partial x^2}z. \tag{3}$$

This vertical velocity in turn generates a correction to the pressure distribution given by

$$h_2^* = \frac{1}{K}\int_z^h w_1 dz = \frac{1}{K}\int_z^h K\frac{\partial^2 h}{\partial x^2}dz = \frac{1}{2}(h^2 - z^2)\frac{\partial^2 h}{\partial x^2}. \tag{4}$$

We note that this pressure correction term is analogous to the one under surface gravity analysis. This is perhaps surprising since deviations from hydrostatic pressure under gravity waves are due to vertical accelerations rather than to vertical velocities as under ground water waves.

From h_2^* we get the following correction term to the horizontal velocity

$$u_2 = -K\frac{\partial h_2^*}{\partial x} = -\frac{K}{2}\frac{\partial}{\partial x}\left[(h^2 - z^2)\frac{\partial^2 h}{\partial x^2}\right] = -\frac{K}{2}\left[2h\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial x^2} + (h^2 - z^2)\frac{\partial^3 h}{\partial x^3}\right] \tag{5}$$

and applying depth integrated continuity now leads to an expanded version of the Boussinesq Equation

$$n\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_0^h (u_1 + u_2)dz = -\frac{\partial}{\partial x}\int_0^h \left(-K\frac{\partial h}{\partial x} - \frac{K}{2}\left[2h\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial x^2} - (h^2 - z^2)\frac{\partial^3 h}{\partial x^3}\right]\right)dz \tag{6}$$

or

$$\frac{\partial h}{\partial t} = \frac{K}{n}\frac{\partial}{\partial x}\left[h\frac{\partial h}{\partial x} + h^2\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial x^2} + \frac{1}{3}h^3\frac{\partial^3 h}{\partial x^3}\right]. \tag{7}$$

Equation (7) may alternatively be written as

$$\frac{\partial h}{\partial t} = \frac{K}{n}\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}h^2 + \frac{1}{3}h^3\frac{\partial^2 h}{\partial x^2}\right). \tag{8}$$

This is the one-dimensional, accretion-free version of Fenton's[7] equation (9) which is a much simplified and corrected version of a result given by Dagan.[5]

Equation (7) also corresponds to the combination of the first and second order equations of Parlange *et al.*[6]

The successive approximations procedure outlined above can be continued to higher orders if desired. It is however also possible to derive the corresponding infinite order result in a different way which is outlined in Appendix 1. The result is

$$\frac{n}{K}\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\sin\left[h(x,t)\frac{\partial}{\partial x}\right]\frac{1}{\cos\left[h(x,t)\frac{\partial}{\partial x}\right]}h(x,t) \qquad (9)$$

where the infinite order differential operators have the meanings described in the appendix.

3 NEW SMALL AMPLITUDE EQUATIONS

By introducing the surface displacement $\eta = h - d$ and making the small amplitude assumption $\eta \ll d$, eqn (7) yields an approximate (second-order in $n\omega d/K$) equation for small amplitude waves in aquifers of intermediate depth

$$\frac{\partial \eta}{\partial t} = \frac{Kd}{n}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{d^2}{3}\frac{\partial^4 \eta}{\partial x^4}\right). \qquad (10)$$

Likewise, eqn (9) yields the infinite-order, small amplitude equation

$$\frac{\partial \eta}{\partial t} = \frac{Kd}{n}\tan\left(d\frac{\partial}{\partial x}\right)\frac{\partial \eta}{\partial x}. \qquad (11)$$

Both of these equations have solutions of the form

$$\eta(x,t) = \eta_0 Re\{e^{-kx}e^{i\omega t}\} = \eta_0 e^{-Re\{k\}x}\cos(\omega t - Im\{k\}x) \qquad (12)$$

in analogy with the small amplitude version of the Boussinesq equation. In the case of the small amplitude Boussinesq equation the wave number k must satisfy the dispersion relation

$$(kd)^2 = i\frac{n\omega d}{K}. \qquad (13)$$

For the second-order equation (10) the corresponding equation is

$$\frac{1}{3}(kd)^4 + (kd)^2 = i\frac{\omega n d}{K} \qquad (14)$$

and it has the four solutions

$$kd = \pm\sqrt{\frac{3}{2}}\sqrt{-1 \pm \sqrt{1 + \frac{4}{3}\frac{i\omega n d}{K}}}. \qquad (15)$$

The four roots described by eqn (15) are each restricted to one quadrant in the complex plane for $0 < n\omega d/K < \infty$, and the usual physical situation of a wave propagating in the positive x-direction with decreasing amplitude corresponds to the one in the first quadrant

$$kd = \sqrt{\frac{3}{2}}\sqrt{-1 + \sqrt{1 + \frac{4}{3}\frac{i\omega n d}{K}}}. \qquad (16)$$

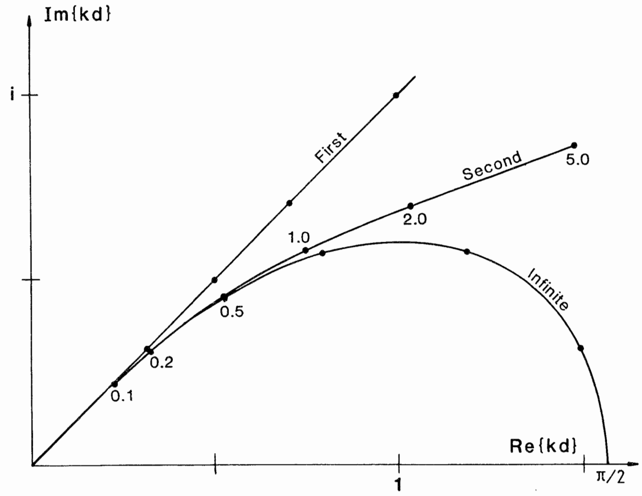

Fig. 2. Position of kd corresponding to eqns (13), (15) and (17) in the complex plane. The numbers on the curves indicate the values of the dimensionless depth $n\omega d/K$.

From the infinite order equation (11) one gets the following dispersion relation

$$kd \tan kd = i\frac{n\omega d}{K} \qquad (17)$$

where we note that eqns (13) and (14) emerge when one and two terms respectively are retained from the Taylor series of the tangent function. Intermediate-order equations may be obtained by analogy. The positions of kd in the complex plane corresponding to eqns (13), (15) and (17) are shown in Fig. 2.

It can be seen that the small amplitude Boussinesq equation is a satisfactory approximation for $0 < n\omega d/K < 0{\cdot}2$ while the second order approximation (15) is satisfactory up to about $n\omega d/K = 1{\cdot}0$.

The higher-order equations (14) and (17) and their intermediate analogies have multiple roots which call for some physical consideration. As mentioned in connection with eqn (15), the usual situation of a wave which propagates in the positive x-direction with decreasing amplitude corresponds to k-values in the first quadrant, $Re\{k\}, Im\{k\} > 0$. The infinite order equation (17) has multiple solutions in the first quadrant two of which are shown in Fig. 3.

In the short-wave (or deep-aquifer) limit the wave numbers given by (17) tend towards a finite limit, i.e., $kd \to \pi/2$ for $n\omega d/K \to \infty$ for the primary mode. This means that these waves become non-dispersive (little change in k as function of ω) with the finite, limiting decay rate $\exp(-\pi x/2d)$. The fact that the $Im(kd) \to 0$ means that the waves become standing rather than progressive in the short wave limit showing no phase

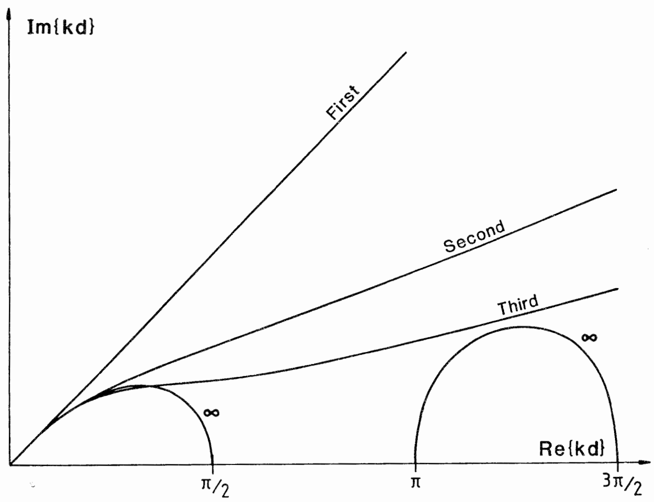

Fig. 3. Position of kd in the complex plane for the primary mode at order 1, 2, 3 and ∞ in $n\omega d/K$. The second mode for order ∞ is also shown. At order ∞ there are infinitely many such modes on similar loops and kd covers the loop as indicated by Fig. 2 for $n\omega d/K$ going from zero to infinity.

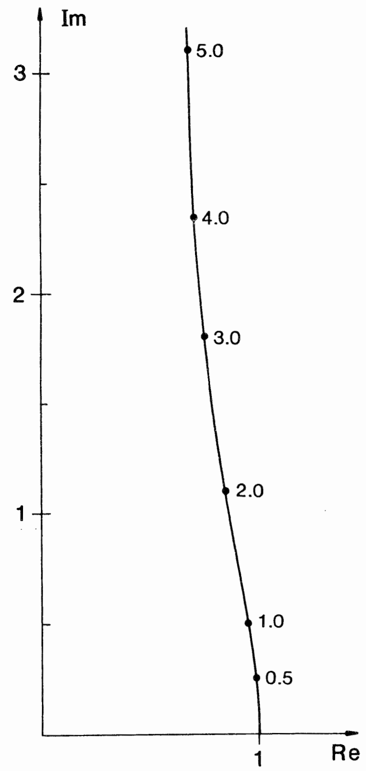

Fig. 4. Position of the bottom pressure amplification $1/\cos kd$ in the complex plane for different values of the dimensionless aquifer depth $n\omega d/K$. The amplification and the forward phase shift indicate a strong influence of the larger oscillations which are taking place at lower x-values.

shift with increasing distance inland. These features are in agreement with the observations by Waddell,[8] Lewandowski and Zeidler[9] and Hegge and Masselink[10] of watertable fluctuations in beaches. They all observed that pressure fluctuations at surf beat frequencies (of the order 0·01 Hz) decayed more slowly than expected on the basis shallow aquifer theory. In addition, Waddell found very little change in phase along a shorenormal transect in accordance with the expected standing wave behaviour for these short waves.

3.1 Exact velocity potential for small amplitude intermediate depth waves

As mentioned above, the infinite-order (in $n\omega d/K$) linear equation (11) has solutions of the form (12). But in order to satisfy Laplace's equation the corresponding potential must then be a trigonometric function of z. One is therefore led to suggest the expression

$$h^* = \eta \frac{\cos kz}{\cos kd} = \eta_0 \frac{\cos kz}{\cos kd} e^{-kx} e^{i\omega t} \tag{18}$$

for the potential (the piezometric head) corresponding to the surface elevation given by (12). It satisfies the Laplace equation and by insertion into the linearised free surface boundary condition $w(x, d, t) = \partial\eta/\partial t$ it yields the dispersion relation (17). Thus, eqn (18) is the velocity potential for small amplitude groundwater waves in aquifers of intermediate depth.

The velocity field which corresponds to the potential (18) is given by

$$u(x, z, t) = -K\frac{\partial h^*}{\partial x} = Kk\eta_0 \frac{\cos kz}{\cos kd} e^{-kx} e^{i\omega t} \tag{19}$$

$$w(x, z, t) = -K\frac{\partial h^*}{\partial z} = Kk\eta_0 \frac{\sin kz}{\cos kd} e^{-kx} e^{i\omega t}. \tag{20}$$

It is a remarkable feature of these waves that the amplitude of the potential variation $h^*(x_0, 0, t)$ at the bottom of the aquifer is greater than the surface oscillations $\eta(x_0, t)$ and leads the surface variation by a phase lead ϕ_0. These features are at first sight surprising but may be understood by considering that the pressure variation at a certain point $(x_0, 0)$ at the base of the aquifer is influenced by the surface variations to the left and to the right as well as directly above. Both the amplification and the phase lead can then be seen as a result of the dominant impact of the larger surface oscillations to the left. The amplification $G = |\cos kd|^{-1}$ and the phase lead $\phi_0 = \mathrm{Arg}\{\cos kd\}$ are illustrated in Fig. 4. We note that both effects disappear in the shallow aquifer limit $n\omega d/K \to 0$.

Groundwater waves with the analogous potential to (18) for very high permeabilities have been studied in the breakwater literature[9] (see Dalrymple *et al.*[11] for a review).

4 COMPARISON WITH EXPERIMENTS

Hele–Shaw cell experiments by Aseervatham[12] with dimensionless depth values in the range $0{\cdot}1 < n\omega d/K < 2{\cdot}0$ and relative amplitudes in the range

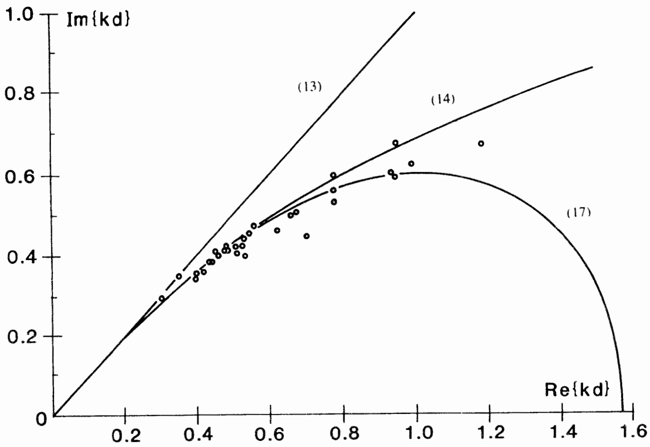

Fig. 5. Experimental values of the complex wave number compared with values prescribed by the dispersion relations (13), (14) and (17).

$0{\cdot}24 < \eta/d < 0{\cdot}48$ agree with the dispersion relations (14) and (17) as shown in Fig. 5.

The data shows a clear departure from (13) which corresponds to the linearised Boussinesq equation but the second-order equation (14) is quite accurate over most of the experimental range. The data in Fig. 5 are all from experiments with a vertical front face in the Hele–Shaw cell but data obtained with an inclined front show the same trend.

While the Hele–Shaw cell data generally agree closely with the form (12) and thus give a well defined, best fit value of k for the full length of the cell, the field measurements of Kang *et al.*[13] at tidal frequencies generally show some deviation from the form (12). That is, the decay rate is slower than exponential and the phase lag grows slower than linearly with distance from the beach (see Aseervatham and Nielsen[14]). It is clear from the tidal field data though that local k-values defined by

$$k = \frac{d \ln |\eta|}{dx} + i\frac{d\phi}{dx} \tag{21}$$

($\phi = \phi(x)$ being the local phase lag) have arguments less than $\pi/4$ like the Hele–Shaw cell data in Fig. 5. This tendency for k to decrease landward from the beach for tidal field data is not totally understood but it may be due to the nature of the boundary condition. That is, it may well be that the tidal signal close to the beach face contains several secondary modes all of which decay faster than the primary mode, *cf.* Fig. 3. This would result in the total amplitude decaying less rapidly further inland where the contribution from the secondary modes dies away.

5 BOUNDARY CONDITIONS

It needs mentioning that the solution (18) is not the immediate result of sinusoidal forcing at a vertical interface between a hydrostatic reservoir and a porous medium. The above has only dealt with the propagation of a time-periodic disturbance inside an aquifer. Interesting boundary problems result from the fact that the pressure distribution under a wave given by (18) is not hydrostatic while the pressure distribution in a driving reservoir would usually be hydrostatic. Hence, the forcing pressure distribution cannot be matched by a single wave mode. What is needed is a suitable combination of some or all of the modes corresponding to the different wave numbers k_j which satisfy (17) for the given value of $n\omega d/K$. This problem is similar to the problem of generating small amplitude surface waves with a vertical flap wave maker.

Consider now the groundwater motion which occurs in a semi infinite, unconfined porous medium which borders on to a hydrostatic reservoir where the surface level oscillates as $\eta_0 \cos \omega t$ with $\eta_0 \ll d$.

We assume that the resulting watertable motion can be written

$$\eta(x,t) = \sum_{j=1}^{\infty} B_j e^{-k_j x} e^{i\omega t} \tag{22}$$

with the corresponding potential (piezometric head)

$$h^*(x,z,t) = \sum B_j e^{-k_j x} \frac{\cos k_j z}{\cos k_j d} e^{i\omega t} \tag{23}$$

In order to find the coefficients B_j we utilise the fact that the functions $\cos k_j z$ are orthogonal with respect to the interval $0 < z < d$. That is, if k_j and k_m satisfy the dispersion relation (17) we have

$$\int_0^d \cos k_j z \cos k_m z \, dz = \begin{cases} 0 & \text{for } j \neq m \\ \dfrac{d}{2} + \dfrac{\sin 2k_j d}{4k_j} & \text{for } j = m \end{cases} \tag{24}$$

and the coefficients B_j are determined in by

$$B_j = \frac{\displaystyle\int_0^d \eta_0 \frac{\cos k_j z}{\cos k_j d} dz}{\left[\dfrac{d}{2} + \dfrac{\sin 2k_j d}{4k_j}\right][\cos k_j d]^{-2}} = \frac{4\eta_0 \sin k_j d \cos k_j d}{2k_j d + \sin 2k_j d}. \tag{25}$$

Cancelling the factor $\cos k_j d$ this leads to the following, general form of the potential (23)

$$h^*(x,z,t) = 4\eta_0 \sum_{j=1}^{\infty} \frac{\sin k_j d}{2k_j d + \sin 2k_j d} \cos k_j z e^{-k_j x} e^{i\omega t} = \eta_0 \sum A_j \cos k_j z e^{-k_j x} e^{i\omega t}. \tag{26}$$

This expression indicates rather slow convergence so that a great many terms could be needed. However, for moderate dimensionless depth, say $n\omega d/K < 2$, the kd-values are densely clustered around the left hand end of the loops in Fig. 3 which means that $\sin k_j d$ and hence

Table 1. Values of the coefficients A_j

$n\omega d/K$	A_1	A_2	A_3	A_4	A_5	A_6
0·1	$1001 + 0{\cdot}017i$	$-0{\cdot}001 - 0{\cdot}020i$	$0{\cdot}000 + 0{\cdot}005i$	$-0{\cdot}000 - 0{\cdot}002i$	$0{\cdot}000 + 0{\cdot}001i$	$-0{\cdot}000 - 0{\cdot}001i$
0·5	$1{\cdot}015 + 0{\cdot}082i$	$-0{\cdot}015 - 0{\cdot}100i$	$0{\cdot}001 + 0{\cdot}025i$	$-0{\cdot}000 - 0{\cdot}011i$	$0{\cdot}000 + 0{\cdot}006i$	$-0{\cdot}000 - 0{\cdot}004i$
1·0	$1{\cdot}059 + 0{\cdot}154i$	$-0{\cdot}062 - 0{\cdot}191i$	$0{\cdot}004 + 0{\cdot}051i$	$-0{\cdot}001 - 0{\cdot}023i$	$0{\cdot}000 + 0{\cdot}013i$	$-0{\cdot}000 - 0{\cdot}008i$
2·0	$1{\cdot}212 + 0{\cdot}213i$	$-0{\cdot}227 - 0{\cdot}287i$	$0{\cdot}017 + 0{\cdot}104i$	$-0{\cdot}003 - 0{\cdot}046i$	$0{\cdot}001 + 0{\cdot}026i$	$-0{\cdot}000 - 0{\cdot}008i$
5·0	$1{\cdot}320 + 0{\cdot}041i$	$-0{\cdot}467 - 0{\cdot}192i$	$0{\cdot}169 + 0{\cdot}233i$	$-0{\cdot}028 - 0{\cdot}125i$	$0{\cdot}008 + 0{\cdot}068i$	$-0{\cdot}003 - 0{\cdot}043i$
10·0	$1{\cdot}288 + 0{\cdot}005i$	$-0{\cdot}474 - 0{\cdot}022i$	$0{\cdot}344 + 0{\cdot}082i$	$-0{\cdot}217 - 0{\cdot}168i$	$0{\cdot}072 + 0{\cdot}165i$	$-0{\cdot}019 - 0{\cdot}100i$
∞	$4/\pi \approx 1{\cdot}273$	$-4/3\pi \approx -0{\cdot}424$	$4/5\pi \approx 0{\cdot}255$	$-4/7\pi \approx -0{\cdot}182$	$4/9\pi \approx 0{\cdot}141$	$-4/11\pi \approx -0{\cdot}116$

the coefficients in (26) are very small for the higher modes, *cf.* Table 1. This is in agreement with the fact that Aseervatham[12] found good agreement between measurements and the single mode form (12) in the parameter range $0{\cdot}04 < n\omega d/K < 2{\cdot}6$.

The wave numbers k_j are generally complex, *cf.* Figs 2 and 3. However, in the limit $n\omega d/K \to \infty$, they are all real valued: $k_j = (2j-1)\pi/2d,\ j = 1, 2, 3, \ldots$ corresponding to the solution (26) becoming

$$h^*(x,z,t) = \frac{4\eta_0}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(2j-1)}\cos\frac{(2j-1)\pi z}{2d}e^{-(2j-1)\pi x/2d}e^{i\omega t}. \tag{27}$$

This is a standing wave with zero surface movements but with finite pressure variations in the interior. For example, the series can be summed for $z = 0$ to give the pressure variation along the base of the aquifer

$$h^*(x,0,t) = \frac{4\eta_0}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(2j-1)}e^{-(2j-1)\pi x/2d}e^{i\omega t}$$
$$= \frac{4\eta_0}{\pi}\tan^{-1}(e^{-\pi x/2d})e^{i\omega t}. \tag{28}$$

This pressure variation is in qualitative agreement with the field observations of Waddell,[8] Lewandowski & Zeidler[9] and Hegge and Masselink.[10]

6 DISCUSSION

The new groundwater flow solutions corresponding to simple harmonic watertable oscillations are able to explain the differences between Hele–Shaw cell measurements and the previously available shallow aquifer solution.

Some features of the new solutions which are somewhat surprising, i.e., the phase lead and the amplification of the potential at the base of the aquifer relative to the watertable oscillation are at closer inspection plausible though not yet experimentally proven.

The availability of the exact small amplitude solution (18) opens the possibility for solving more general, linear, non-steady problems in terms of Fourier transforms. It also opens the possibility of constructing finite amplitude solutions in analogy with Stokes waves. The advantage of this approach compared to that of Dagan[5] or Parlange *et al.*[6] is that the problem becomes limited to one of satisfying the free surface boundary condition. A sum of components of the form (18) will automatically satisfy the Laplace equation and Darcy's law in the interior.

The standing wave behaviour with moderate decay rates which is found in the limit $n\omega d/K \to \infty$ is in agreement with field observations of high frequency (of the order 0·01 Hz) oscillations of beach groundwater pressure.

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APPENDIX: EXACT DIFFERENTIAL EQUATION FOR PLANAR FREE SURFACE SEEPAGE FLOW OVER A HORIZONTAL BED

Consider the Rayleigh expansion about the horizontal bottom $z = 0$ for the piezometric head $h^*(x, z, t)$:

$$h^*(x,z,t) = h_0^*(x,t) - \frac{z^2}{2!}\frac{\partial^2 h_0^*}{\partial x^2}(x,t) + \frac{z^4}{4!}\frac{\partial^4 h_0^*}{\partial x^4}(x,t) - \cdots, \qquad (A.1)$$

where $h_0^*(x, t)$ is the piezometric head on the bottom $z = 0$. This form satisfies Laplace's equation and the no-flow condition on the bottom boundary identically. It was originally introduced by Rayleigh to study the propagation of water waves over a flat bed. In the form of equation (A.1) the expansion provides a convenient frame for series approximations in terms of the shallowness, which is (depth/length of disturbances).[2] Here, however, we write the full series as the exact operator equation, as done by Fenton (1972)[15] to obtain high-order solutions for water waves:

$$h^*(x,z,t) = \cos\left(z\frac{\partial}{\partial x}\right)h_0^*(x,t) \qquad (A.2)$$

where the cosine differential operator has significance only as its power series expansion to infinite order, and in which $(z\partial/\partial x)^j$ is interpreted as $z^j\partial^j/\partial x^j$, such that:

$$\cos\left(z\frac{\partial}{\partial x}\right) \equiv 1 - \frac{z^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{z^4}{4!}\frac{\partial^4}{\partial x^4} - \cdots \qquad (A.3)$$

which gives equation (A.1) to infinite order. Now, substituting in the surface boundary condition that on the free surface $z = h(x, t)$ the pressure is zero, such that the piezometric head is $h^* = h$, equation (A.2) gives:

$$h(x,t) = \cos\left(h(x,t)\frac{\partial}{\partial x}\right)h_0^*(x,t). \qquad (A.4)$$

This equation may be reverted to give an expression for the piezometric head on the bottom in terms of that on the free surface. We write this as:

$$h_0^*(x,t) = \frac{1}{\cos h(x,t)\frac{\partial}{\partial x}}h(x,t) \qquad (A.5)$$

where the operator $1/\cos$ is not interpreted as the sec operator, but as the power series of $\cos(h(x, t)\partial/\partial x)$ raised to the power -1, and expanded using the binomial theorem, but where the individual nonlinear differential operators do not commute. That is

$$\frac{1}{\cos h(x,t)\frac{\partial}{\partial x}}h(x,t) = \left(1 - \frac{1}{2}h(x,t)^2\frac{\partial^2}{\partial x^2} + \frac{1}{24}h(x,t)^4\frac{\partial^4}{\partial x^4} + \cdots\right)^{-1} h(x,t)$$
$$= \left(1 + \frac{1}{2}h(x,t)^2\frac{\partial^2}{\partial x^2} - \frac{1}{24}h(x,t)^4\frac{\partial^4}{\partial x^4} + \frac{1}{2}h(x,t)^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}h(x,t)^2\frac{\partial^2}{\partial x^2}\right) + \cdots\right)h(x,t). \qquad (A.6)$$

Substituting equation (A.5) into equation (A.2) gives the exact expression for the piezometric head in terms of the water depth and all its spatial derivatives:

$$h^*(x,z,t) = \cos\left(z\frac{\partial}{\partial x}\right)\frac{1}{\cos(h(x,t)\partial/\partial x)}h(x,t). \qquad (A.7)$$

Differentiating this expression with respect to x gives an expression for the horizontal velocity:

$$u(x,z,t) = -K\frac{\partial h^*}{\partial x}(x,z,t) = -K\frac{\partial}{\partial x}\cos\left(z\frac{\partial}{\partial x}\right)\frac{1}{\cos(h(x,t)\partial/\partial x)}h(x,t), \qquad (A.8)$$

where K is the coefficient of permeability.

This result can be used to obtain the governing partial differential equation using the exact mass conservation equation, equation (5) of Fenton (1990),[7] for the particular case here of planar flow over a flat bed with no accretion:

$$n\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,z,t)\,dz = 0. \qquad (A.9)$$

Substituting equation (A.8), and performing the integration with respect to z between 0 and $h(x, t)$, in which it can be shown that the symbolic formalism goes through, gives the exact but infinite order governing equation in terms of the surface elevation:

$$\frac{n}{K}\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\sin\left(h(x,t)\frac{\partial}{\partial x}\right)\frac{1}{\cos(h(x,t)\partial/\partial x)}h(x,t) \qquad (A.10)$$

in which the derivative operations proceed in order from the innermost to the outermost, from right to left in this notation. For example, to obtain the second-order equation (8) it is necessary to expand the series operators to second order, starting with the form:

$$\frac{n}{K}\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h(x,t)\frac{\partial}{\partial x} - \frac{h(x,t)^3}{3!}\frac{\partial^3}{\partial x^3} + \cdots\right)\left(h(x,t) + \frac{h(x,t)^2}{2!}\frac{\partial^2 h(x,t)}{\partial x^2} - \cdots\right). \qquad (A.11)$$