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Numerical methods for nonlinear waves

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Abstract

This Chapter gives a survey of numerical methods for solving fully-nonlinear problems of wave propagation in coastal and ocean engineering. While low-order theory may give insight, for accurate answers fully-nonlinear methods are becoming the norm. Such methods are often simpler than traditional methods, partly because the full equations are simpler than some of the approximations which are widely used.

A lengthy description of the Fourier approximation method is given, which is the standard numerical method used to solve the problem of steadily-propagating waves. This may be used to provide an approximate solution for waves in rather more general situations, or, as is often the case, to give initial conditions for methods which go on to simulate the propagation of waves over more general topography. The family of such propagation methods is then described, including Lagrangian methods, marker-and-cell methods, finite difference methods – including some exciting recent developments, boundary integral equation methods, spectral methods, Green-Naghdi Theory, and local polynomial approximation. Finally a review is given of methods for analysing laboratory and field data and extracting wave information.

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1. Introduction

The first statement that should be made about the use of fully-nonlinear numerical methods for waves is to emphasize just how powerful low-order theories have been in coastal and ocean engineering. They can describe most of the physical phenomena associated with the propagation of waves. As they enable analytical solutions in many cases, they have been able to provide much insight into the nature of wave motion, such as the refraction, diffraction, reflection and dispersion of waves. The ability of mathematics to provide solutions to idealized problems and to provide explicit solution forms to less-idealized problems has provided a huge corpus of knowledge which can be used in coastal and ocean engineering, as can be gathered from a reading of Wehausen and Laitone (1960).

Possibly the real role of low-order theory is, to invert the motto of a popular book on numerical methods: *The Purpose of Low-order Theory is Insight, Not Numbers*. (We have mischievously replaced the word ”Computing” in Hamming's 1973 original by ”Low-order Theory” and turned its entire meaning around). Linear and low-order theory provide insight and understanding of maritime problems and approximate answers where approximate answers are sufficient. However, the slightest geometrical complication renders much theory unable to be applied in simple explicit terms. Real problems in coastal and ocean engineering usually have to be solved numerically. If that is the case, it is a noteworthy fact that a fully-nonlinear formulation of the computational problem is often simpler than lower-order formulations, albeit less revealing. In the usual mathematical approximations of the nonlinear physical world there are no derivatives higher than the second. Only when we start to approximate do higher derivatives and other complications creep in.

So, we begin to have some feeling for the attractions of fully-nonlinear computational solutions to coastal and ocean engineering problems – not only are low-order solutions often simply not accurate enough, but also for non-trivial geometries the nonlinear formulation is usually simpler and less arbitrary. There is another powerful psychological reason which spurs researchers in the area, exemplified by the famous response from the mountaineer G. L. Mallory when asked why he wanted to climb Mt Everest: ”Because it is there”. The goal of obtaining accurate solutions has attracted many people in the past, often because of some spirit of personal achievement – and competition.

There is another cultural dialectic which is a microcosm of that described above. This arises in the differences between computational methods for nonlinear problems, and is possibly to be compared with the contrasts between Classicism and Romanticism. Some methods are simple: simple to present, to learn, and to program, but they are computationally expensive, maybe proportional to the cube of the number of computational points; yet they have been implemented time after time after time. Other methods are complicated; a great deal of intelligence and culture has gone into their development; remarkably, some can solve the underlying elliptic problems with a computational effort proportional only to the number of points and the level of approximation; and, hitherto, they have hardly been applied other than by their progenitors.

Possibly in the choice of a method for nonlinear wave computation, people are instinctively attracted by the Principle of Occam's Razor as stated by the mediaeval logician and theologian,William of Ockham. A commonly-accepted view of his Razor is that we should be concerned with *simplicity* of description. However, what he was really saying was that we should be concerned with *adequacy* of explanation and not necessarily mere simplicity. It is in this sense that we might appeal to the Razor to justify the use of fully-nonlinear methods. In many cases, for an adequate explanation, or for a method which we know is going to give uniformly valid results, we have to resort to a full nonlinear description. At the same time, however, we should strive to keep things as simple as possible, and to be intolerant of methods which unnecessarily complicate or writings which unnecessarily obfuscate.

Most importantly, for a moment turning against the tide of the preceding discussion, we should also know when it is *not* necessary to use fully-nonlinear methods. That is because we should know when not to waste human time and effort rather than from a fear of computational expense. Nonlinear methods are more computationally expensive than low-order methods, but when compared with other engineering costs, computer time is relatively cheap.

In this Chapter we describe computational methods which are feasible for solving problems in coastal and ocean engineering which generally require no essential analytical approximations, and which are capable of providing results of high accuracy. Scant mention will be made of lower order theories or computational methods. The most important exclusion in this case is that of Boussinesq models for the propagation of waves. This is a large and thriving area of research and application, and a separate Chapter in this volume deals with such methods. Although there have been many papers written on the subject of computing the nonlinear wave motion of the sloshing of liquid in containers, because of the often-idealized geometry and limited physical scale of such problems they form a different area of research from the scope of this book and we have chosen not to include them. Similarly we will present few methods most suited to problems of the generation of waves by ships or the scattering of waves by bluff bodies.

An article of this nature will not have included some works it might have. If that is the case, there may have been a good reason akin to the immediately-preceding ones. There are some recent review articles in the field which play a similar role to this one, but in each case the approach is different from our own. Interested readers might like to consult Peregrine (1990) and Tsai and Yue (1996).

2. The governing equations

Here we consider the equations as they are used in a wide variety of applications. A rather more detailed and general presentation is given in Yeh (1995). Most applications in coastal and ocean engineering are of such a large physical extent that detailed solution of the primitive equations of fluid mechanics throughout the flow field has not been feasible. Sometimes this has been done, which will be described below – see, for example, the recent papers by Lin and Liu (1998a, 1998b). However, in most methods used two assumptions are made which enable the problem to be reduced to that of solving Laplace's equation. In this case several methods exist which solve the whole flow field by summary methods, such as recognizing that the solution depends on values only around the boundary, or by eigenfunction expansions. Here we describe the theory which enables that approach.

The first assumption made is that motion of the fluid is irrotational, which is typically a good approximation in maritime situations where the motion may be relatively quiescent and waves propagate over a fluid which is essentially at rest, and where there are no dominant effects of viscosity or boundary layers. Simple fluid dynamic theory shows that if a fluid is initially irrotational, such as one would be if it were at rest, and if there is no viscosity, then the fluid remains irrotational (Batchelor, 1967, #5.3). In this case, there exists a velocity potential ϕ (Batchelor, #2.7) such that the velocity vector $\mathbf{u}$ is given by the gradient $\mathbf{u} = \nabla\phi$, which identically satisfies the condition for irrotationality, $\nabla \times \mathbf{u} = 0$. In cartesian co-ordinates (x, y, z) the gradient operator is $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, which gives the velocity components $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$, and $w = \partial\phi/\partial z$. Throughout this work we will choose the co-ordinates in the horizontal plane to be x and z, with y the vertical co-ordinate.

We also assume that the fluid is incompressible, in which case mass conservation is satisfied by the equation $\nabla.\mathbf{u} = 0$, so that substituting the condition for irrotationality, we obtain

$$\nabla^2\phi = 0, \tag{2.1}$$

showing that the velocity potential must satisfy Laplace's equation throughout the flow domain. In cartesian co-ordinates this becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.2}$$

It might be thought that this is a remarkable result, that the Euler equations, the set of three nonlinear dynamic equations of fluid dynamics can be reduced to a single partial differential equation, linear in the dependent variable ϕ. The boundary conditions of the flow problem must be introduced so that it can be solved. Solutions to equations such as this which are elliptic in nature possess no real characteristics such as are encountered in hyperbolic systems such as the long wave equations. The potential at any point depends continuously on the value of ϕ or its derivative normal to the boundary at *all* points around the boundary. While the linearity of this equation is crucial in the development of theoretical solutions to wave problems, it is this mutually-dependent nature, and not the nonlinearity of the boundary conditions, which is the main problem as far as the numerical calculation of the propagation of waves is concerned.

In two-dimensional problems and where the motion can be rendered steady by a Galilean transformation, introducing a co-ordinate system travelling at the speed of the waves, it is more convenient to use a stream function ψ, but this will be described below when the steady travelling wave problem is discussed.

2.0.1 Kinematic boundary conditions

The boundary conditions for solid boundaries and the free surface can be obtained in general terms. If a material surface, one which consists always of the same fluid particles and moves with them, is specified geometrically by $F(\mathbf{x}, t) =$ constant, where $\mathbf{x}$ is the position vector: $\mathbf{x} = (x, y, z)$ in cartesian co-ordinates, then F is invariant for a fluid particle on the surface, and so the material derivative DF/Dt, the rate of change of F at a fluid particle is zero (Batchelor, 1967, #2.1). As $D/Dt = \partial/\partial t + \mathbf{u}.\nabla$, the general boundary condition for any surface bounding the fluid is

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{u}.\nabla F = 0 \quad \text{on the surface } F(\mathbf{x}, t) = \text{ constant.}$$

Substituting $\mathbf{u} = \nabla\phi$ we have the general kinematic condition

$$\frac{\partial F}{\partial t} + \nabla\phi.\nabla F = 0 \quad \text{on the surface } F(\mathbf{x}, t) = \text{ constant.} \tag{2.3}$$

The quantity ∇F can be found simply in terms of the geometry and velocity of the local boundary, as follows for solid boundaries. However, it is in this form (2.3) that is most suitable for practical application and for the sea surface. On solid boundaries such as the sea-bed or sea-walls or structures, under the irrotational assumption used, this expresses the fact that fluid particles may move along the boundary but never across it.

Solid boundaries: In the case of the sea bed or sloping structures which are stationary solid boundaries where the local elevation of the boundary can be given as a function of the horizontal co-ordinates such as $y = h(x, z)$, then we let $F = y - h(x, z)$, which is identically zero on the boundary. In cartesian co-ordinates, $\nabla F = (-\partial h/\partial x, 1, -\partial h/\partial z)$, and (2.3) gives

$$\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial h}{\partial z} = 0 \quad \text{on any solid boundary } y = h(x, z). \tag{2.4}$$

The general statement of the boundary condition in the form of (2.3) can be related to the velocity of the boundary as well. If the boundary is moving such that we can write

$$F(\mathbf{x}, t) = f(\mathbf{x} - \mathbf{U}t) = \text{Constant}, \tag{2.5}$$

where $\mathbf{U}$ is the local velocity of the boundary and we have a local time t, then $\partial F/\partial t = -\mathbf{U}.\nabla f$ and $\nabla F = \nabla f$ in the limit $t \to 0$ or where the boundary is translating such that $\mathbf{U}$ is independent of position, then (2.3) becomes

$$\nabla\phi.\nabla f = \mathbf{U}.\nabla f. \tag{2.6}$$

As a fundamental property of the gradient operator is that ∇f is normal to the surface on which f is constant, the unit vector normal to the boundary $\hat{\mathbf{n}}$, is given by $\hat{\mathbf{n}} = \nabla f / |\nabla f|$ and dividing (2.6) by the scalar $|\nabla F|$, it can be written

$$\nabla\phi.\hat{\mathbf{n}} = \mathbf{U}.\hat{\mathbf{n}}\,, \tag{2.7}$$

which, because taking a scalar product with a unit vector gives the component in that direction, can be written

$$\frac{\partial\phi}{\partial n} = U_n, \tag{2.8}$$

i.e. the derivative of ϕ normal to the boundary is equal to U_n, the velocity of the local boundary normal to itself. In the case of a stationary boundary, (2.8) becomes

$$\frac{\partial \phi}{\partial n} = 0. \tag{2.9}$$

Even though this interpretation in terms of the velocity of the boundary and the normal derivative of ϕ has a simple physical significance, and in the case of a vertical wall where (2.4) cannot be used we simply obtain $\partial\phi/\partial x = 0$, in general it is easier to use the form of (2.4).

Free surface: The above procedure can also be used to obtain simply the kinematic free surface boundary condition, where, because fluid particles on the free surface remain on the free surface, then there is also some quantity F which is constant for all particles on that surface. The free surface elevation is denoted by $y = \eta(x, z, t)$, and we let $F = y - \eta(x, z, t)$ so that (2.3) gives

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z} = 0 \quad \text{on the free surface } y = \eta(x, z, t), \tag{2.10}$$

which is the required boundary condition. There is an important difference between these superficially-similar conditions: equation (2.4) for a fixed solid boundary is linear in the dependent variable ϕ, while in this equation (2.10) we have introduced another dependent variable η and products of the dependent variables (or their derivatives) η and ϕ occur, so that the equation is nonlinear and the location of the free surface is also an unknown. Immediately one of the great complications of water wave theory has been introduced, that many of the simple techniques of mathematics, such as the superposition of solutions, are no longer available to us.

Lagrangian description of surface using material marker particles: For computational purposes, many methods allow the surface, and possibly solid boundaries, to be defined by hypothetical marker particles that are material points on the boundary and free to move. The computational process must include some means of treating this, although this is not usually the most demanding part of the process, either theoretically or computationally. Even though the surface equations contain products of dependent variables, they are really classified more as quasilinear rather than nonlinear, as all terms involving values of ϕ and η and their spatial derivatives can be evaluated, giving numerical values of the time derivatives, which can then be incorporated into a time-stepping procedure using standard methods for ordinary differential equations following a quite linear process.

Equation (2.10) is not useful in this context. Rather, the simple differential equations equating the rate of change of position of a body to the fluid velocity at that point may be used. For a particle denoted by m, the three differential equations governing its position $\mathbf{x}_m = (x_m, y_{m,} z_m)$ are given by the components of the vector equation

$$\frac{d\mathbf{x}_m}{dt} = \nabla\phi(\mathbf{x}_m). \tag{2.11}$$

In some methods the values of the normal derivative $\partial\phi/\partial n$ and tangential derivatives are calculated where otherwise cartesian co-ordinates are used, in which case simple geometry can be used to calculate the derivatives in that frame.

2.0.2 Dynamic boundary condition on the free surface

The pressure equation or unsteady Bernoulli equation for unsteady irrotational flow of an incompressible fluid, such as we have postulated here, can be obtained simply from the governing Euler's equation (Yeh, 1995, #3.4). The result is that, throughout the fluid,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p}{\rho} + gy = C(t), \tag{2.12}$$

where p is the fluid pressure, ρ is the fluid density, and $C(t)$ is a function of time only which can be brought to zero by a redefinition of ϕ. On the free surface $y = \eta(x, z, t)$, for the problems in which we are interested in where there is no generation of waves or surface tension effects, the pressure can be taken to be a constant which we set to zero, giving the dynamic boundary condition

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2+\left(\frac{\partial \phi}{\partial z}\right)^2\right)+g\eta=C(t), \quad \text{on the free surface } y=\eta(x,z,t), \quad (2.13)$$

where we have shown explicitly the components of the $|\nabla\phi|^2$ term, whose presence here means that this boundary condition too is a nonlinear equation.

Equation (2.13) is not useful by itself in the context of time-stepping calculations, as it gives $\partial\phi/\partial t$ at a point in space which the surface momentarily occupies. Two different representations of the free surface have to be allowed for here:

1. Surface marker particles move vertically only: In some methods where the description of overturning is excluded it is convenient to define the surface by particles which do not move horizontally. We introduce the symbol ϕ_s for the velocity potential on the free surface:

$$\phi_s\left(x,z,t\right)=\phi\left(x,\eta\left(x,t\right),z,t\right). \quad (2.14)$$

Use of elementary calculus shows that (2.13) is simply modified here to compute the rate of change of ϕ_s at a vertically-sliding surface marker:

$$\frac{\partial \phi_s}{\partial t}=C(t)-g\eta-\frac{1}{2}\left|\nabla\phi\right|^2+\frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial t} \quad \text{on } y=\eta(x,z,t), \quad (2.15)$$

where $\partial\eta/\partial t$ is obtained from (2.10).

2. Material (Lagrangian) marker particles Again, elementary partial differential calculus tells us that ϕ_m, the value of ϕ at the marker particle is given by

$$\frac{d\phi_m}{dt}=\frac{\partial \phi}{\partial t}+\mathbf{u}\cdot\nabla\phi=\frac{\partial \phi}{\partial t}+\left|\nabla\phi\right|^2 \quad \text{at } (x_m,\eta_m,z_m), \quad (2.16)$$

so that (2.13) becomes

$$\frac{d\phi_m}{dt}=C(t)-g\eta+\frac{1}{2}\left|\nabla\phi\right|^2 \quad \text{at } \mathbf{x}=\mathbf{x}_m(t), \quad (2.17)$$

such that (2.11) and (2.17) provide the necessary ordinary differential equations to track the movement and the change of potential at the particle.

2.0.3 Lateral boundary conditions

There remains one type of boundary condition to describe, and this is the type at lateral boundaries, where we have both open boundary conditions, allowing for waves to enter and/or leave the field of interest, as well as wavemaker-type boundary conditions, where the motion of a boundary might be specified. For the moment we will keep the description more general and will allow these to be specified by the horizontal velocity condition

$$\frac{\partial \phi}{\partial x}=G(x,y,z,t) \quad \text{on lateral boundaries}, \quad (2.18)$$

where $G(x,y,z,t)$ is some specifiable velocity field. In many cases, allowing for the entry and exit of waves from the computational domain is a demanding task, and we will defer discussion of this until later specific examples.

3. Periodic waves

In many wave computations in coastal and ocean engineering the motion is periodic in time, corresponding to the cyclic input of waves of constant height and period. There is a large family of problems where the motion is also periodic in space. This periodicity is assumed at the outset and the waves are often taken to be models of more complicated flow situations. Usually this means assuming a relatively simple geometry such as a horizontal bed. Waves of this type include those which propagate steadily without change, and standing waves. In such cases of space and time periodicity a spectral approach is adopted, so that a series solution may be assumed, each term of which satisfies the field equation such as (2.2) and the coefficients of the series may be found analytically by a succession of linear problems, which is the traditional approach, or numerically by solving a set of nonlinear equations. For problems of a more general nature, where motion is not periodic, or the geometry is not trivial, the problems are *quasilinear*, as the nonlinear terms are such that they are able to be evaluated at a point in time, and the process of calculating the evolution in time can proceed by linear methods. These problems will be described in Section 4 and all subsequent Sections.

3.1 The steady travelling wave problem

In many problems of the propagation of highly nonlinear waves it is necessary to have initial or boundary conditions which correspond to a solution of the full nonlinear equations such as the steady propagation of waves over a horizontal bed, before they encounter shoaling topography, or where it might be necessary to simulate that motion by using the steady propagating waves as a boundary condition, specifying the fluid motions at one open boundary as the waves enter. In this case a useful model is that of the steady propagation of waves, satisfying the full nonlinear boundary conditions, but within the irrotational approximation.

In many other more practical situations it is convenient to overlook the possibly-complicated nature of a wave propagation problem, such as where the waves might be propagating over water of possible non-uniform density which might be flowing on a shear current and over varying permeable or deformable topography. Uncertainties of the full problem render its solution too difficult. A convenient set of approximations is to assume that locally at least the bed is impermeable and flat, that the propagation of disturbances is collinear and they are of infinite length transverse to the direction of propagation such that the flow is two-dimensional, that the fluid is homogeneous and incompressible, and that the boundary layer is small such that inviscid flow theory can be used. Under these approximations it is still desirable to obtain solutions which correspond to a single periodic wave train which propagates steadily without change of form. This is the steady wave problem, and a great deal of attention has been given to it as it has been considered to be an important and convenient model for more general wave propagation problems. The common analytical theories applied to the problem are (1) Stokes theory, most suitable for waves which are not very long relative to the water depth, and (2) Cnoidal theory, suitable for the other limit where the waves are long. In the case of high waves, those theories lose their accuracy, although Fenton (1990) showed that their accuracy was greater than realized, and fifth-order versions of both theories were accurate enough for practical problems.

3.1.1 Literature survey

The search for a method to solve the problem of steadily-progressing waves seems to have been something similar to the search for the Holy Grail. It has attracted the attention of many mathematicians and engineers over the last 150 years. In the era before computers it was remarkable that some very good approximate solutions were obtained even for the highest waves. There are many works which could be referred to. Some of the methods are very sophisticated, such as those which treat the singularity near the crest of waves approaching the highest. However we will not describe them specifically here as most of them could not be classified as providing a consistent computational method for nonlinear waves which is the general thrust of this review. A review article which does describe such approaches has been given by Schwartz and Fenton (1982). Those two authors came together after having worked independently on the problem at the same time, both devising computer-assisted algebra manipulation methods, but from different ends of the wavelength/depth ratio.

Waves in deeper water – Stokes theory: The more general approach was that of Schwartz (1974), who used Stokes' original method for waves in water of finite depth, well-known to best suited to waves which are not too long. All variation in the horizontal is represented by Fourier series and the coefficients in these series can be written as perturbation expansions in terms of a parameter which increases with the ratio of wave height to length. Substitution of the high order perturbation expansions into the governing equations and manipulation of the series yields the solution. This had been carried (incorrectly) to fifth-order by hand computation, but Schwartz used computer manipulation of the expansions to generate very high-order results. This method was used by Cokelet (1977) to produce a number of results for physical quantities and by Williams (1981), who gained extra accuracy by incorporating the crest singularity analytically. However, the results from all these programs based on Stokes' expansions were produced using an inverse plane, so that substantial modifications would be necessary to calculate engineering quantities such as the fluid velocity at a particular point.

Waves in shallower water – cnoidal theory: At the other end of the length scale, for waves which are much longer than the water depth, the fundamental solution is obtained from cnoidal theory, an appellation given by Korteweg and de Vries in 1895 from the name of the Jacobian elliptic function which is the basis of the solution for those waves. Fenton (1979) produced an explicit higher-order cnoidal theory. This was calculated to ninth-order but the complexity of the solution meant it could be presented to fifth-order only. Subsequently, that approach was reviewed (Fenton, 1990), and when the series were recast in terms of shallowness rather than wave height, much better results for engineering quantities were obtained. A useful approximation was introduced by recognizing that the so-called "hyperbolic" approach of Iwagaki (1968) could be systematized and made presentation and use of the theory rather simpler. It was proposed that this be called the "Iwagaki approximation". This presentation has been improved and presented in a review and presentation of all cnoidal theory in Fenton (1998**?**).

Theory for the solitary wave: The limiting case of cnoidal theory for waves of infinite length is the solitary wave, theoretically of infinite length, but where almost all of the disturbance is confined to a finite length. Fenton (1972) obtained a high-order theory using computer manipulation of high-order expansions, but in terms of the ratio of water depth to wavelength. The same method, but adding computer-enhancement of the series, was used by Longuet-Higgins and Fenton (1974) to produce a number of results for physical quantities. Although Witting (1975) and Witting and Bergin (1981) were critical of the series approach of Fenton, Pennell and Su (1984) took the method, with some mathematical enhancements, to 17th order. However, their results did not agree with Hunter and Vanden-Broeck (1983), who recast the problem as an integro-differential equation which they could solve numerically accurately, and whose results were close to, but more accurate than, Witting and Bergin's. A similar method was used by Tanaka (1986) who then went on to examine the linear stability about that solution. For the case of water in infinite depth, Tanaka (1983, 1985) solved the steady wave problem accurately and then examined the linear stability.

3.1.2 Fourier approximation methods

Both the high-order Stokes theories and cnoidal theories suffer from a similar problem, that in the inappropriate limits, shallower water for Stokes theory and deeper water for cnoidal theory, the series become slowly convergent and ultimately do not converge. An approach which overcomes this is one which abandons any attempt to produce series expansions based on a small parameter and obtains the solution numerically, not by solving for the flow field numerically, but by using a spectral approach, where a series is assumed, each term of which satisfies the field equation, and then the coefficients are found numerically. The usual method, suggested by the basic form of the Stokes solution, is to use a Fourier series which is capable of accurately approximating any periodic quantity, provided it is sufficiently continuous. A reasonable procedure, then, instead of assuming perturbation expansions for the coefficients in the series as is done in Stokes theory, is to calculate the coefficients numerically by solving the full nonlinear equations. This approach would be expected to be more accurate than either of the perturbation expansion approaches described above, because its only approximations would be numerical ones, and not the essential analytical ones of the perturbation methods. Also, increasing the order of approximation would be a relatively trivial numerical matter without the need to perform extra mathematical operations.

This Fourier approximation approach originated with Chappelear (1961). He assumed a Fourier series in which each term identically satisfied the field equation throughout the fluid and the boundary condition on the bottom. The values of the Fourier coefficients and other variables for a particular wave were then found by numerical solution of the nonlinear equations obtained by substituting the Fourier series into the nonlinear boundary conditions. He used the velocity potential ϕ for the field variable and instead of using surface elevations directly he used a Fourier series for that too. By using instead the stream function ψ for the field variable and point values of the surface elevations Dean (1965) obtained a rather simpler set of equations and called his method "stream function theory". A more powerful method but with less practical utility was developed by Vanden-Broeck and Schwartz (1979) who formulated the problem as a pair of nonlinear integro-differential equations on an inverse plane with the spatial co-ordinates as the unknowns. Chaplin (1980) put the earlier approaches on a more sound theoretical basis by using Schmidt orthogonalization of the matrices used. In these approaches the solution of the equations proceeded by a method of successive corrections to an initial estimate such that the least-squares errors in the surface boundary conditions were minimized. Rienecker and Fenton (1981) presented a collocation method which gave slightly simpler equations, where the nonlinear equations were solved by Newton's method, such that the equations were satisfied identically at a number of points on the surface rather than minimizing errors there. The presentation emphasized the importance of using methods which maintain the full realizable accuracy throughout and the important practical detail that it is necessary to know the current on which the waves travel if the wave period rather than length is specified as a parameter, and whether that current was a current measured at a point or a depth-averaged value.

A simpler method and computer program have been given by Fenton (1988), where the necessary matrix of partial derivatives necessary is obtained numerically. In application of the method to waves which are high, in common with other versions of the Fourier approximation method, it was found that it is sometimes necessary to solve a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. For very long waves these methods can occasionally converge to the wrong solution, that of a wave one third of the length. This problem can be avoided by using a sequence of height steps.

Results from these numerical methods show that accurate solutions can be obtained with Fourier series of 10 to 20 terms, even for waves close to the highest, and they seem to be the best way of solving any steady water wave problem where accuracy is important. Sobey (1989), made a comparison between different versions of the numerical methods. He concluded that there was little to choose between them, but that it was an advantage to include the possibility of using depth-averaged current, as in the presentation of the theory below.

It is possible to obtain nonlinear solutions for waves on shear flows for special cases of the vorticity distribution. For waves on a constant shear flow, Dalrymple (1974a), and a bi-linear shear distribution (Dalrymple, 1974b) used a Fourier method based on the approach of Dean (1965). Dalrymple (1996) has made available programs for these cases on the Internet, as well as the more general one of waves on a piecewise-linear shear current. The ambiguity caused by the specification of wave period in the absence of knowledge of the current seems not to have been emphasized, however.

Solution for more general shear flows is difficult. In many cases the details of the shear flow are not known, and the irrotational model seems to be adequate for many situations in the absence of any other information. An exception is the delightful paper by Peregrine (1974) who obtained first-order solutions for so-called "shear waves" where the flow field is a fast-moving thin sheet of fluid at the surface with arbitrary shear and still water beneath it. Teles da Silva and Peregrine (1988) subsequently obtained high-order solutions for steady progressing waves where the vorticity in the shear layer is constant.

Theory: Here we present an outline of the theory which has been widely used to provide solutions in a number of practical and theoretical applications, giving fluid velocities and pressures for engineering design. The presentation of the theory follows Fenton (1988). The method provides accurate solutions for waves up to very close to the highest, and in the context of this Chapter its main application might be to providing reliable input data at a boundary where a more general unsteady method might take over the computations for a rather more general geometry. The problem considered is that of two-dimensional

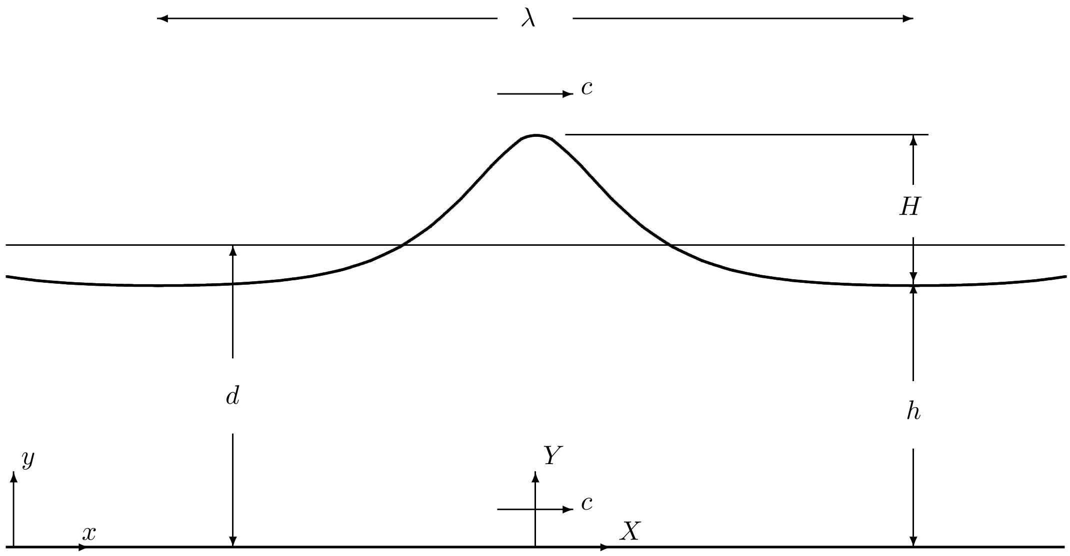

Figure 3-1. Wave train, showing important dimensions and co-ordinates

periodic waves propagating without change of form over a layer of fluid on a horizontal bed, as shown in Figure 3-1. A co-ordinate system (x, y) has its origin on the bed, and waves pass through this frame with a velocity c in the positive x direction. It is this stationary frame which is the usual one of interest for engineering and geophysical applications. Consider also a frame of reference (X, Y) moving with the waves at velocity c, such that $x = X + ct$, where t is time, and $y = Y$. The fluid velocity in the (x, y) frame is (u, v), and that in the (X, Y) frame is (U, V). The velocities are related by $u = U + c$ and $v = V$. It is easier to solve the problem in this moving frame in which all motion is steady and to use the stream function formulation. If the fluid is incompressible, in two dimensions a stream function $\psi(X, Y)$ exists such that the velocity components are given by

$$U = \partial\psi/\partial Y, \quad \text{and} \quad V = -\partial\psi/\partial X.$$

If motion is irrotational and $\nabla \times \mathbf{u} = \mathbf{0}$, as established in Section 2, it follows that ψ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} = 0. \tag{3.1}$$

The kinematic boundary conditions to be satisfied are

$$\psi(X, 0) = 0 \quad \text{on the bottom, and} \tag{3.2}$$

$$\psi(X, \eta(X)) = -Q \quad \text{on the free surface,} \tag{3.3}$$

where $Y = \eta(X)$ on the free surface and Q is a positive constant denoting the volume rate of flow per unit length normal to the flow underneath the stationary wave in the (X, Y) co-ordinates. In these co-ordinates the apparent flow is in the negative X direction. The dynamic boundary condition to be satisfied is that pressure is zero on the surface so that Bernoulli's equation (or, equation (2.13) for steady flow) becomes

$$\frac{1}{2}\left(\left(\frac{\partial\psi}{\partial X}\right)^2 + \left(\frac{\partial\psi}{\partial Y}\right)^2\right) + g\eta = R \quad \text{on the free surface,} \tag{3.4}$$

where R is a constant.

The basis of the method is to write the analytical solution for ψ in the spectral form

$$\psi(X,Y) = -\bar{U}\,Y + \sqrt{\frac{g}{k^3}}\sum_{j=1}^{N} B_j \frac{\sinh jkY}{\cosh jkd}\cos jkX, \tag{3.5}$$

where $\bar{U}$ is the mean fluid speed on any horizontal line underneath the stationary waves, the minus sign showing that in this frame the apparent dominant flow is in the negative x direction. The $B_1, \ldots, B_N$ are dimensionless constants for a particular wave, and N is a finite integer. The truncation of the series for finite N is the only mathematical or numerical approximation in this formulation. The quantity k is the wavenumber $k = 2\pi/\lambda$ where λ is the wavelength, which may or may not be known initially, and d is the mean depth as shown on Figure 3-1. Each term of this expression satisfies the field equation (3.1) and the bottom boundary condition (3.2) identically. It might be thought that the use of the denominator $\cosh jkd$ is redundant, but it serves the useful function that for large j the B_j do not have to decay exponentially, thereby making solution rather more robust. Possibly more importantly, it also allows for the treatment of deep water, such that if we introduce a vertical co-ordinate Y_* with origin at the mean water level such that $Y = d + Y_*$, then in the limit as $kd \to \infty$,

$$\frac{\sinh jkY}{\cosh jkd} \sim e^{jkY_*}, \tag{3.6}$$

which can be used as the basis for variation in the vertical.

If one were proceeding to an analytical solution, the coefficients B_j would be found by using a perturbation expansion in wave height. Here they are found numerically by satisfying the two nonlinear equations (3.3) and (3.4) from the surface boundary conditions, which become, after dividing through to make them dimensionless:

$$-\bar{U}\sqrt{k/g}\,k\eta(X) + \sum_{j=1}^{N} B_j \frac{\sinh jk\eta(X)}{\cosh jkd}\cos jkX + Q\sqrt{\frac{k^3}{g}} = 0, \quad \text{and} \tag{3.7}$$

$$\frac{1}{2}\left(-\bar{U}\sqrt{k/g} + \sum_{j=1}^{N} jB_j \frac{\cosh jk\eta(X)}{\cosh jkd}\cos jkX\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{N} jB_j \frac{\sinh jk\eta(X)}{\cosh jkd}\sin jkX\right)^2 + k\eta(X) - Rk/g = 0, \tag{3.8}$$

both to be satisfied for all x. To solve the problem numerically these two equations are to be satisfied at a sufficient number of discrete points so that we have enough equations for solution. If we evaluate the equations at $N+1$ discrete points over one half wave from the crest to the trough for $m = 0, 1, \ldots, N$, such that $x_m = m\lambda/2N$ and $kx_m = m\pi/N$, and where $\eta_m = \eta(x_m)$, then (3.7) and (3.8) provide $2N+2$ nonlinear equations in the $2N+5$ dimensionless variables: $k\eta_m$ for $m = 0, 1, \ldots, N$; B_j for $j = 1, 2, \ldots, N$; $\bar{U}\sqrt{k/g}$; kd; $Q\sqrt{k^3/g}$; and Rk/g. Three extra equations are necessary for solution. One is the expression for the dimensionless mean depth kd in terms of the dimensionless depths $k\eta_m$ evaluated using the trapezoidal rule:

$$\frac{1}{N}\left(\frac{1}{2}(k\eta_0 + k\eta_N) + \sum_{m=1}^{N-1} k\eta_m\right) - kd = 0. \tag{3.9}$$

For quantities which are periodic such as here, the trapezoidal rule is very much more accurate than usually believed. It can be shown that the error is of the order of the last (Nth) coefficient of the Fourier series of the function being integrated. As that is essentially the approximation used throughout this work (where it is assumed that the series can be truncated at a finite value of N) this is in keeping with the overall accuracy.

The remaining two equations necessary could be provided by specifying numerical values of any two of the parameters introduced. However in practice it is often the physical dimensions of wavelength λ, mean water depth d and wave height H which are known, giving a numerical value for the dimensionless wave height kH for which an equation can be provided connecting the crest and trough heights $k\eta_0$ and $k\eta_N$ respectively: $H = \eta_0 - \eta_N$,which we write in terms of our dimensionless variables as

$$k\eta_0 - k\eta_N - kd\frac{H}{d} = 0, \tag{3.10}$$

because in some problems we know the wave period rather than the wavelength and we do not know kd initially. If we do know the wavelength, we have a trivial equation for kd:

$$kd - 2\pi\frac{d}{\lambda} = 0. \tag{3.11}$$

There are now $2N + 5$ equations in the $2N + 5$ dimensionless variables, and the system can be solved. Some formulations of the problem (*e.g.* Dean, 1965) allow more surface collocation points and the equations are solved in a least-squares sense. In general this would be thought to be desirable, but in practice seems not to make much difference, and the system of equations appears quite robust.

Specification of wave period and current: In many problems it is not the wavelength λ which is known but the wave period τ as measured in a stationary frame. The two are connected by the simple relationship

$$c = \frac{\lambda}{\tau}, \tag{3.12}$$

where c is the wave speed, however it is not known *a priori*, and in fact depends on the current on which the waves are travelling. In the frame travelling with the waves at velocity c the mean horizontal fluid velocity at any level is $-\bar{U}$, hence in the stationary frame the time-mean horizontal fluid velocity at any point denoted by $\bar{u}_1$, the mean current which a stationary meter would measure, is given by

$$\bar{u}_1 = c - \bar{U}. \tag{3.13}$$

In the special case of no mean current at any point, $\bar{u}_1 = 0$ and $c = \bar{U}$, which is Stokes' first approximation to the wave speed, usually incorrectly referred to as his "first definition of wave speed", and is that relative to a frame in which the current is zero. Most wave theories have presented an expression for $\bar{U}$, obtained from its definition as a mean fluid speed. It has often been referred to, incorrectly, as "the wave speed".

A second type of mean fluid speed or current is the depth-integrated mean speed of the fluid under the waves in the frame in which motion is steady. If Q is the volume flow rate per unit span underneath the waves in the (X, Y) frame, the depth-averaged mean fluid velocity is $-Q/d$, where d is the mean depth. In the physical (x, y) frame, the depth-averaged mean fluid velocity, the "mass-transport velocity", is $\bar{u}_2$, given by

$$\bar{u}_2 = c - Q/d. \tag{3.14}$$

If there is no mass transport, such as in a closed wave tank, $\bar{u}_2 = 0$, and Stokes' second approximation to the wave speed is obtained: $c = Q/d$. In general, neither of Stokes' first or second approximations is the actual wave speed, and the waves can travel at any speed. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed. To apply the methods of this section, where wave period is known, to obtain a unique solution it is also necessary to specify the magnitude and nature of that current.

We now eliminate c between (3.12) and (3.13) or (3.14) so that they can be re-written in terms of the physical variables $\tau\sqrt{g/d}$ and $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$ which have to be specified. Equations (3.13) and

(3.14) respectively become

$$\sqrt{kd}\,\bar{U}\sqrt{k/g} + kd\,\frac{\bar{u}_1}{\sqrt{gd}} - \frac{2\pi}{\tau\sqrt{g/d}} \;=\; 0 \quad \text{and} \tag{3.15}$$

$$\frac{Q\sqrt{k^3/g}}{\sqrt{kd}} + kd\,\frac{\bar{u}_2}{\sqrt{gd}} - \frac{2\pi}{\tau\sqrt{g/d}} \;=\; 0. \tag{3.16}$$

This discussion has shown how problem-specific are the ideas associated with current, wave speed and wave period. In any general presentation of results it is better to use the physical dimensions of the wave and recognize that the steady wave problem possesses a two-parameter family of solutions in terms of kH and kd only, relative wave height and water depth respectively.

There are $2N+5$ equations: two free surface equations (3.7) and (3.8) at each of $N+1$ points, the mean depth condition (3.9), the wave height condition (3.10), and either (3.11) if the relative wavelength is known or (3.15) or (3.16) if the wave period and current are known. The $2N+5$ variables to be solved for are the $N+1$ values of surface elevation $k\eta_m$, the N coefficients B_j, $\bar{U}\sqrt{k/g}$, kd, $Q\sqrt{k^3/g}$, and Rk/g. The input data requires values of H/d and either λ/d or values of dimensionless period $\tau\sqrt{g/d}$ and one of the known mean currents $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$.

Computational method: The system of nonlinear equations can be iteratively solved using Newton's method. If we write the system of equations as

$$F_i(\mathbf{x}) = 0, \quad \text{for } i = 1, \ldots, 2N+5, \tag{3.17}$$

where F_i represents equation i and $\mathbf{x} = \{x_j, j = 1, \ldots, 2N+5\}$, the vector of variables x_j (there should be no confusion with that same symbol as a space variable), then if we have an approximate solution $\mathbf{x}^{(n)}$ after n iterations, writing a multi-dimensional Taylor expansion for the left side of equation i obtained by varying each of the $x_j^{(n)}$ by some increment $\delta x_j^{(n)}$:

$$F_i\left(\mathbf{x}^{(n+1)}\right) \approx F_i\left(\mathbf{x}^{(n)}\right) + \sum_{j=1}^{2N+5}\left(\frac{\partial F_i}{\partial x_j}\right)^{(n)} \delta x_j^{(n)}. \tag{3.18}$$

If we choose the $\delta x_j^{(n)}$ such that the equations would be satisfied by this procedure such that $F_i\left(\mathbf{x}^{(n+1)}\right) = 0$, then we have the set of linear equations for the $\delta x_j^{(n)}$:

$$\sum_{j=1}^{2N+5}\left(\frac{\partial F_i}{\partial x_j}\right)^{(n)} \delta x_j^{(n)} = -F_i\left(\mathbf{x}^{(n)}\right) \quad \text{for } i = 1, \ldots, 2N+5, \tag{3.19}$$

which is a set of equations linear in the unknowns $\delta x_j^{(n)}$ and can be solved by standard methods for systems of linear equations. Having solved for the increments, the updated values of all the variables are then computed for $x_j^{(n+1)} = x_j^{(n)} + \delta x_j^{(n)}$ for all the j. As the original system is nonlinear, this will in general not yet be the required solution and the procedure is repeated until it is.

It is possible to obtain the array of derivatives of every equation with respect to every variable, $\partial F_i/\partial x_j$ by performing the analytical differentiations, however as done in Fenton (1988) it is rather simpler to obtain them numerically. That is, if variable x_j is changed by an amount ε_j, then on numerical evaluation of equation i before and after the increment (after which it is reset to its initial value), we have the numerical derivative

$$\frac{\partial F_i}{\partial x_j} \approx \frac{F(x_1, \ldots, x_j + \varepsilon_j, \ldots, x_{2N+5}) - F(x_1, \ldots, x_j, \ldots, x_{2N+5})}{\varepsilon_j}. \tag{3.20}$$

The complete array is found by repeating this for each of the $2N+5$ equations for each of the $2N+5$ variables. Compared with the solution procedure, which is $O(N^3)$, this is not time consuming, and gives a rather simpler program.

To begin this procedure it is necessary to have some initial estimate for each of the variables. It is relatively simple to use linear wave theory assuming no current. If the wave period is known, it is necessary to solve for the wavenumber. The solution from that simple theory is

$$\sigma^2 = gk\tanh kd, \tag{3.21}$$

where the angular frequency $\sigma = 2\pi/\tau$. This equation which is transcendental in kd could be solved using standard methods for solution of a single nonlinear equation, however Fenton and McKee (1990) have given an approximate explicit solution:

$$kd \approx \frac{\sigma^2 d}{g}\left(\coth\left(\sigma\sqrt{d/g}\right)^{3/2}\right)^{2/3}. \tag{3.22}$$

This expression is an exact solution of (3.21) in the limits of long and short waves, and between those limits its greatest error is 1.5%. Such accuracy is adequate for the present approximate purposes. Having solved for kd linear theory can be applied:

$$\begin{aligned}
k\eta_m &= kd + \frac{1}{2}kH\cos\frac{m\pi}{N}, \quad \text{for } m = 1,\ldots,N,\\
\bar{U}\sqrt{k/g} &= \sqrt{\tanh kd},\\
B_1 &= \frac{1}{2}\frac{kH}{\sqrt{\tanh kd}}, \quad B_j = 0 \text{ for } j = 2,\ldots,N,\\
Q\sqrt{k^3/g} &= kd\sqrt{\tanh kd},\\
Rk/g &= kd + \frac{1}{2}\frac{\bar{U}^2 k}{g}.
\end{aligned}$$

In the limit of very long waves the spectrum of coefficients becomes broad-banded and more terms have to be taken, as the region over which the effective wave is concentrated is but a small fraction of the overall wavelength. Similarly, as the highest waves are approached the crest becomes more and more sharp, causing the spectrum to become broader and convergence to solution more difficult. The numerical cnoidal theory described below could be used, but for many applications, say for wavelengths as long as 50 times the depth, the Fourier method provides good solutions. More of a problem is that it is difficult to get the method to converge to the solution desired for high waves which are moderately long. In many cases the tendency of the method is to converge to a solution with a wavelength 1/3 of that desired. The results should be monitored, and if that has happened the problem is simply remedied by solving for two lower waves and using the results to extrapolate upwards to provide better initial conditions for the solution at the desired height.

Post-processing to obtain quantities for practical use: Once the solution has been obtained in these dimensionless variables oriented towards then quantities rather more useful for physical calculations can be evaluated. Often it is more convenient to present them in terms of the water depth as being the fundamental length scale. Here we assume that all physical quantities are available, for example, having solved for all the dimensionless variables $k\eta_m$ the numerical value of k has been used to calculate all the η_m, and so on.

It can be shown from (3.5) and the Cauchy-Riemann equations, here considering the physical frame, the now unsteady velocity potential $\phi(x,y,t)$ is given by

$$\phi(x,y,t) = \left(c - \bar{U}\,\right)x + \sqrt{\frac{g}{k^3}}\sum_{j=1}^{N} B_j \frac{\cosh jky}{\cosh jkd}\sin jk\,(x - ct) + C(t), \tag{3.23}$$

where we have shown the additional function of time $C(t)$ for purposes of generality. The velocity components anywhere in the fluid are given by $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$: and the velocity components anywhere in the fluid are given by:

$$u(x,y,t) = c - \bar{U} + \sqrt{\frac{g}{k}} \sum_{j=1}^{N} jB_j \frac{\cosh jky}{\cosh jkd} \cos jk\,(x-ct)\,, \tag{3.24}$$

$$v(x,y,t) = \sqrt{\frac{g}{k}} \sum_{j=1}^{N} jB_j \frac{\sinh jky}{\cosh jkd} \sin jk\,(x-ct)\,. \tag{3.25}$$

In some situations, such as simulating the propagation of a single wave of elevation as approximating a solitary wave entering still water, it would be necessary to add a linear function of x so that the velocities at the wave extremities were correct, so that they can be matched to the region of otherwise undisturbed water.

Should they be needed, expressions for the acceleration components can be obtained simply from these expressions.

To compute the free surface elevation at an arbitrary point requires more effort, as we only have it at discrete points η_m. We take the cosine transform of the $N+1$ surface elevations:

$$E_j = \sum_{m=0}^{N}{}'' \eta_m \cos\frac{jm\pi}{N} \quad \text{for } j = 1, \ldots, N, \tag{3.26}$$

where $\sum''$ means that it is a trapezoidal-type summation, with factors of 1/2 multiplying the first and last contributions. This could be performed using fast Fourier methods, but as N is not large, simple evaluation of the series is reasonable. It can be shown that the interpolating cosine series for the surface elevation is

$$\eta(x,t) = 2\sum_{j=0}^{N}{}'' E_j \cos jk\,(x-ct)\,, \tag{3.27}$$

which can be evaluated for any x and t.

The pressure at any point can be evaluated using Bernoulli's theorem, but most simply in the form from the steady flow, but using the velocities as computed from (3.24) and (3.25):

$$\frac{p(x,y,t)}{\rho} = R - gy - \frac{1}{2}\Big((u(x,y,t)-c)^2 + v^2(x,y,t)\Big)\,. \tag{3.28}$$

3.1.3 Numerical cnoidal theory

To address the problem described above, where the Fourier spectrum becomes broad for long waves, Fenton (1995) introduced a variant of the above methods, using cnoidal functions as the fundamental means of approximation, so that very long waves could be treated without any special measures. It was found that the method could be used for waves whose length is greater than eight times the water depth, and gave accurate results for all waves longer than this. For physically-realizable wave heights it is very accurate, but if the wave height is approaching that of the theoretical maximum, the accuracy is degraded to approximate engineering accuracy. The method has not yet been extended to the case where the wave period instead of wavelength is specified.

Similar to Fourier methods, a spectral approach is used, whereby series of functions are generated which satisfy the field equations identically. Then the coefficients of those series are found numerically. In this case all functions of X, the horizontal co-ordinate in the travelling frame, are approximated by polynomials of degree N in terms of the square of the Jacobian elliptic function $\mathrm{cn}^2(\theta|m)$ for the surface elevation and bottom velocity of the form suggested by conventional cnoidal theory. The argument θ is actually a scaled X co-ordinate: $\theta = \alpha X/h$, where α is a parameter which is related to the shallowness (depth/wavelength)^2 . The quantity m is the parameter of the elliptic functions. Conventional cnoidal theory uses α as the expansion parameter in the same way that Stokes theory uses the wave steepness

kH. The depth h is actually the water depth underneath the wave troughs (see Figure 3-1) and it is simpler to work with this quantity rather than mean depth d in this method. The two series introduced are

$$\frac{\eta}{h} = 1 + \sum_{j=1}^{N} Y_j \operatorname{cn}^{2j}(\theta|m), \tag{3.29}$$

$$\frac{f'}{Q/h} = W_0 + \sum_{j=1}^{N} W_j \operatorname{cn}^{2j}(\theta|m), \tag{3.30}$$

where f' is the negative of the horizontal fluid velocity on the bed, and the Y_j and W_j are dimensionless coefficients. The form of these series has been suggested by previous analytical cnoidal theories.

Following the pioneering work of Boussinesq and Rayleigh we assume an expansion for ψ of the form:

$$\psi = -Y\frac{df}{dX} + \frac{Y^3}{3!}\frac{d^3 f}{dX^3} - \ldots = -\left(\sin Y \frac{d}{dX}\right) . f(X), \tag{3.31}$$

where the series can be easily shown by term-by-term differentiation to satisfy Laplace's equation. The series can be written as a formal summation, which has to be used for the detailed evaluation in computer programs to high order, but using the trigonometric shorthand has a certain appeal - and several elementary operations of algebra and calculus go through (see Fenton, 1972). It can be shown by term-by-term operations and writing down the results in terms of trigonometric operators that

$$U = \frac{\partial\psi}{\partial Y} = -\left(\cos Y \frac{d}{dX}\right) . f'(X), \tag{3.32}$$

$$V = -\frac{\partial\psi}{\partial X} = \left(\sin Y \frac{d}{dX}\right) . f'(X), \tag{3.33}$$

and further differentiation shows that the expansions satisfy the condition of irrotationality that $\partial U/\partial Y - \partial V/\partial X = 0$. Introducing the scaled variable $\theta = \alpha X/h$ and substituting $Y = \eta$ for the surface, the velocity components there (U_s, V_s) are then given by

$$\begin{aligned} \frac{U_s}{Q/h} &= -\left(\cos\alpha\frac{\eta}{h}\frac{d}{d\theta}\right) . \frac{f'}{Q/h}, \\ \frac{V_s}{Q/h} &= \left(\sin\alpha\frac{\eta}{h}\frac{d}{d\theta}\right) . \frac{f'}{Q/h}. \end{aligned} \tag{3.34}$$

On substituting these into equations (3.3) and (3.4) we have two nonlinear algebraic equations valid for all values of X. The equations include the following unknowns: α, m, gh^3/Q^2, Rh^2/Q^2, plus a total of N values of the Y_j for $i = 1 \ldots N$, and $N + 1$ values of the W_j for $i = 0 \ldots N$, making a total of $2N + 5$ unknowns. For the boundary points at which both boundary conditions are to be satisfied we choose $M + 1$ points equally spaced in the vertical such that:

$$\operatorname{cn}^2\left(\alpha\frac{x_i}{h}|m\right) = 1 - i/M, \quad \text{for} \quad i = 0 \ldots M, \tag{3.35}$$

where $i = 0$ corresponds to the crest and $i = M$ to the trough. This has the effect of clustering points near the wave crest, where variation is more rapid and the conditions at each point will be relatively different from each other. If we had spaced uniformly in the horizontal, in the long trough where conditions vary little the equations obtained would be similar to each other and the system would be poorly conditioned. We now have a total of $2M + 2$ equations but so far, none of the overall wave parameters have been introduced. It is known that the steady wave problem is uniquely defined by two dimensionless quantities: the wavelength λ/d and the wave height H/d. As discussed above, in many practical problems the wave period is known, but here we consider only those where the dimensionless wavelength λ/d is known. It can be shown that λ/d is related to α using the following expression from the theory of elliptic functions, which we term the Wavelength Equation:

$$\alpha\frac{\lambda}{d}\frac{d}{h}-2K(m)=0, \tag{3.36}$$

where $K(m)$ is the complete elliptic integral of the first kind, and where the equation has introduced another unknown d/h, the ratio of mean to trough depth.

The equation for this ratio is obtained by taking the mean of equation (3.29) over one wavelength or half a wavelength from crest to trough:

$$\frac{d}{h}=1+\sum_{j=1}^{N}Y_j\,\overline{\mathrm{cn}^{2j}\left(\theta|m\right)}. \tag{3.37}$$

If we denote the mean value of the $2j$th power of the cn function over a wavelength by I_j:

$$I_j=\overline{\mathrm{cn}^{2j}\left(\theta|m\right)}, \tag{3.38}$$

the I_j can be computed from the recurrence relations

$$I_{j+2}=\left(\frac{2j+2}{2j+3}\right)\left(2-\frac{1}{m}\right)I_{j+1}+\left(\frac{2j+1}{2j+3}\right)\left(\frac{1}{m}-1\right)I_j,\quad \text{for all } j. \tag{3.39}$$

where, (Gradshteyn & Ryzhik, 1965, #5.13): $I_0=1$, $I_1=(-1+m+e(m))/m$, where $e(m)=E(m)/K(m)$, and $E(m)$ is the complete elliptic integral of the second kind.

Equation (3.37) can be written

$$1+\sum_{j=1}^{N}Y_j\,I_j-\frac{d}{h}=0, \tag{3.40}$$

thereby providing one more equation, the Mean Depth Equation.

Finally, another equation which can be used is that for the wave height:

$$\frac{H}{h}=\frac{\eta_0}{h}-\frac{\eta_M}{h}, \tag{3.41}$$

which, on substitution of equation (3.29) at $x=x_0=0$ where $\mathrm{cn}(0|m)=1$ and, because $\mathrm{cn}(\alpha x_M|m)=0$ from equation (3.35), gives

$$\frac{H}{d}\frac{d}{h}-\sum_{j=1}^{N}Y_j=0, \tag{3.42}$$

the Wave Height Equation.

We write the system of equations, similar to the Fourier method, as

$$F_i(\mathbf{x})=0,\quad \text{for } i=1,\ldots,2M+5, \tag{3.43}$$

where F_i is the equation with reference number i, the $2M+2$ equations described above plus the three equations (3.36), (3.40), and (3.42), and where the variables which are used are the $2N+5$ unknowns described above plus d/h:

$$\mathbf{x}=\left\{x_j,\ j=1\ldots 2N+6\right\}, \tag{3.44}$$

Whereas the parameter m has been used in cnoidal theory, it has the unpleasant property that it has a singularity in the limit as $m\to 1$, which corresponds to the long wave limit, and as a gradient method is used to solve the nonlinear equations this might make solution more difficult. It is more convenient to use the ratio of the complete elliptic integrals as the actual unknown, which we choose to be the first:

$$x_1 = \frac{K(m)}{K(1-m)}. \tag{3.45}$$

The solution of the system of nonlinear equations follows that described previously, using Newton's method in a number of dimensions, where it is simpler to obtain the derivatives by numerical differentiation.

As the number of equations and variables can never be the same ($2M + 5$ can never equal $2N + 6$ for integer M and N), we have to solve this equation as a generalized inverse problem. Fortunately this can be done conveniently by the Singular Value Decomposition method (for example, Press *et al.*, 1992, #2.6) so that if there are more equations than unknowns, $M > N$, the method obtains the least squares solution to the overdetermined system of equations. In practice this was found to give a certain rugged robustness to the method, despite the equations being rather poorly conditioned.

This poor conditioning comes about because the set of functions $\left\{\mathrm{cn}^{2j}(\theta|m), j = 0 \ldots N\right\}$ used to describe spatial variation in the horizontal do not form an orthogonal set, unlike terms in a Fourier series, and they all tend to look like one another. That result, although apparently an esoteric mathematical property, has the important effect that the system of equations is not particularly well-conditioned, and numerical solutions show certain irregularities and a relatively slow convergence with the number of terms taken in the series. This meant that it was difficult to obtain solutions for $N > 10$. The Fourier methods, however, using the robustly orthogonal trigonometric functions, do not seem to have this problem. Fortunately, however, in the case of the numerical cnoidal theory, good results could be obtained with few terms.

3.2 Other periodic wave systems

The two-dimensional steady travelling wave is a convenient approximation to the wave field in many areas in coastal and ocean engineering, where relatively little might be known about the topography or indeed the wave field; it provides a convenient approximation which at least models many waves as they are observed – they are long-crested and apparently of permanent form, propagating without much change. There are other solutions to the same idealized equations which are of more theoretical interest, which should be mentioned here, as they constitute evidence that there are other solutions to the governing equations.

3.2.1 Plane waves with period doubling and cyclic waves

The first such solution was by Chen and Saffman (1980) who found steadily travelling two-dimensional waves in deep water such that only every second or third wave was exactly the same, such that the period actually doubled or tripled. Bryant obtained a number of fully-nonlinear solutions for periodic waves, both of a travelling and a standing nature. In Bryant (1983) he obtained solutions for unsteady periodic gravity waves in deep water, whose shape changes cyclically as they propagate. He showed that they are closer to breaking than are steady permanent waves of the same height and wavelength. Both families are more of a mathematical curiosity than of great practical importance.

3.2.2 Doubly-periodic ("short-crested") waves

It is possible to obtain steadily propagating solutions corresponding to a doubly-periodic set of waves which when viewed from above, form a diamond-shaped pattern, leading to the name "short-crested waves". Meiron *et al.* (1982) obtained steadily propagating three-dimensional deep-water waves. For waves in finite depth Roberts (1983) used a Stokes-type expansion in wave steepness with computer manipulation, similar to Schwartz (1974) for progressing waves. He found some interesting mathematical phenomena in the solution coefficients. That these were merely a result of the solution technique was verified by Roberts and Schwartz (1983) who obtained accurate numerical solutions by using the same type of Fourier/Newton method described above for steady planar waves. Bryant (1985) solved the problem for deep water, establishing existence boundaries for the system of waves.

The diamond-shaped wave pattern problem can be considered to be the same as that of the reflection of waves obliquely incident on a vertical wall. In one limit of angle of incidence where the waves approach parallel to the wall these are steadily-progressing waves, and in the other limit, normal to the wall, they are planar standing waves, which are treated below. Fenton (1985c) used the same method as Roberts and Schwartz to solve the problem numerically, but his main aim was to calculate the forces and moments on the wall, so as to verify some earlier interesting analytical results for that problem. This was also studied by Marchant and Roberts (1987) using Roberts' 1983 method. Tsai *et al.* (1994) also used the Fourier/Newton method and presented results for the almost-highest waves.

It is worth noting that most of these papers solved the nonlinear equations using Newton's method, similar to that described above in detail for steady planar waves. Bryant showed and the experience of the present author is that he was right, that a more robust set of equations is obtained if the equations themselves are discrete Fourier transformed, so that each of the resulting set of equations is dominated by a particular (unknown) Fourier component. This follows from the orthogonal nature of the basis functions – for a linear system that procedure would eliminate all other variables but the particular component. This procedure does not seem generally to have been used, but the author does recommend.

3.2.3 Standing waves

The problem of accurately computing the problem of two-dimensional standing waves in deep water was solved by Schwartz and Whitney (1977, 1981) using a perturbation series method similar to Schwartz (1974) where the coefficients were found numerically. A direct numerical approach similar to the Fourier/Newton method described above was adopted by Saffman and Yuen (1979). Vanden-Broeck and Schwartz (1981) used a numerical Fourier method, but it was relatively low order, while the later work of Tsai and Jeng (1994) computed the solutions for higher waves rather more accurately and gave results for the highest waves. They confirmed earlier hypotheses that the angle at the sharp-crested highest wave is 90^{o}.

Bryant went on to obtain some more interesting solutions, including progressive free waves in a circular basin of finite depth (Bryant, 1989), and jointly with Stiassnie the problem of standing waves in a deep square basin (Bryant and Stiassnie, 1995).

4. More general wave propagation problems

As can be observed particularly in shallower water, there is a notable tendency for waves to be more coherent and permanent in their form than linear theory would predict, and the spatially-periodic models of the previous Section 3 have been widely used in practice as convenient approximations to more general problems. In general, however, problems of wave propagation are in more complicated geometries, the waves are not spatially periodic, and one cannot assume simple analytic functions which satisfy the field equation for all time and space. It would, however, be possible to build temporal periodicity into models by assuming a Fourier series in time and solving a succession of nonlinear problems for the spatial variation, in a manner similar to periodic problems above in Section 3. For more general problems the whole computational domain would have to be approximated and generally a potential problem solved over the whole region, possibly for each computational instant over the wave period. This might be computationally demanding, but it is still intriguing that it is a path which seems not to have been followed at all.

The usual way in which more general problems have been solved is to compute the evolution in time, even with a periodic input. Although the boundary conditions on the surface, (2.10) and (2.13) are nonlinear, in that they contain products of quantities to be determined as part of the solution, in many coastal and ocean engineering applications they are rather more quasilinear in their nature, such that time derivative terms are linear. All nonlinear terms involving spatial derivatives can be evaluated at a particular instant, leaving the time derivatives able to be computed and the solution advanced in time using linear methods. This large Section 4 is devoted to the plethora of methods which have been used to solve those important general problems.

4.1 Methods which solve at points throughout the flow field

The slightly clumsy title to this section is an attempt to group a number of methods which otherwise have little to do with each other. They include Lagrangian methods, Marker and Cell methods, finite difference methods, which describe some remarkable recent work on modelling the whole flow field with a turbulence-closure method, and finite element methods.

4.1.1 Lagrangian methods

The method of Brennen and Whitney (Brennen,1970) solved the Lagrangian equations of motion for the inviscid planar flow of a homogeneous or inhomogeneous fluid. The fluid was divided up into cells in true Lagrangian space, such that the location of the free surface is known, and solution of the resulting equations was by successive over-relaxation. They simulated the generation of a wave by a wall and its running up over a sloping beach. This avenue seems not to have been followed.

A rather different approach is that of particle methods. Smoothed Particle Hydrodynamics (Monaghan, 1994), is Lagrangian in that it follows particles and does not require a grid. It is an unusual and phenomenological method which is a technique modified from astrophysics applications. The real fluid is approximated by an artificial fluid which is more compressible. The article referred to here shows how it can be applied to wave shoaling and dam-break problems, where there is a free surface and where there are solid boundaries. Another moving particle method has been developed by Koshizuka *et al.* (1998). In both these papers the results presented seem to be qualitative rather than quantitative, but they are both interesting to read.

4.1.2 Marker and Cell methods

Chan and Street (1970) modified the original Marker and Cell technique so that it could be applied to free surface problems more successfully. In this method, the flow domain is divided up into rectangles (cells) while the surface is defined by massless marker particles which are free to move. The Navier-Stokes equations are satisfied by finite difference equations throughout the field of flow. In the vicinity of the surface some quite irregularly-dimensioned computational modules arise. The propagation of a solitary wave and its reflection by a vertical wall was successfully simulated. This problem was also modelled by Funakoshi and Oikawa (1982) using a Marker and Cell method, and they presented some interesting results for the phase shift, the amount of time delay caused by the wave travelling up the wall and down again.

Miyata (1986) described the further development of a Marker and Cell technique where the treatment of the free surface and especially regions of high curvature were more carefully treated. A number of problems of the interaction between waves and bluff bodies were described, where the interactions led to wave breaking.

Chen *et al.* (1997) describe the development of the ”surface marker and micro cell method” for the solution of two-dimensional problems with a free surface. Considerable effort has gone into the formulation so that multi-valued free surfaces, the impact of free surfaces with solid obstacles and converging fluid fronts may be studied. The method may be able to simulate wave slam problems, but no such results are presented.

4.1.3 Finite difference methods

Laplace's equation: The above Marker and Cell methods could be described as finite difference methods, as such expressions are used on rectangular cells throughout the flow, and even the irregular computational stars on the free surface make use of finite difference expressions. Given the traditional success of finite difference methods in solving Laplace's equation in many areas, it is noteworthy that they seem to have been seldom applied to wave propagation problems. This may be because the method has to describe the flow on the length scale of the crest of a wave and also the length scale of its region of propagation. This leads to huge numbers of points necessary and then if one uses Laplace's equation, a large computational cost.

Finite difference methods have been used rather more for problems of a rather more compact nature. Yeung and Vaidhyanathan (1992) developed a numerical two-dimensional wave tank and applied it to the problem of the nonlinear interaction of waves with a submerged cylinder. They used boundary-fitted co-ordinates such that the co-ordinates moved as the free surface moved. A variational method was used for the grid generation, which could handle steep and multi-valued free surface profiles. Lagrangian marker particles were used to define the surface, and problems of wave breaking over the cylinder could be simulated.

A recent paper by Li and Fleming (1997) has introduced an interesting and powerful method, which used a simple σ transform to map the flow from an irregular domain to one which is bounded by two horizontal planes so that boundary conditions can be satisfied on a known domain. In these co-ordinates all finite difference expressions become more complicated and Laplace's equation no longer has constant coefficients. However, because of the regular grid in the transformed co-ordinates, only small storage of point geometry was required. A multigrid method was used, and the method applied to a well-known standard wave shoaling and diffraction problem and obtained good agreement with experimental results. The model was claimed to be fast and to run on a personal computer, even though 130,395 grid points were used. A run time was 10 hours.

Navier-Stokes equation: Although outside a stated boundary of this Chapter, that little attention will be given to problems of a more naval architectural nature, in the nature of a signpost we mention a paper by Yeung and Ananthakrishnan (1992), in which the Navier-Stokes equation was solved for the viscous flow problem associated with the heaving motion of a floating body. A finite difference method was used, based on the boundary-fitted co-ordinates of Yeung and Vaidhyanathan (1992) mentioned above. The Reynolds number of the flow problem was 1000, scarcely applicable to coastal or ocean engineering.

Turbulent flow equations: A remarkable method has been introduced by Lin and Liu (1998a, 1998b). The two papers describe the development of a numerical model for studying the evolution of a wave train with shoaling and eventual breaking, right through the surf zone. Unlike almost all methods described in this Chapter, which assume irrotational flow, the model solves the Reynolds equations for the mean flow field and the $k - \epsilon$ equations for turbulence closure using finite difference methods. To track free-surface movements, the Volume of Fluid method is used. Care was taken to verify the accuracy of each component of the numerical model by comparison with analytical solutions and experimental results. The method (or methods) have been able to describe the shoaling and spilling breaking of long waves on a beach, as described in the first paper, where a number of other physical results were presented. In the second paper attention turned to the simulation of a plunging breaker, and again a number of physical results are presented. These works seems to form a major step forward in both computational methods and physical understanding of the phenomena of wave breaking.

4.1.4 Finite element methods

Eatock Taylor (1996, p51) wrote: "It is perhaps surprising that the finite element method has not been adopted more widely in computational hydrodynamics", which is a statement with which the present author can readily agree. At first blush the method, widely used throughout mechanics with its ability to handle complicated geometries and difficult constitutive relations, would seem ideally suited to wave propagation problems in coastal and ocean engineering. In an interesting series of papers, Wu and Eatock Taylor and co-workers have described the development of the method and its comparison with boundary integral equation methods (Wu and Eatock Taylor, 1994; Wu and Eatock Taylor, 1995; Greaves *et al.* 1997). At the end of that corpus of work, however, the method seems to have been damned with relatively faint praise, and the contributions of several other methods, described in the course of this Chapter, seem to far outweigh any potential that finite elements might have.

The possible advantages of finite elements for simulating wave propagation include –

1. their ability to handle complicated geometries and a moving free surface;

2. the finite-width banded structure of the matrices generated, leading to economies of storage and computational time; and
3. their ability to handle field equations of almost any complexity.

In the case of (1) boundary integral equation methods are every bit as good and have the advantage that they have to approximate one fewer physical dimension. Finite element methods suffer from the problem of having to generate a mesh of elements at each time step. In the case of finite differences the use of boundary fitted co-ordinates and volume of flow methods has enabled them to handle non-trivial geometries, but in general finite elements should be better at this, but at some cost.

Point (2) is where finite elements do have an advantage over boundary integral equation methods, but local polynomial approximation and spectral methods also have this advantage and so do finite difference methods for which, if computations can be performed on a rectangular grid, storage is trivial.

Finally, in the case of (3), certainly finite difference methods can be successfully used, even for turbulent flow simulations as reported above. In any case, in wave propagation problems Laplace's equation is a good model for many situations, and the other main methods can be used.

The sequence of finite element papers mentioned succeeded in developing models which could solve the two-dimensional problems of a wave tank whose length was 40 times the water depth, with a vertical plate wavemaker; a circular cylinder oscillating below the free surface with computations extending as far as 50 times the cylinder diameter; and standing waves in a rectangular tank with various submerged bodies. None of these could be described as challenging for other methods. They compared computation times with a boundary integral equation method and found the finite element program took much less time. However, the gradual growth of error seemed to be a problem. If one looks at the results presented, there is little to compare with the achievements of boundary integral equation, finite difference, spectral, and local polynomial approximation methods in being able to handle large regions with high and possibly breaking waves.

4.2 Boundary integral equation methods

Numerical methods for wave propagation using boundary integral equations (BIEs) have been proposed for some two decades. In principle, such methods are almost ideal for they can handle irregular geometries simply, and can even treat the overturning of waves as they break, as the free surface is usually described in a Lagrangian sense. Their disadvantage is that they can require large amounts of computational effort, as a full matrix equation must be solved at each step in time. A noteworthy feature of the history of BIE methods is how many papers have been published which have repeated the canonical theory and numerical method and have then described the application to a particular problem or problems.

The boundary integral equation can be obtained by either using a Green function or using Cauchy's integral theorem. Elements of a simple theory are presented below. If one knows the potential distribution around the boundary, the velocity of a point on the boundary may be calculated, and Bernoulli's equation can be used to calculate the rate of change of the potential on the boundary. These give differential equations for each boundary point, and the solution and the wave may be advanced in time. Then at the next step it is necessary to solve the integral equation for the updated potential around the boundary and the whole process is repeated successively.

The application of BIE methods was initiated by Longuet-Higgins and Cokelet (1976) for the study of waves in deep water. They used a Green function method, which set up a boundary integral equation with a logarithmic kernel. As they were using infinitely-deep water with assumed lateral periodicity they could conformally map onto a more convenient domain. Time integration was by an Adams-Bashforth-Moulton scheme. They found sawtooth oscillations on the free surface which had to be regularly smoothed, but they were successful in simulating the overturning of waves.

A different approach was introduced by Vinje and Brevig (1981), who used the Cauchy integral theorem in terms of a complex potential function as the integral equation valid around the boundary. It can be shown using complex analysis that the Longuet-Higgins and Cokelet formulation can be obtained from this. The complex formulation tends to have more diagonally-dominant matrices, with faster solution methods.

Baker *et al.* (1982) introduced a different approach which, although considerably more powerful and efficient, seems not to have been used by other workers. Perhaps that is because the presentation of the method departs from the canonical approach commenced by Longuet-Higgins and Cokelet – and is non-trivial in the mathematical operations involved. The method can be used in both two and three dimensions, can be used for layered flows, and requires only $O(N^2)$ arithmetic operations per time step. Although it was necessary to assume lateral periodicity which would limit its application in general, the authors managed to solve several problems of wave propagation, including wave breaking over varying bottom topography.

New *et al.* (1985) modified the method of Longuet-Higgins and Cokelet (1976) so as to study the propagation of waves on water of constant finite depth. They assumed lateral periodicity and mapped the region onto an irregular annular region. This enabled high resolution computations, and by assuming a linear wave solution as initial condition they were able to demonstrate the overturning of the wave crest and to calculate the details of the flow field.

A problem which originated in naval architecture, but which may have some utility in studying the breaking of a wave near a vertical wall, is the use by Grosenbaugh and Yeung (1989) of Vinje and Brevig's formulation to solve the problem of the generation of a plunging breaker by the motion of a bluff floating body.

Dold and Peregrine (1984, 1986) briefly describe a very accurate method which is presented in detail by Dold (1992). This was another important computational development. A complex formulation of the boundary integral equation was used, with a singularity subtraction technique so that the integrand to be approximated was continuous. A point-label parameter was introduced such that approximations to derivatives and integrals could be made using equi-spaced formulae. The time stepping was highly accurate, as Taylor series expressions were used. The resolution in time was 5th order and space discretization 10th order; it is stable and fast.

The original development was for the case of a flat bottom with lateral periodicity, which limited its application somewhat. This can be overcome *via* conformal mapping. Tanaka *et al.* (1987) applied the method to examine the stability of a solitary wave. It was found that the stability results depended on the sign of the perturbation about the steady solution. If the unstable perturbation has one sign, the wave soon breaks, but if it is of the other sign, it loses height and approaches a lower solitary wave of almost equal energy. Cooker *et al.* (1990) extended the method to include the scattering of solitary waves by a semi-circular cylinder, using conformal mapping to give a flat bed for computations. Passoni (1996) also used the method with conformal mapping to study the reflection of a wave by a vertical wall with a small sloping section in front of it. The method has been used by Yasuda *et al.* (1990) to study the propagation of a solitary wave over a submerged rectangular obstacle. Later Yasuda *et al.* (1997) studied the kinematics of the breaking in considerable detail. Liu, Hsu and Lean (1992) applied the method with a conventional free-space logarithmic Green function for the integral equation, but also introduced another Green function consisting of a series of free-space functions so that it yielded solutions laterally periodic in space which meant that only the free surface need be included in numerical integrations. Cooker *et al.* (1997) used the method of Dold and Peregrine as adapted by Tanaka *et al.* (1987) to solve the problem of the reflection of a solitary wave by a vertical wall, and gave a number of interesting results.

Brorsen and Larsen (1987) used the Green function approach. They concentrated on the generation of waves inside the computational domain by a vertical line of pulsating sources within the model, which allowed the generation of any wave form desired and provided the basis for a general open-boundary formulation.

Grilli *et al.* (1989) developed a method based on the Green function formulation, but used a high-order time-stepping method in the spirit of Dold and Peregrine. Considerable effort went into the spatial approximation of the integral equation. Wave generation was performed by simulating wavemaker movement or by imposing extended periodicity conditions. Wave absorption was for constant wave shapes. It was found that no smoothing was necessary. They simulated several problems including the propagation of a steady periodic wave, the breaking of an initial sine wave, the generation and run-up on a slope of a solitary wave, and a transient wave generated by an articulated wavemaker.

Subsequently, Grilli and Horrillo (1997) gave more attention to boundary conditions, using both numerical flap-type wavemaker and exact wave generation using accurate solutions for steady waves. They went to some lengths to obtain a zero mass flux boundary condition, although this would have been trivially obtained had they used equation (3.16) as part of their Fourier approximation method. An absorbing beach was modelled at the end of the tank in which free-surface pressure was used to absorb energy from high-frequency waves and a piston-like condition for low-frequency waves.

Dommermuth *et al.* (1988) compared theory and experiment for the propagation, overturning and breaking of a high wave. They used the Vinje and Brevig scheme for the boundary integral equation with a fourth-order predictor-corrector scheme for the time-stepping. A number of interesting results were obtained, including the necessity to re-grid occasionally to guard against instability. One conclusion was, comparing theory and experiment, that potential theory is satisfactory right up to the point of entry of the breaker tip. A computational run took 30hrs on a supercomputer.

Ohyama and Nadaoka (1991) developed a robust model with open boundaries which they applied to a number of practical studies of wave propagation over real topography (Ohyama and Nadaoka, 1994, Ohyama *et al.* (1994, 1995)).

Beale *et al.* (1996) in a more mathematically-oriented study considered the formulation of Baker *et al.* for an infinitely-deep two-dimensional fluid. They proved nonlinear stability and convergence of the method as long as the solution remained regular.

It is clear that the computational expense of boundary integral equation methods can be high. This can be offset by vertically sub-dividing the computational domain into sub-domains and requiring continuity across the boundaries. In this way, instead of a full matrix corresponding to every computational point influencing every other point explicitly and the whole matrix being full, the matrix will be composed of full blocks centred on the diagonal which will be considerably smaller, with all terms outside those blocks being zero and superfluous for storage or calculation. Wang *et al.* (1995) used this approach to develop a two-dimensional numerical wave tank which was 110 times the wavelength of the simulated waves. Their paper describes a number of details, including how the domain decomposition was done, and how they installed a damping mechanism for longer waves which developed. The integral equation method was that using a Green function, time stepping was generally fourth-order, and smoothing was occasionally necessary. Plunging breakers were computed.

De Haas and Zandbergen (1996) describe a similar two-dimensional method involving domain decomposition. A Green function method was used, with fourth-order time-stepping. They computed the propagation of waves over a bar, a computation which, with four subdomains, took less than 8 hours on a 90Mhz Personal Computer and 14 minutes on a supercomputer. A larger problem solved was the propagation of irregular waves over an uneven bottom. They noted that the method is being extended into the three-dimensional model described above (Broeze *et al.*, 1993).

4.2.1 Three-dimensional methods

All methods described to here have been for planar wave propagation. Although outside our self-imposed limitation of not treating problems of calculating wave effects on structures and ships, we should mention the early and remarkable achievement of Isaacson (1982) in developing a method which enabled the calculation of the three-dimensional propagation and interaction of fully-nonlinear waves with both fixed and floating bodies. In two dimensions the Green function is a logarithmic function: $1/2\pi \times \log r$, where r is the distance between two points $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2$ similarly, such that $r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. In three dimensions the Green function is $1/4\pi r$, where $r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$, and where the notation is obvious. In three dimensions of course, the appropriate integral equation is written over the *surface* enclosing the domain of interest, which is divided into a number of surface panels. Isaacson introduced an image function which automatically satisfied the flat-bottom boundary condition. Although the detail of tracking both the surface of the body and the sea surface must have been considerable, he used his method as a working tool for ocean engineering problems.

A group working in the Netherlands has developed a method and obtained solutions for three-dimensional breaking waves. The approach began with Romate and Zandbergen (1989), who developed the boundary integral formulation in three dimensions. The original paper was for linear problems with a flat upper boundary. After writing a couple of useful papers on absorbing boundary conditions for such numerical models (Romate, 1992; Broeze and Romate, 1992), they applied the methods developed to nonlinear wave propagation and breaking in three dimensions, and showed the overturning breaking of the tip of a breaker plunging over topography (Broeze *et al.*, 1993). The programs ran on a supercomputer.

Celebi *et al.* (1998) also developed a nonlinear three-dimensional numerical method. They used the same three-dimensional singularity as that described above, but they claimed to ”de-singularize it” by placing such sources (rather than using it as a Green function) outside the computational domain. In this way it was hoped to avoid errors caused by integrating across a singularity. This presumably comes at some computational cost in that if an iterative scheme is used for numerical solution the matrices will not be so diagonally-dominant or rapidly convergent. They used a fourth/fifth order Runge-Kutta-Fehlberg scheme for advancing the solution in time. It was found necessary to re-grid the solution occasionally to avoid the occurrence of a sawtooth instability. Results were produced for three cases (a) the generation of waves by a piston-type wavemaker and their subsequent propagation., (b) diffraction by a truncated vertical cylinder inside a rectangular tank with side walls, and (c) diffraction by a bottom mounted vertical cylinder in the open sea. A supercomputer was used, with a computational time of 20 hours. Results were satisfactory, especially interesting was the result that for nonlinear diffraction the fully nonlinear transient results obtained agreed with experiment better than did results from a second-order diffraction program.

4.2.2 A two-dimensional method which exploits periodicity around the boundary

The author (Fenton, 1993) has developed a method for wave propagation in two dimensions which has some unusual features. It is included here for several reasons – to give the flavor of the mathematics for boundary integral equation methods; to present a simple means of subtracting the singularity in the complex boundary integral equation presentation; and, as it has not been made widely available, to present the really unique feature of the method, that it recognizes that if one passes around a boundary, then all quantities are periodic in position and Fourier methods may be used to obtain simple and accurate differentiation and integration formulae (Fenton, 1992, 1996). The resulting method seems to work quite well for shoaling problems, although it has not been exhaustively tested (Fenton and Kennedy, 1996).

A non-singular boundary integral equation: Consider a two-dimensional region such as that shown in Figure 4-1 containing an incompressible fluid which flows irrotationally, in which case a scalar potential function ϕ exists and satisfies Laplace's equation: $\nabla^2\phi = 0$. A typical boundary value problem is where the value of ϕ or its normal derivative $\partial\phi/\partial n$ or a combination of the two is known at all points on the closed boundary C.

Consider Cauchy's integral formula (see any book on functions of a complex variable):

$$\pi_m \, i \, w(z_m) = \oint \frac{w(z)}{z - z_m} \, dz, \tag{4.1}$$

where $i = \sqrt{-1}$, $z_m = x_m + iy_m$ is the complex co-ordinate of a reference point on the boundary as shown in Figure 4-1, $z = x + iy$ is the general point shown, and $w = \phi + i\psi$, the complex potential, where ψ is the conjugate function, the stream function. In Vinje and Brevig's formulation they used either the real or the imaginary part of (4.1) depending on whether the point m was on a part of the

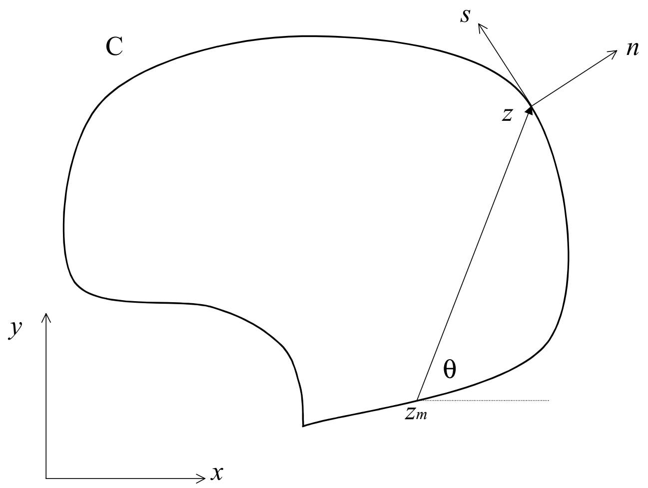

Figure 4-1. Computational domain showing important points and co-ordinates

boundary where ϕ or ψ was known. In this way the algebraic equation obtained by approximating the integral equation is more dominant in the unknown at that point and its numerical properties are better than Green function formulation, although one can be shown to be equivalent to the other.

Here we introduce an alternative expression which effectively subtracts the singularity in (4.1). If the complex function $w = \phi + i\psi$ is analytic, then Cauchy's integral theorem may be invoked for the function $(w(z) - w(z_m))/(z - z_m)$, giving

$$\oint \frac{w(z) - w(z_m)}{z - z_m}\, dz = 0. \tag{4.2}$$

In this equation the integrand is everywhere continuous, even at $z = z_m$, where it becomes the finite value $dw/dz|_m$, and its numerical approximation should be simpler. In this form it is not necessary to calculate the angle π_m at each point. Equation (4.1) is trivially obtained from (4.2), but this form has a number of advantages.

Numerical scheme using periodicity around the contour: A feature of two-dimensional boundaries is that around the boundary all variation is periodic, for in a second circumnavigation of the boundary the integrand is the same as in the first, and so on. This suggests the use of methods that exploit periodicity to gain handsomely in accuracy. A continuous co-ordinate j is introduced here, which is 0 at some reference point on the boundary, and after a complete circumnavigation of the boundary has a value N, which will be taken to be an integer. This was introduced by Dold and Peregrine (1984) who exploited the ability to use finite-difference formulae with equi-spaced values in j. Here we go further and exploit periodicity as well. The integral in equation (4.2) can be written

$$\int_0^N \frac{w(z(j)) - w(z_m)}{z(j) - z_m} \frac{dz}{dj}\, dj = 0. \tag{4.3}$$

Now a numerical approximation is introduced to transform the integral equation into an algebraic one in terms of point values. The integral in equation (4.3) is replaced by the trapezoidal rule approximation:

$$\sum_{j=0}^{N-1} \frac{w(z_j) - w(z_m)}{z_j - z_m}\, z_j' = 0, \tag{4.4}$$

where $z_j = z(j)$ and $z_j' = dz(j)/dj$, but in which after the differentiation, j takes on only integer values. In this case the trapezoidal rule has reduced to the simple sum as the end contributions are from the same point, $z_0 = z_N$ because of the periodicity. This is a particularly simple scheme when compared with some such as Gaussian formulae which have been used to approximate boundary integrals. Where the integrand is periodic, as it is here, the trapezoidal rule is capable of very high accuracy indeed. If it is periodic and has a continuous kth derivative, and if the integral is over a period, then (Abramowitz and Stegun, 1965, #25.4.3):

$$\text{Error} \leq \frac{\text{Constant}}{N^k}. \tag{4.5}$$

This accuracy flows from the nature of the Fourier series which interpolates a function of such a degree of continuity over N points. For functions that are of low degrees of continuity, where k might be 0, 1 or 2 say, the accuracy will be comparable to traditional low-level polynomial approximation of the integrals, however if high degrees of continuity exist, the method should be very accurate indeed. One will have to be careful with the point-spacing at corners, to ensure sufficient continuity.

In the form of equation (4.4), the expression is not yet useful, as the point $j = m$ has to be considered. It is easily shown that in this limit, the integrand (and hence the summand) becomes $dw(m)/dm$, and extracting this term from the sum gives the expression with a "punctured sum" $j \neq m$:

$$\frac{dw}{dm}(m) + \sum_{j=0,\ j\neq m}^{N-1} \frac{w_j - w_m}{z_j - z_m}\, z'_j = 0, \tag{4.6}$$

for $m = 0,\ 1,\ 2, \ldots,\ N-1$, and where the obvious notation $w_j = w(j)$ etc. has been introduced. The notation $dw(m)/dm$ means differentiation with respect to the continuous variable m, evaluated at integer value m. It is convenient here to introduce the symbol Ω_{mj} for the geometric coefficients:

$$\Omega_{mj} = \alpha_{mj} + i\beta_{mj} = \frac{z'_j}{z_j - z_m}, \tag{4.7}$$

whose real and imaginary parts are the coefficients α_{mj} and β_{mj}. Equation (4.6) becomes

$$\frac{dw}{dm}(m) + \sum_{j=0,\ j\neq m}^{N-1} \Omega_{mj}(w_j - w_m) = 0. \tag{4.8}$$

It is easily shown, writing $z(j)$ in complex polar notation as $z(j) = z_m + r(j)e^{i\theta(j)}$, that

$$\alpha_{mj} = \frac{1}{r}\frac{dr}{dj} = \frac{d}{dj}(\log r) \quad \text{and} \quad \beta_{mj} = \frac{d\theta}{dj}, \tag{4.9}$$

such that

$$\Omega_{mj} = \frac{d}{dj}(\log(z_j - z_m)), \tag{4.10}$$

also able to be obtained from equation (4.7). These weighting coefficients can be seen to have a relatively simple physical significance.

One is free to use either the real or imaginary part of the integral equation and of the sums which approximate it, equation (4.6) or (4.8). The two parts can be extracted to give

$$\frac{d\phi}{dm}(m) + \sum_{j=0,\ j\neq m}^{N-1} \left[\alpha_{mj}(\phi_j - \phi_m) - \beta_{mj}(\psi_j - \psi_m)\right] = 0 \tag{4.11}$$

and

$$\frac{d\psi}{dm}(m) + \sum_{j=0,\ j\neq m}^{N-1} \left[\alpha_{mj}(\psi_j - \psi_m) + \beta_{mj}(\phi_j - \phi_m)\right] = 0. \tag{4.12}$$

Either of these equations can be used at each of the N computational points, provided either $d\phi/dm$

or $d\psi/dm$ is known at that point, which can be done from the boundary conditions as described above. Each equation is written in terms of the $2N$ values of ϕ_j and ψ_j. If N of these are known, specified as boundary conditions, then there are enough linear algebraic equations and it should be possible to solve for all the remaining unknowns.

It can be shown that in these equations, the dominant coefficients are the sum $\sum_{j=0,j\neq m}^{N-1}\beta_{mj}$, the coefficient of ψ_m in (4.11) and ϕ_m in (4.12), and the equations are nearly diagonally dominant in those quantities. This is fortunate, for as equation (4.11) can be used on the free surface where $d\phi/dm$ can be evaluated and where ψ_m is the unknown and 4.12 on the sea bed where $d\psi/dm = 0$, and where ϕ_m is unknown, the system of equations is nearly diagonally dominant, which suggests a certain computational robustness, and the possibility of iterative solution.

Although this formulation is likely to be rather more accurate than schemes which approximate the integrand by low-order polynomials, the complex formulation of Vinje and Brevig also possesses this property of diagonal dominance.

Numerical computation of coefficients: Now we have the problem of determining the derivatives with respect to the arc-variable dz/dj and dw/dj. In problems of wave shoaling, the boundary of the computational region, including the sea bed and the free surface, is quite irregular. The periodicity around the boundary may be exploited to give a simple scheme for computing the necessary derivatives around the boundary. The main problem is to compute values of the z_j'. Also, it is convenient to be able to use a means of interpolation between the computational points for plotting purposes which has the same accuracy as the underlying numerical method. Both can be accomplished simply and economically using Fourier approximation.

Suppose the position of each of the N boundary points z_j, $j = 0, 1, \cdots, N-1$, is known. Consider the discrete Fourier transform of the points:

$$Z_m = \frac{1}{N}\sum_{j=0}^{N-1} z_j\, e^{-i2\pi mj/N} = D(z_j\,;m), \tag{4.13}$$

which is a sequence of the complex Fourier coefficients Z_m, for $m = -N/2, \cdots, +N/2$. The Fourier series which interpolates the z_j is

$$z(j) = {\sum_{m=-N/2}^{+N/2}}'' Z_m\, e^{+i2\pi mj/N}, \tag{4.14}$$

where the sum $\sum''$ is interpreted in a trapezoidal rule sense, with a value of 1/2 multiplying the end contributions at $\pm N/2$. For the case of integer j, this is the inverse discrete transform, denoted by the symbol D^{-1}:

$$z_j = D^{-1}(Z_m\,;j), \tag{4.15}$$

although in keeping with the approach of this paper we have not yet adopted integer values for the j in equation (4.14). It can be differentiated to give:

$$z_j' = \frac{i2\pi}{N}{\sum_{m=-N/2}^{+N/2}}'' mZ_m\, e^{+i2\pi mj/N} = \frac{i2\pi}{N}D^{-1}(mZ_m\,;j). \tag{4.16}$$

In this way, if fast Fourier transform programs are available, the z_j' may be computed by taking the discrete Fourier transform of the points z_j, multiplying each coefficient by m and inverting, all of which can be done in $O(N\log N)$ operations.

The values z_j' can now be used in (4.7). The derivatives of the complex potential which appear in (4.11) and (4.12) are accomplished using the same method.

The linear algebraic equations approximating the integral equations have been expressed relatively simply in terms of the coordinates of the computational points z_j and the derivative around the boundary, z_j'. The accuracy of the method depends on how continuous the latter are, and in Fenton (1996) some effort was spent in ensuring continuity across corners of the boundary. In fact it was found that even if no special spacing was used, the accuracy was still surprisingly high. When it was applied to moving boundary problems such as the shoaling of waves, as the boundary points moved, the most sophisticated schemes for point spacing became the most inappropriate, as the accuracy of the schemes were quickly destroyed by the movement of the points. It was found that the most robust schemes obtained simply used equally-spaced points.

Set-up and solution of system of equations: When the z_j' have been obtained, the coefficients $\Omega_{mj} = \alpha_{mj} + i\beta_{mj}$ can be calculated and used in expressions (4.11) and (4.12), one for each point at which an unknown exists. As the equations are nearly diagonally dominant, however, it should be possible to exploit the simple Gauss-Seidel iterative procedure, particularly for timestepping problems such as those for wave propagation, and in practice this was found to work very well indeed. The computational effort is $O(N^2)$ per iteration, and the happy result was found in the present work, that as all boundary points are interpreted as Lagrangian particles, and carry the geometry of the problem with them, then the coefficients are very slowly varying, and a forward extrapolation of previous results gave such an accurate initial estimate that only one iteration was usually necessary each time step to achieve an accuracy of seven figures.

Much programming detail can be avoided if the step of assembling into a matrix is bypassed. In this case, equations (4.11) and (4.12) may simply be rewritten: for points on the free surface

$$\psi_m = \frac{-d\phi(m)/dm - \sum_{j=0,\, j\neq m}^{N-1} \left(\alpha_{mj}(\phi_j - \phi_m) - \beta_{mj}\psi_j\right)}{\sum_{j=0,\, j\neq m}^{N-1} \beta_{mj}}, \tag{4.17}$$

and for points on the sea bed

$$\phi_m = \frac{d\psi(m)/dm + \sum_{j=0,\, j\neq m}^{N-1} \left(\alpha_{mj}(\psi_j - \psi_m) + \beta_{mj}\phi_j\right)}{\sum_{j=0,\, j\neq m}^{N-1} \beta_{mj}}. \tag{4.18}$$

In practice, a procedure of over-relaxation can be adopted to give faster convergence. It was found convenient in the present work where the coefficients changed slowly, not to store all the coefficients α_{mj} etc., as this requires storage of $O\left(N^2\right)$, but to generate the coefficients necessary for each equation every time it had to be evaluated such that the storage was $O(N)$, and large numbers of points could be used. Overall, the implementation of the scheme in this iterative form was particularly simple and rapid.

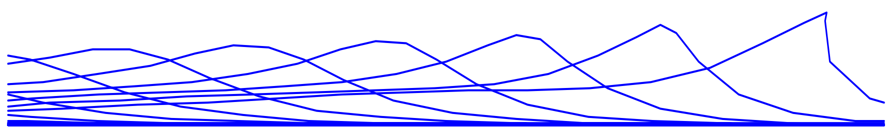

Figure 4-2. High wave on a steeply-shelving beach showing overturning

The only results reported here are for a wave height $H/d = 0.25$ and a length 25 times that of the depth. The initial conditions were computed using an accurate Fourier method (Fenton, 1988(Fenton 1988)).

The wave was allowed to propagate across a shelf with a cosine profile, which shoaled to $1/4$ the depth in a distance of roughly half the horizontal length scale of the wave, a rather abrupt case, and corresponding to the shoaling of a wave on a coral reef. Results are shown in Figure 4-2, and they show some of the interesting phenomena associated with this nonlinear problem. After the wave travelled almost right across the shelf, it started to grow in height quite quickly, while travelling over water of constant shallower depth, and the large feature of a shelf developed behind the wave, which seemed to be in the process of separating from the main wave and possibly becoming part of an oscillatory tail. At the final stage a sharp crest began to form, which turned over as shown, the surface particles in this latter stage experiencing very large accelerations. When compared with detailed pictures of plunging breakers which have been alluded to above, this looks fairly modest, however, so were the computing requirements, which were several hours on a 33MHz Personal Computer, and the author has made no effort to capture the last instants of the breaking; the wave plunged to breaking before the next profile was plotted.

4.3 Spectral and pseudospectral methods

The essential idea behind spectral and pseudo-spectral methods is that global approximation can be used, such that a field of flow is represented by a series of mathematical functions where each of those functions possesses properties such as identically satisfying the field equation and certain boundary conditions, leaving other boundary conditions to determine the coefficients of the functions in the series. The graph of those coefficients is the spectrum of that which is being described, leading to the adjective ”spectral”. It is well-known that spectral approximation is capable of high accuracy. The family of methods described in Section 3 as Fourier methods for steady waves depend on this, where the problem there was to determine the relatively few coefficients in the series for stream function.

The distinction between ”spectral” and ”pseudospectral” is not particularly important. Spectral methods are generally those where all computations can be performed in terms of the spectral coefficients without having to re-compute the physical quantities which are being described. Whereas this is often the case in partial differential equations with simple quadratic nonlinearities, the highly nonlinear nature of water wave boundary conditions where the location of the boundary may itself be an unknown, is such that it is usually necessary to pass repeatedly between the spectral representation and what it is representing. All such methods are pseudospectral.

4.3.1 Early approaches

Early methods using global approximations were simple and accessible – and relatively inefficient computationally. More recent developments have seen remarkably efficient methods developed, but at considerable cost in terms of complexity. Multer (1973) originated the use of spectral methods applied to the full unsteady nonlinear equations. He considered a tank with vertical ends, one of which was allowed to move as a piston wavemaker. He defined the free surface by Lagrangian marker particles which were allowed to move, similar to the procedure used in BIE methods. Fourier coefficients were found by a least-squares procedure. Through the use of relatively low-order numerical integration and the accruing of rounding errors in an orthogonalization process it was found that the method was of limited robustness, yet it was a significant innovation.

Fenton and Rienecker (1982) used the following Fourier series, similar to Multer, rewriting them in a slightly different form here for compatibility with similar Fourier sums in this chapter such as (3.23):

$$\phi(x,y,t) = \bar{U}\,x + \sqrt{gd^3}\sum_{j=-N/2}^{+N/2}{}^{\prime\prime} A_j(t)\frac{\cosh jky}{\cosh jkd}\; e^{ijkx}, \tag{4.19}$$

$$\eta(x,t)/d = \sum_{j=-N/2}^{+N/2}{}^{\prime\prime} Y_j(t)\; e^{ijkx}, \tag{4.20}$$

where d is a depth scale, which was taken to be the undisturbed depth. All derivatives with respect to x, y and t used in the nonlinear free surface boundary conditions (2.10) and (2.13) can be simply obtained by differentiation of these expressions. Equation (4.19) satisfies the two-dimensional version of Laplace's equation (2.2) identically, and the bottom boundary condition (2.4) for the special case of a flat bed, $\partial\phi/\partial y = 0$ on $y = 0$. An additional term involving $\sinh jky$ was introduced in an appendix for the case of more general bottom topography when the bottom boundary condition would have to be satisfied by collocation methods, similar to BIE methods. All variation with time is contained in the coefficients $A_j(t)$. Values of ϕ are known on the surface particles at an initial time t, but to advance the solution it was necessary to know the $A_j(t+\Delta)$. To do this it was necessary to solve the matrix equation obtained from (4.19) written for each of the surface particles. The matrix is full, and, unlike some BIE methods, not diagonally dominant, although the underlying Fourier structure meant that it was not badly conditioned and solution was quite robust. The operation count of this method is $O(N^3)$ at each time step, in common with most BIE implementations.

A leapfrog scheme was used for the time stepping. Differently from Multer, the surface marker particles were not Lagrangian but were constrained to move vertically only. In this way fast Fourier methods could be used for all spatial interpolation and differentiation and the trapezoidal rule could be used for numerical integration with the same accuracy as the underlying Fourier accuracy. An advantage of this was also that at the stage of solving the matrix equation at each time step, the horizontally-equispaced points gave matrix coefficients which retained the character of the underlying orthogonality of the Fourier series on such points, and meant that the solution was robust and accurate. As the free surface was also represented by a Fourier series, this meant that it could only be a single-valued function of the horizontal space dimension, and the method could not describe overturning waves. In practice the Fourier series would require many terms to describe such motion and would probably not be as efficient as BIE methods for such problems anyway. The concentration on accuracy allowed computations of the propagation of solitary waves to proceed for many time steps and to provide interesting results for the reflection of a solitary wave by a wall and the nonlinear overtaking interaction of two solitary waves of different heights.

4.3.2 Fast methods

A more powerful approach was developed independently by Dommermuth and Yue (1987) and West *et al.* (1987), based on conventional analyses of waves by modal and perturbation expansions. This allows a pseudospectral treatment of the nonlinear free-surface conditions, so that the computational effort is proportional to the number of modes N and the order M of the analysis. As the method depends on an expansion about the mean water level, it would have difficulty in treating overturning waves.

Let the potential on the surface be $\phi_s(x, z, t) = \phi(x, \eta, z, t)$. Introducing the horizontal plane gradient operator $\nabla_2 = (\partial/\partial x, \partial/\partial z)$, then substituting into the free surface boundary conditions (2.10) and (2.13) become

$$\frac{\partial\eta}{\partial t} + \nabla_2\phi_s \cdot \nabla_2\eta - (1 + \nabla_2\eta \cdot \nabla_2\eta)\frac{\partial\phi}{\partial y} = 0 \quad \text{on the free surface } y = \eta, \tag{4.21}$$

and

$$\frac{\partial\phi_s}{\partial t} + g\eta + \frac{1}{2}\nabla_2\phi_s \cdot \nabla_2\phi_s - \frac{1}{2}(1 + \nabla_2\eta \cdot \nabla_2\eta)\left(\frac{\partial\phi}{\partial y}\right)^2 = 0, \quad \text{also on } y = \eta. \tag{4.22}$$

These two equations can be integrated in time for the evolution of ϕ_s and η if $\partial\phi/\partial y)_{y=\eta}$ can be evaluated. To do this it is assumed that ϕ and η are $O(\varepsilon) \ll 1$ and ϕ is written as the series

$$\phi(x, y, z, t) = \sum_{m=1}^{M} \phi^{(m)}(x, y, z, t), \tag{4.23}$$

where $\phi^{(m)} = O(\varepsilon^m)$. The free surface potential is then expanded in a Taylor series about $y = 0$ (the level of the undisturbed free surface):

$$\phi_s(x,z,t) = \phi(x,\eta,z,t) = \sum_{m=1}^{M}\sum_{k=0}^{M-m}\frac{\eta^k}{k!}\left.\frac{\partial^k\phi^{(m)}}{\partial y^k}\right|_{(x,0,z,t)}. \tag{4.24}$$

At each order $\phi^{(m)}$ satisfies a boundary value problem for $z < 0$ subject to a Dirichlet condition on $z = 0$ given by (4.24), given η and ϕ_s. These are solved recursively from $m = 1, 2, \ldots$. Typically the $\phi^{(m)}$ are represented by an eigenfunction expansion

$$\phi^{(m)}(x,y,z,t) = \sum_n \phi_n^{(m)}(t)\,\psi_n(x,y,z) \quad \text{for } m = 1,2,\ldots,M, \tag{4.25}$$

where the ψ_n are basis functions satisfying the field equation and conditions on the bottom and side boundaries, which will, in general limit the method's applicability. Substituting (4.25) into the sequence of Dirichlet conditions (4.24) the modal amplitudes $\phi_n^{(m)}(t)$ are obtained using pseudospectral collocation and fast transforms. Finally the vertical velocity on the free surface is given by

$$\frac{\partial\phi}{\partial y}(x,\eta,z,t) = \sum_{m=1}^{M}\sum_{k=0}^{M-m}\frac{\eta^k}{k!}\sum_n\phi_n^{(m)}(t)\left.\frac{\partial^{k+1}\psi_n}{\partial y^{k+1}}\right|_{(x,0,z)}. \tag{4.26}$$

In practice rapid convergence with both M and N was obtained, and for wave heights lower than 80% of the maximum convergence was exponential.

The original application was to deep water, Liu, Dommermuth, and Yue (1992) have applied it to interactions with submerged bodies and varying bottom topography. It is a very powerful method. Tanaka (1993) used it to study the Mach reflection of high solitary waves. He found a number of results which contradicted an earlier simple theory. Most notably, he found that the theoretical prediction that the Mach stem had an amplitude four times that of the incident wave simply did not occur, and that effects of regular reflection were much more likely to ameliorate the Mach stem and its effects. The development of the Mach stem is such a slow process that wave tank experiments have not been able to be used to resolve the differences between theory and computation, and the numerical wave tank in this case has proved to be more definitive.

Craig and Sulem (1993) introduced another high-order spectral scheme based on an expansion about the undisturbed level. The method was applied to planar flow problems with a flat bed, although they are not necessary limitations of the method. Spatially-periodic boundary conditions were assumed.

4.4 Green-Naghdi Theory

This is a very different approach to the problem, and was initiated by Green, Laws and Naghdi (1974) and Green and Naghdi (1976a, 1976b). The theory has its origins in nonlinear plate and shell theory, and used concepts of directed surfaces, known as Cosserat surfaces, with the main aim of reducing the dimensionality of the problem by one. As such, this approach could be lumped in with Boussinesq theories and the local polynomial approximation methods subsequently described here, as theories which assume a particular variation in the vertical, thereby removing that dimension from the computations. In its original formulation, the theory is difficult and the results presented were of first order only.

In subsequent papers its potential in nonlinear wave computations became more fully realised. Ertekin *et al.* (1986) applied it to the notable phenomenon observed experimentally that a ship in a restricted waterway can periodically generate two-dimensional solitons which move away ahead of the vessel. That work required only the assumptions that neither horizontal velocity varied with elevation and that vertical velocity varied linearly with it, no more than the classical long wave theory. Yet the satisfaction of the boundary conditions in the Green-Naghdi formulation meant that the problem could be solved.

Shields and Webster (1988) brought the theory into the context of coastal and ocean engineering. They derived the equations differently, using a variational approach for unsteady inviscid flow. Like the previous work, a fundamental aspect was that a form was assumed for variation in the direction to be eliminated (the vertical), chosen such that boundary conditions could be satisfied. Then a variational procedure was used to minimize the error when this was substituted into the equations, giving algebraic equations. A σ transformation was used which mapped the flow domain of a relatively thin fluid body, such as a sea of infinite depth, to a region between two parallel planes:

$$s(x,y,z,t)=\frac{2z-(\eta+h)}{\eta-h},$$

so that on the surface $z=\eta$, $s=+1$, while on the bed $z=h$, $s=-1$. Then for the velocity field a polynomial variation with s was assumed. An approximation was then made, that equations such as the momentum equations were conserved in a 'weak' sense, being multiplied by a power of s and integrated over the fluid depth. This was enough to determine the coefficients necessary.

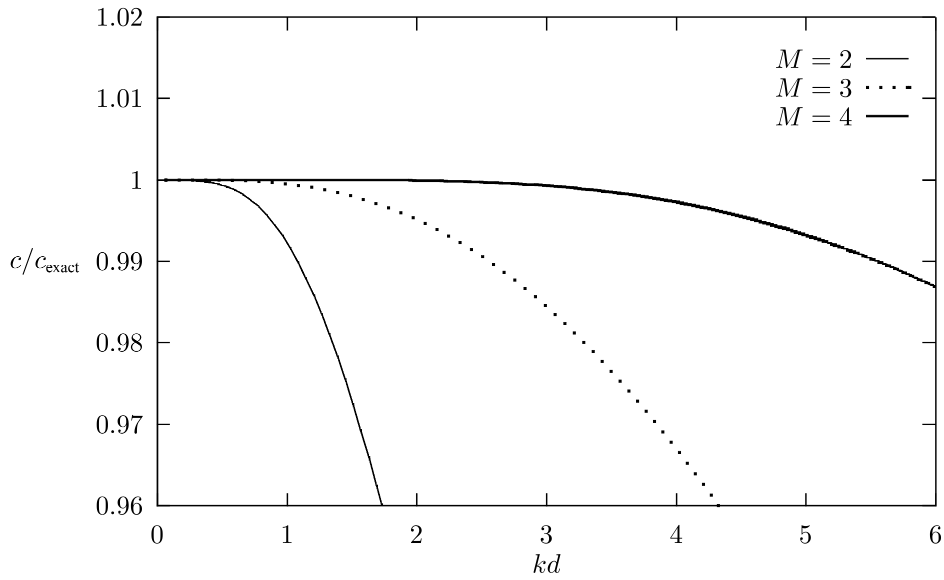

Figure 4-3. Approximations for linear phase velocity

Next a number of problems were examined, for steady two-dimensional waves. The equations quickly become very complicated, such that a reasonable level for presentation was third level, showing cubic variation. These were applied to the solitary wave, and results were obtained which were very rapidly convergent with the level of approximation, much more so than results from perturbation theory which are in the form of power series which are slowly convergent (Fenton, 1972).

Shields and Webster then studied periodic waves over a flat bed and obtained results which, although not as accurate as high-order perturbation methods in the limit of high waves, converged very quickly and gave usable results over much of the range of possible waves. As an example, consider the linearised analytical expressions they obtained for the linear wave speed c in terms of the depth d and wavenumber k, for which the exact linearised expression is $c_{\text{exact}}^2/gd=\tanh kd/kd$:

$$\frac{c^2}{gd}=\frac{1}{1+\frac{1}{3}(kd)^2},\qquad(4.27)$$

$$\frac{c^2}{gd}=\frac{1+\frac{1}{10}(kd)^2}{1+\frac{13}{30}(kd)^2+\frac{1}{80}(kd)^4},\qquad(4.28)$$

$$\frac{c^2}{gd}=\frac{1+\frac{13}{105}(kd)^2+\frac{1}{420}(kd)^4}{1+\frac{16}{35}(kd)^2+\frac{3}{140}(kd)^4+\frac{1}{6300}(kd)^6}.\qquad(4.29)$$

It can be seen that these are in the form of Padé approximants, rational functions of polynomials. If these are expressed as power series, they agree with the power series expansion of the exact expression to second, fourth and sixth order. When compared with the exact expression as shown in Figure 4-3, where M is the order of approximation, it can be seen that these are remarkably accurate and quickly convergent expressions, and are much more so than the equivalent power series. They are accurate even for water deeper than the nominal deep water limit $kd = \pi$. These results were also obtained by Kennedy (1997) using polynomial approximation applied to the conventional formulation of the problem, as described below.

4.5 Local polynomial approximation

Local Polynomial Approximation (LPA) methods were originally conceived as a more consistent way of analyzing sub-surface pressure records (Fenton, 1986). Traditionally for such problems a global approximation method has been used, where the wave field over a finite period is approximated by a Fourier series. The approximation is of very high accuracy and is valid throughout the region of interest. However the equations in which that representation is then used are only those of linear theory. This not only leads to a discrepancy between the low accuracy by which the physical system is modelled and the high numerical accuracy used in that model, but also severe limitations as to the boundary geometries which can be considered, as the theory is valid for a flat bed.

LPA methods, on the other hand, are an attempt to use approximation by locally-valid polynomials where the level of approximation is arbitrary and specifiable, with a finite order of accuracy, but able to be used with the full nonlinear equations. They are an attempt to turn in the direction of conventional numerical methods for the simulation of nonlinear waves, so that the level of physical and numerical approximation is consistent, and that as far as possible routine computational techniques can be used. Their local nature leads to the fortunate result that the computational effort is less than global methods such as BIE methods. In common with those methods, they can be used to satisfy other boundary conditions locally, so that irregular boundaries can be treated, including problems of wave interactions with solid boundaries of a possibly abrupt nature or of a gentle nature such as an irregularly varying bottom.

4.5.1 Two-dimensional computational methods

Local polynomial approximation methods can be used as a means of spatial approximation in performing the unsteady computations for the propagation of waves over varying topography. Whereas in spectral and pseudospectral methods Laplace's equation is satisfied by series of eigenfunctions; and by satisfying a boundary integral equation in those eponymous methods, in the methods to be presented here it is satisfied by polynomial functions which are valid locally. Kennedy (1997), in his PhD thesis and in a series of papers (Kennedy, 1996), Kennedy and Fenton (1996, 1997), and Fenton and Kennedy (1996), has developed the method for both two and three-dimensional waves and has shown that it is an accurate and efficient way of solving many wave propagation problems. It is accurate, even for waves which are approaching what is normally thought of as deep water, and its computational cost is proportional only to the number of computational points ("$O(N)$"), and to the order of approximation. It is capable of treating irregular geometries, however it has not yet been modified to allow for the case of a bluff body such as a cylinder in the flow. As it depends on representing the flow field by polynomials, it would have some difficulty in describing an overturning wave, but as a general workhorse for a number of problems of wave shoaling it may have much to offer.

In this section, on two-dimensional flow fields, two main variants are summarized here: a fully nonlinear model which can provide highly accurate results, and a model which uses Taylor expansions about the undisturbed surface to increase speed, but which sacrifices some accuracy for high waves. For both methods, the expense of solution at each time step is directly proportional to the number of computational subdomains, which allows wave evolution to be computed over relatively large regions with a reasonable computational cost.

Solution of Laplace's equation: For both the fully nonlinear and expansion LPA methods for one dimension in plan, the basic method of solution for Laplace's equation is very similar. As shown in Figure 4-4, the computational domain is divided into subdomains extending vertically from the free

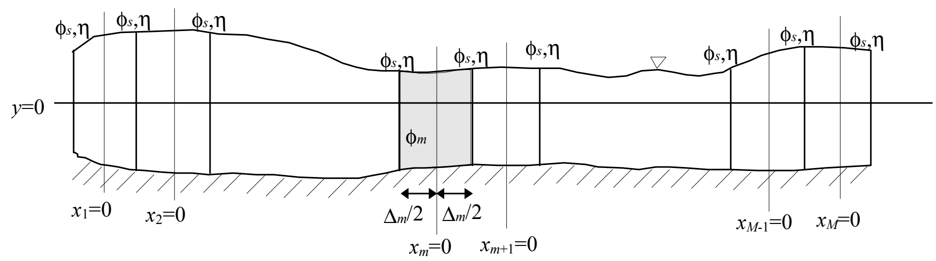

Figure 4-4. Definition sketch for local polynomial approximation

surface to the bed. In any typical subdomain, m, the velocity potential ϕ_m at any point (x_m, y) is represented by the polynomial, similar to (5.4):

$$\phi_m(x_m, y, t) = \left[A_{0R} + \mathrm{Re} \left(\begin{array}{c} \left(\sum_{j=1}^{n-1} (A_{jR} + iA_{jI})(x_m + iy)^j \right) + \\ (x_m + iy)^n \left\{ \begin{array}{ll} A_{nR}, & n \text{ odd} \\ iA_{nI}, & n \text{ even} \end{array} \right. \end{array} \right) \right]_m \tag{4.30}$$

where n is an integer ≥ 3 which controls the level of approximation, $i = \sqrt{-1}$, $\mathrm{Re}(\ldots)$ means taking the real part. The coefficients A_j are functions of time. For any given n, it is these coefficients which must be chosen to best satisfy the boundary value problem. Equation (2.2) is identically satisfied, as in the full complex form of (4.30) the right side is an analytic function of $z_m = x_m + iy$. With the introduction of subdomains, two additional constraints are introduced: the velocity potential, ϕ, and its normal derivative, $\partial\phi/\partial x$, must be continuous across subdomain boundaries.

The velocity potential ϕ may be made analytically continuous across subdomain boundaries through a transformation of basis functions which, in addition, almost halves the number of independent coefficients. However, $\partial\phi/\partial x$ will still be discontinuous across boundaries. (Details of the transformation may be found in Kennedy and Fenton, 1995.) In a domain with M subdomains, the revised basis functions may now be though of as having n independent coefficients defined at each internal boundary between subdomains, plus n coefficients at each of the left and right global boundaries, for a total of $(M-1)\,n+2n = (M+1)\,n$ independent coefficients. The constraints on these are as follows. At each internal boundary between subdomains, the free surface velocity potential is set to the specified value. Next, the two-dimensional form of the bottom boundary condition (2.4) is imposed, using the average value of $\partial\phi/\partial x$ across the boundary. The remaining $n-2$ constraints at each internal boundary match the horizontal velocity, $\partial\phi/\partial x$, across the boundary at $n-2$ discrete points. For overall continuity, these collocation points are here set to the Gauss-Legendre points for level $N = n-2$, using the free surface (or still water level for the expansion method) and bed as limits. At each of the left and right global boundaries, ϕ is also specified at the surface and (2.4) at the bed. However, instead of a velocity match as with the internal boundaries, the horizontal velocity at the boundary, $\partial\phi/\partial x$, is instead set to the known value at $n-2$ collocation points.

All of these constraints result in a set of block banded linear equations. These may be solved using any banded or block banded matrix solver, both of which have a computational cost which is directly proportional to the number of subdomains, M.

Linear dispersion characteristics: Here, as a test of the ability of polynomials to describe the flow field, we consider what results they give for the linear phase speed, compared with traditional approximation by periodic functions in x and hyperbolic functions in y. As subdomain lengths go to zero, a set of differential equations for the velocity potential results, which may be easily solved for the case of small amplitude waves over a level bed. Figure 4-5 shows the LPA small amplitude phase speed relative to the exact relationship for the levels $n = 3, 4, 5, 7$, with collocation points set to the Gauss-Legendre points for $N = n-2$. Accuracy for the level of approximation $n = 3$ is poor in anything other than shallow water but increasing to $n = 4$ gives usable small amplitude results past the nominal deep water limit of $kd = \pi$ $(L/d = 2)$. The level of approximation $n = 5$ (usually used with the LPA expansion method) has good dispersion characteristics even for very short waves with $kd = 2\pi$ $(L/d = 1)$, while with $n = 7$, phase speeds remain excellent past a dimensionless wavenumber of $kd = 3\pi$ $(L/d = 2/3)$.

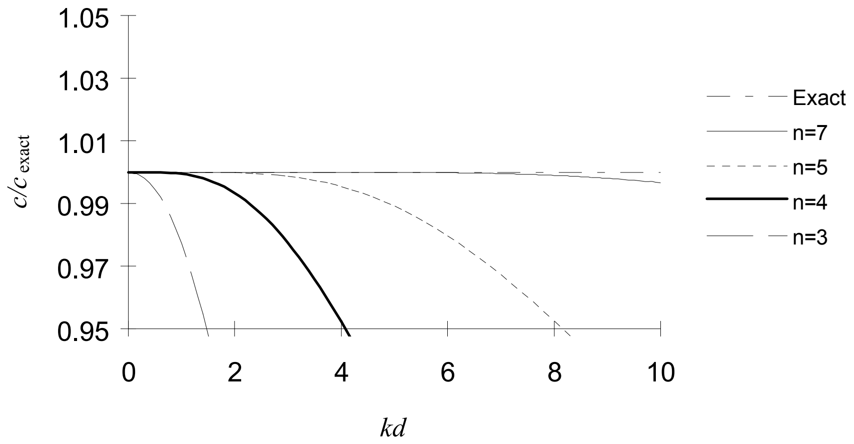

Figure 4-5. LPA linear phase speed

Fully nonlinear LPA: The fully nonlinear version of LPA can provide excellent accuracy for finite depth potential flow calculations. There are two main steps: Laplace's equation is solved exactly as described earlier, and a time stepping method (here usually third or fourth order Adams-Bashforth) is then used to solve the evolution equations (2.10) and (2.15) to advance the solution to the next time step. If the Gauss-Legendre points of level $N = n - 2$ are used as collocation points, then the first $n - 3$ weighted moments of flow will be conserved between subdomains, as well as having velocity matches at the collocation points. For an accurate potential flow method, computations are also quite efficient. For a very large computational run with 900 subdomains and 4000 time steps, total run time for the level $n = 7$ is about 3.5 hours on a 150MHz Personal Computer. Figure 4-6 shows the shoaling of a solitary wave of initial height $H/d = 0.15$ as it propagates onto a shelf of depth $0.5d$. The classical fissioning into multiple solitons is clearly evident, with the leading wave reaching a final dimensionless height on the shelf of 0.507. As an independent estimate of computational accuracy, relative energy fluctuations were less than 2×10^{-4}.

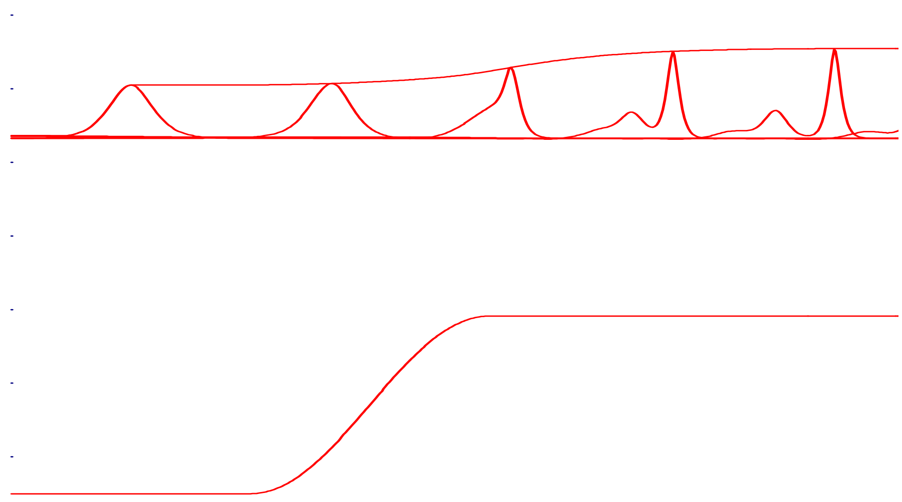

Figure 4-6. Results from the LPA method for the propagation of a wave over a shelf of depth 50%.

LPA free surface expansion method: The free surface expansion method is somewhat more complex, with two major differences from the fully nonlinear version. The first difference is that, instead of solving Laplace's equation using the free surface and the bed as limits, it is instead solved between the still water level and the bed. The mode coupling free surface expansion of Dommermuth and Yue (1987) is then used to relate the value of ϕ at the free surface to the value of ϕ at the still water level. The order of expansion may be easily changed to accommodate the level of nonlinearity of the problem considered. Accuracy is less than the fully nonlinear version for higher waves, but there is one major advantage: since the upper limit of the computational domain remains constant through time, a matrix equation must only be filled and decomposed once, rather than at each time step as with fully nonlinear LPA. This decomposed matrix is then solved with different right hand sides at each time step, which is much faster.

It is worthwhile to implement the second major change only if the computational domain is invariant with time, as is the case here. This involves another change of basis functions, so that there is only one independent variable per computational point. Details of this transformation may be found in Kennedy (1997). With the new basis functions, all conditions but (2.14) are automatically satisfied, so this constraint is used at every computational point to generate a new set of linear matrix equations for the LPA solution to Laplace's equation. The new matrix is purely banded and has both fewer variables and a smaller bandwidth than with the previous basis functions. Computational speeds are therefore further increased. A reasonable analogy may be made between the new basis functions and B-splines, as both are piecewise continuous polynomials which use a set of interpolation conditions to reduce the number of independent computational variables to one per computational point. These new basis functions could also be computed for the fully nonlinear version, but to retain full accuracy, they would have to be recomputed at each time step as the free surface moves. This would slow down computations, which is why they were not used.

The two parameters which control the accuracy of the LPA expansion method are the degree of the polynomials, n, and the order of free surface expansion, J. While the fully nonlinear LPA was developed to calculate potential flow problems with very high accuracy, the LPA expansion method is viewed as a tool for more practical problems. To give reasonable nonlinear accuracy, which is mostly controlled by the order of free surface expansion, J, and good frequency dispersion, which is only affected by the LPA level, n, the parameters $J = 3$ and $n = 5$ were generally used.

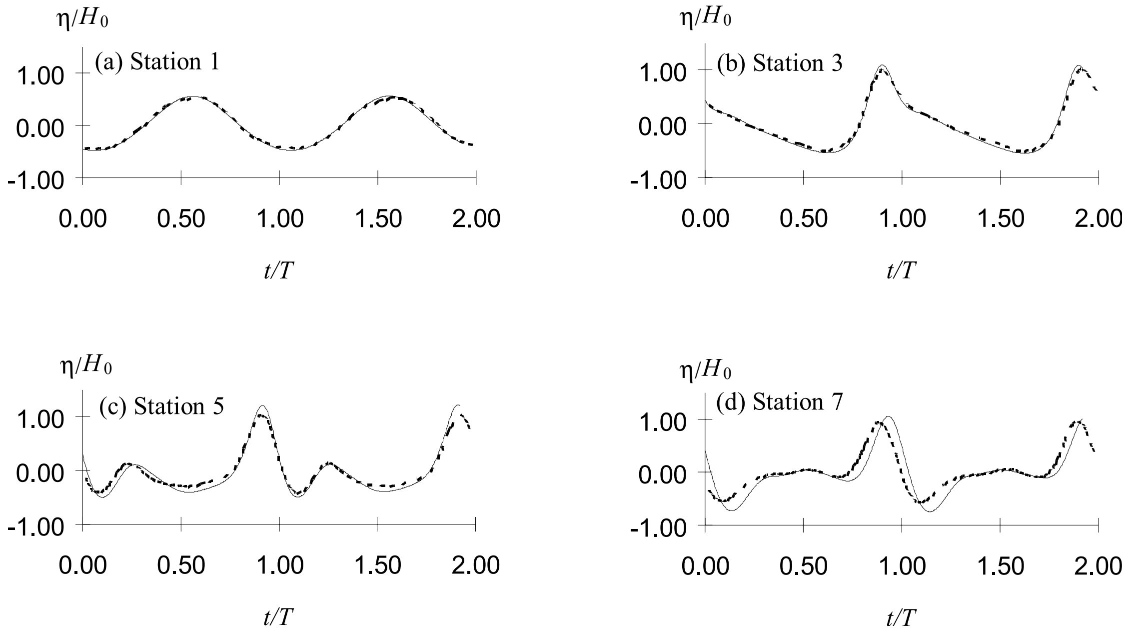

Figure 4-7. Measured and computed time series, $H_0/d = 0.05$, $T\sqrt{g/d} = 9.903$.

For an example of the capabilities of the method, computations here will be compared with the experimental results of Beji and Battjes (1993) as reported by Ohyama *et al.* (1994). In this experiment regular waves were propagated over a two dimensional bar-trough setup and time series of surface elevations were taken at various points. Results are presented here for a single wave train which was initially of reasonably small amplitude in intermediate depths, but became significantly nonlinear over the bar. The initial height was $H_0/d = 0.05$ with a period of $T\sqrt{g/d} = 9.903$. Figure 4-7 shows a comparison between computed and experimental values at Stations 1, 3, 5 and 7, which are, respectively, just before the bar, on the bar crest, on the downslope and in the trough. Agreement is quite good, with the LPA expansion model accurately predicting the steepening of the wave as it progresses up and sheds secondary waves on the bar, and its decomposition into higher harmonics on the downslope. As the wave progresses, computations begin to overestimate wave heights slightly due to the lack of dissipation in the model, and a small phase difference appears. However, similar differences were also noted in the fully nonlinear boundary element computations of Ohyama *et al.* (1994). Overall, the expansion model predicts wave evolution quite well, and may be relied on to provide a good estimate of nonlinear wave evolution for a wide range of waves. Computations are quite fast. For example a computation with 900 subdomains and 4000 time steps would have a total run time on a 150MHz Personal Computer of about 10 minutes for the LPA level $n = 5$ and expansion level $J = 3$, compared with 3.5 hours for the fully nonlinear version using $n = 7$.

4.5.2 Three-dimensional computational methods using local polynomial approximation

In this section the development and application of a method developed by Kennedy is described, following his PhD thesis (1997), and summarized in Kennedy and Fenton (1996). This method uses some of the ideas developed above for two dimensions extended to three dimensions. Again, the velocity potential is represented in the vertical by a polynomial of arbitrary degree, but here a set of differential equations results for the local polynomial approximation to the exact solution. This is shown to provide excellent linear and nonlinear results for a wide range of waves. The degree of polynomial may also easily be changed to give the level of accuracy desired for a particular problem.

Solution of Laplace's equation: The velocity potential used here assumes a polynomial variation in the vertical such that

$$\phi(x, y, z, t) = \sum_{j=0}^{M} A_j(x, z, t)\, y^j, \tag{4.31}$$

where $M \geqslant 2$. The Boussinesq approach would be to write a Taylor series expansion about some point in the water column which satisfies the bottom boundary condition (2.4) to the order of accuracy desired. However, we want to distribute error more evenly than is possible with a Taylor series, where error quickly increases away from the expansion point. The obvious approach would be some sort of finite element method, but this would involve volume integrals which are slow to compute and unless higher order elements were used, convergence would be slow and computational costs would rise significantly. A different approach is used which is simple in concept, and distributes the error in satisfying (2.1) over the water column. First, constraints are imposed so that the velocity potential satisfies (2.4). Next the mean and first $M - 2$ weighted averages of 2.1 over the water column are set to zero such that

$$\int_h^\eta y^l \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) dy = 0, \quad \text{for } l = 0, \ldots, M - 2. \tag{4.32}$$

The appropriate global horizontal boundary conditions finish the specification of the problem and a set of linear equations results which may be solved.

Once the flow field is known, the free surface elevations and velocity potential may be updated in time using the evolution equations (2.10) and equation 2.15 for updating the value of potential on a marker able to move vertically only.

Linear properties: An important property of any approximate model is its ability to describe the flow structure in the vertical, whether or not the waves are large. This was tested in two ways, one by using linearized versions of the free surface conditions to give analytical approximations to the linear dispersion relation connecting wave speed c and wavelength.. The results obtained are exactly the same as the three expressions (4.27), (4.28) and (4.29) as obtained by Shields and Webster (1988) using Green-Naghdi shallow water theories I, II and III. and shown in Figure 4-3. Clearly, even in what is widely considered to be water so deep relative to the wavelength that the effects of the bottom are negligible, $\lambda/d \approx 2$, $kd \approx \pi$, the accuracy is high, possibly surprising for such a case where the vertical distribution is exponential but we are approximating by a polynomial. The second test was that of the vertical distribution of fluid velocities. The accuracy of an integral quantity such as wave speed is not always maintained when it comes to point quantities, but the method proved to predict fluid velocities accurately when compared with linear wave theory, as reported in the references cited, up to a limit of about $kd = \pi$.

It was concluded that the level of LPA approximation $M = 2$ is only suitable for waves in shallow and mildly intermediate depths, and it was suggested that this was not enough. The approximation $M = 3$ gave better results through to the nominal deep water limit, while $M = 4$ gave accurate results through to quite deep water. It was concluded that an increase to an LPA level greater than this was not justified.

Nonlinear properties: The properties of the governing equations were investigated by comparing their solutions for steady nonlinear waves of propagation with those obtained from Fourier methods. Fully nonlinear LPA solutions were found by assuming a steady travelling wave as with the Fourier methods, and determining coefficients by solving the resulting set of nonlinear equations. The results were excellent, with both LPA levels $M = 3$ and 4 able to describe the wave accurately to within about 10% of the waves of maximum height, when the numerical methods could not solve the equations. It was concluded that the ability to describe nonlinear waves gave additional confidence in the accuracy of the three-dimensional LPA method.

Time stepping solutions: The stepping forward in time is the relatively simple part, with either a leapfrog scheme and a third-order Adams-Bashforth method used to update the free surface elevations and surface velocity potentials, after a Runge-Kutta technique for the first time steps to start them. To solve Laplace's equation, all coefficients were represented using fifth-degree two-dimensional B-splines, which are simply the product of one-dimensional B-splines in x and z which have the same centre. Errors in the LPA solution are then proportional to δ^4, where δ is the mesh size. The banded system of linear equations which results was solved using a line-by-line successive under-relaxation technique. Several problems of three-dimensional wave propagation were solved, as now described.

The run-up of a focussed solitary wave on a vertical wall: For the first test a solitary wave of height

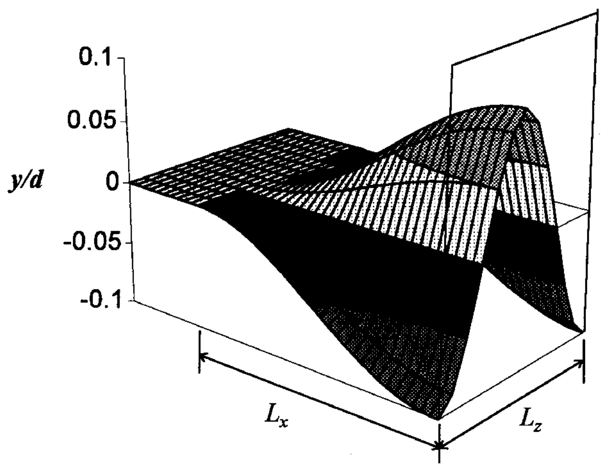

Figure 4-8. Bottom topography for solitary wave reflection

$H/d = 0.2$ was propagated over topography which tended to focus the wave, which was then reflected by a vertical wall. Computations were performed with $M = 4$. Figure 4-8 shows the topography which consisted of a flat bed followed by a double cosine variation in the x and z directions with an amplitude of $0.1\,d$. Figure 4-9 shows the instantaneous surface profile shortly after maximum run-up. Conservation

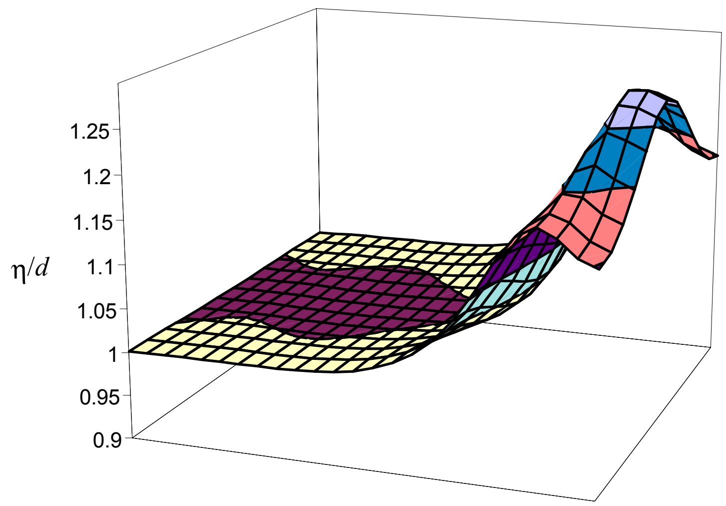

Figure 4-9. The reflection of a focussed solitary wave by a vertical wall

of energy was used as an independent check on the accuracy. For all tests, the maximum fluctuation in relative total energy at any time was less than 2×10^{-4}.

The propagation of regular waves over Whalin's topography: In a series of experiments, Whalin (1971) propagated waves over a semi-circular shoal which tended to focus waves on the flat behind the shoal. Many investigators have since performed computations over the same topography. In the LPA computations regular waves were generated from one boundary using as input the time series of velocity from the fully nonlinear LPA solutions for steady waves. On the transmitting boundary a radiation-type boundary condition was used and solved in finite difference form using an upwinding scheme. Throughout the computations $M = 3$ was used. Figure 4-10 shows the experimental and computational harmonic amplitudes along the centreline for the highest waves. The shoaling topography begins at a distance of 8m. Also shown on the figure are results from different Boussinesq models. For the longest wave with $\tau = 3$s, both computational models overpredict the amplitude of the first harmonic and moderately underpredict the amplitude of the second and third harmonics. This is surprising, for both computational models should operate best in this range. However, the experimental topography was actually composed of a series of steps, whereas the computations approximated this by a smooth slope and so dissipation might have been significant in the experiments.

For the next wave computations agree somewhat better with experimental data. Both LPA and Boussinesq models predict similar features. A feature of this wave is that the initial amplitude of the first harmonic used to calculate the incoming waves appears to be too high. The third figure shows LPA results for a wave of what seems to be the actual input height, and agreement is much better. The final wave tested, with a period of 1 second, had an initial length to depth ratio of 3.27. This was the best predicted of all the waves. The computational values of the first harmonic are still slightly high on the final shoal, but apart from some initial noise the second harmonic is well predicted. The Boussinesq equations tended to underpredict this harmonic. Kennedy concluded that agreement with experiment was not as good as might be hoped, but as different computational methods exhibited similar behavior, that might be too self-critical a judgement.

5. The nonlinear analysis of field and laboratory wave data

An important subset of problems is that of determining the nature of waves or the flow field underneath waves from recorded data, such as a time history of the free surface at a point or the recorded pressure

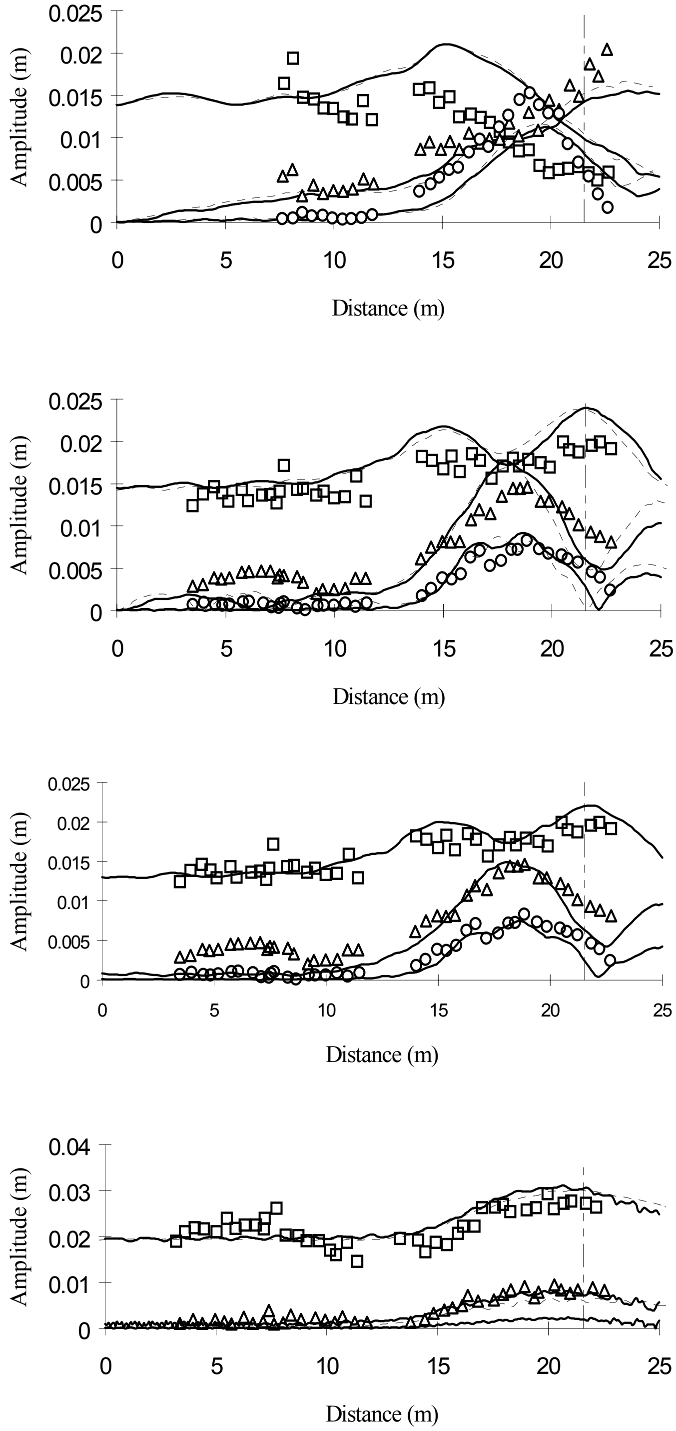

Figure 4-10. Harmonic amplitudes along centreline of Whalin's topography. Solid lines – Kennedy's results; dashed lines – Boussinesq numerical solution; symbols – experimental results; vertical chain-dashed line – beginning of dissipating beach, not reproduced in computations. (a) $\tau = 3$s, $A_1 = 0.0146$m, (b) $\tau = 2$s, $A_1 = 0.00.0149$m, (c) $\tau = 2$s, $A_1 = 0.0149$m (expt), 0.0135m (comp), (d) $\tau = 1$s, $A_1 = 0.0195$m.

history at a sub-surface point. It is strange that these problems, although apparently important, have received relatively little attention when compared with the effort which has gone into devising methods for the simulation of wave propagation as described above.

5.1 The wave kinematics problem

The problem here is, given the recorded time history of the free surface at a point, such as might be obtained from a wave staff, to calculate the velocity structure underneath the waves. This is important in the estimation of wave loading on marine structures. Dean (1965) tackled this problem in the same paper where he made his contributions to solving the steady wave problem (Section 3.1). He assumed that the stream function ψ could be written as a finite Fourier series, but, *cf.* equation (3.5), there should be sine functions as well, because with experimental results one will not have just even functions as when a wave is symmetric about a crest:

$$\psi(X,Y) = -\bar{U}\,Y + \sum_j \sinh jkY \left(A_j \cos jkX + B_j \sin jkX\right), \tag{5.1}$$

remembering that $X = x - ct$ is a local variable moving with the wave. From a procedure similar to that for Fourier approximation theory described in Section 3, given a time history of elevation between the troughs of a single wave it was possible to use (5.1) and satisfied at enough computational points that the coefficients could be found by a least-squares method. He presented results for four waves which showed it to have worked well.

Dean's method assumed that all time and horizontal space variation could be brought together in the form of $x - ct$, namely that all harmonics are bound to the main wave and travel at the same speed, c. Lambrakos (1981) allowed a more general assumption, where harmonics were allowed to travel at various speeds (but without a mean current). As in this case the whole motion cannot be rendered steady by the subtraction of wave speed, it is easier to use velocity potential ϕ as it appears in the unsteady surface dynamic equation (2.12)

$$\phi(x,y,t) = \sum_i \sum_j \cosh jky \left(A_{ij} \cos\left(jkx - i\omega t\right) + B_j \sin\left(jkx - i\omega t\right)\right), \tag{5.2}$$

where i here is an integer and ω is the fundamental frequency of the input signal. The coefficients were to be found mainly by substituting into the kinematic free surface equation (2.10) and satisfying that on a least-squares basis. The dynamic equation (2.12) was used subsequently to give the surface elevation at other points on the wave. The results obtained seem quite good.

Forristall (1985) also took as his primary data the readings of surface elevation as a function of time, and to extract the kinematics chose to satisfy the surface kinematic boundary condition, assuming a planar wave problem. He noted that if the free surface history is known, then the bottom and the kinematic surface boundary condition define a Neumann problem which can be solved for the velocity potential, which could be found through finite difference methods, using irregular computational modules at the surface. He noted that using the linear dispersion relation is surprisingly effective in calculating the surface evolution, but that using second order interaction equations gives better results. The results were excellent, however he noted that because of directional spreading the method should be extended into three dimensions for field problems. He subsequently did this (Forristall, 1986), in which case the computations were very intensive, but good agreement with demanding field experiments was obtained.

Sobey (1992) developed a method based on approximation by a *local* Fourier series, where, similar to (3.23), variation is assumed to be as

$$\phi(x,y,t) = \bar{u}x + \sum_{j=1}^{N} A_j \frac{\cosh jk(y+d)}{\cosh jkd} \sin j\left(kx - \omega t\right), \tag{5.3}$$

but where this is intended to apply only over a local segment of record, with relatively few terms in the series used. As with local polynomial approximation as implemented above, the assumption has also been made that all components are travelling at the same speed. The assumption of trigonometric variation in the horizontal, in the context of local approximation, might be thought to be somewhat arbitrary, as one is approximating a finite length of record of arbitrary variation with no periodicity apparent. With reported used values of series length of only $N = 3$, some of the advantages of Fourier approximation, such as high accuracy and orthogonality of the basis functions, are not brought into play. However, in deeper water the resultant variation with y may make the approximation more economical than polynomial methods. Sobey reported good agreement with data obtained from numerical solution of regular waves, however there were problems with the analysis of laboratory data, including difficulties with convergence of nonlinear optimization methods.

Baldock *et al.* (1996) describe an experimental investigation in a wave flume where a large number of water waves were focused at one point in space and time to produce a large transient wave group. Measurements of the water surface elevation and the underlying kinematics were compared with linear wave theory and a second-order solution based on the sum of wave-wave interactions. The latter gave an improved description, but many of the wave-wave interactions were found to occur at a higher order of wave steepness. The paper did not refer to an earlier one from the same group and based on the same experiments whose contributions in the present context are rather greater. Baldock and Swan (1994) modified the method of Lambrakos (1981) to analyse the results. Whereas Lambrakos concentrated on the kinematic free surface condition, Baldock and Swan found the coefficients in the expansion by minimizing errors in both surface boundary conditions. They noted that the least-squares fit to the boundary conditions was dominated by the large number of spatial locations at which measured data was not provided, and so they introduced an arbitrary weighting function which gave much more weight to results from the measurement point. The computations were intensive, taking two hours on a workstation. The paper is very interesting and provides important insight into the processes around the focused wave group, however, in view of the computational intensity and a certain arbitrariness with the weighting function, the method used does not seem to be a routine way of analysing records. For more general situations where there is no focusing, local methods of analysis might be preferred.

5.2 Analysis of sub-surface pressure and velocity measurements

5.2.1 Introduction

Often in coastal and ocean engineering wave data are recorded by means of a sub-surface pressure transducer. The information obtained by a pressure transducer is $p(t_n)$, the pressure at a finite number of instants t_n, $n = 1, 2, ..., N$. From this record it is desired to infer the properties of the wave which is passing overhead. The conventional approach based on linear wave theory is to take the signal $p(t_n)$, obtain its discrete Fourier transform P_j, for $j = 0, \pm 1, ..., \pm N/2$, use linear wave theory to find the corresponding harmonic components of the surface elevation and fluid velocity, and then to obtain the actual surface elevations and velocities by inverse Fourier transforms. A limiting feature of linear theory is that all components of the waves travel at the speed corresponding to each component, and in the face of all experience, that there is no tendency for higher harmonics to be swept along at the speed with which the dominant wave feature is travelling. Particularly in near-shore regions, with the observed tendency of long waves to travel as (nonlinear) waves of translation, this is an unnecessarily limiting assumption. Also, it seems strange and unnecessary that the *whole* record is used to determine conditions at one point.

A more severe problem is the fundamental ill-conditioning of the problem, where fluid motions are inferred from solution of an elliptic equation (Laplace) from boundary data specified on one level only, that of the pressure transducer. Usually it is desired to find the surface elevations $\eta(t_n)$, from the calculated spectrum Y_j of the free surface elevation. The transfer function connecting Y_j and P_j is proportional to $\cosh k(j)d \,/\, \cosh k(j)y_p$, where $k(j)$ is the wavenumber given by the linear dispersion relation for the jth harmonic of the signal, and y_p is the elevation of the transducer above the bed. For higher frequency components the transfer function varies like $\exp(k(j)(d - y_p))$, and using the short wave approximation for the linear dispersion relation, this varies as $\exp(j^2\omega^2(d - y_p))$, where ω is 2π divided by the total time of the record. It is clear that the transfer function grows remarkably quickly with j, corresponding to higher frequency components. Even for smooth records with spectra P_j which decay quite quickly in j, this exponential growth of the transfer function with j completely destroys any accuracy for harmonic components shorter than the water depth. The method is really only suited to long waves in shallow water. Unfortunately it is for these conditions that linear wave theory is not particularly appropriate, as the waves are likely to be nonlinear and to be long, giving rise to the presence of higher harmonics with their attendant ability to destroy the meaningful part of the signal.

5.2.2 Local sine-wave approximation

Nielsen (1989) originated the use of local approximations. His method cannot be described as ”nonlinear” and hence it falls outside the limits of this Chapter, nevertheless we mention it here, as it is a simple and charming idea and seems to work quite well in many problems. It is based on the fitting of a single sine wave to three pressure measurements equi-spaced in time and using linear theory to calculate the corresponding surface elevation at the central point in time. The present author has always been of the view that it should not work. However, it does seem to work better than theory would suggest, even for large amplitude and nonlinear waves. However it does suffer from a certain lack of robustness, and is not yet a tool for routine analysis of records. Its performance has been studied rather more completely by Townsend (Townsend and Fenton 1996, Townsend,1997, Townsend and Fenton, 1999) in the context of the methods now to be presented here.

In a paper to appear Sobey and Hughes (1998) have taken the idea further and have incorporated horizontal velocity measurements into the formulation. They applied it to the smooth data obtained from Fourier approximation methods for steady waves, for which they reported rather disappointing results. They went on to use the local Fourier approximation method of Sobey (1992) described above. This is presented further below.

5.2.3 Local polynomial approximation methods

The use of local low-degree polynomial approximation does not overcome the fundamental ill-conditioning of the problem, but in water of finite depth the approach is much less susceptible to the problems of ill-conditioning described above.

Basic theory: In this section it is assumed that the waves are travelling over an impermeable bed which is locally flat, that all motion is two-dimensional, and that the fluid is incompressible and the fluid motion irrotational such that a complex velocity potential w exists, $w = \phi + i\,\psi$, where ϕ is velocity potential and ψ is stream function, which is an analytic function of the complex coordinate $z = x + i\,y$. The coordinate origin is taken to be on the bed, beneath the pressure probe. As the entire discussion is based on local approximation we can introduce a local time t, which is zero at the instant at which the pressure reading is taken. The velocity components (u, v) are given by $u - i\,v = dw/dz$. The approximation is made here that the motion *locally* is propagating without change in the x direction with a speed c, which is as yet unknown. Hence, variation with x and t can be combined in the form given by X, a co-ordinate moving with the wave such that $X = x - ct$. Locally, this is a reasonable assumption, as the time scale of distortion of the wave as dispersion and nonlinearity take effect is considerably larger than the local time over which the theory is required to be valid.

A principle of local polynomial approximation is adopted, such that in the vicinity of the pressure probe, throughout the depth of fluid, the complex velocity potential $w(x, y, t)$ and the free surface $\eta(x, t)$ are given by polynomials of degree M in the complex variable $Z = z - ct = x - ct + iy$ moving with the wave such that

$$w(x, y, t) = \sum_{j=0}^{M} \frac{a_j}{j+1} (z - ct)^{j+1}, \tag{5.4}$$

and

$$\eta(x, t) = \sum_{j=0}^{M} b_j (x - ct)^j. \tag{5.5}$$

As w is an analytic function of z, the expansion (5.4) satisfies Laplace's equation (2.2) identically throughout the flow. The bottom boundary condition $v(x,0,t) = 0$ is satisfied if the coefficients a_j are real only, as the b_j are. It remains to satisfy the boundary conditions on the free surface, that the pressure is constant and that particles remain on the surface. It is expected that this type of approximation will be best for long waves, as it is well-known that for short waves, variation in the vertical is exponential, and the use of expressions like (3.5) would be more appropriate.

Here we use the approximation again that motion is steady in a coordinate system $(x-ct, y)$. The steady kinematic equation that the value of ψ is constant on the surface $y = \eta$ is

$$\operatorname{Im} w = -Q, \tag{5.6}$$

where Q is a constant, the volume flux per unit span under the waves. The steady Bernoulli equation is

$$\frac{1}{2}\left|\frac{dw}{dZ}\right|_S^2 + g\eta = R, \tag{5.7}$$

where R is a constant, and the subscript S denotes the surface $y = \eta$.

Also, in the frame in which motion is steady Bernoulli's equation can be written for the point $(0, y_p)$, where the pressure probe is located, and as the pressure around that point can be expressed as a Taylor series in $x - ct$ we can write

$$\frac{p}{\rho}(x, y_p, t) = R - \frac{1}{2}\left|\frac{dw}{dZ}\right|_P^2 + g\eta - gy_p = \sum_{j=0}^{M} p_j (x-ct)^j. \tag{5.8}$$

The coefficients p_j can be found from a sequence of the local pressure readings $p(t_n)$ by interpolation, or more likely, approximation.

Substitution of the series (5.4) and (5.5) into equations (5.6), (5.7) and (5.8) gives polynomials in $x - ct$. These polynomials must be valid locally for all values of $x - ct$, hence the coefficients of each power of $x - ct$ must agree across the equation. This gives a system of nonlinear equations in the unknown coefficients a_0, a_1, a_2, ..., and b_0, b_1, b_2, ... The equations are in terms of the coefficients p_0, p_1, p_2, ..., calculated from the pressure readings. It is feasible to produce the equations by hand calculation for $M = 2$. However for larger values of M the amount of calculation becomes prohibitive, and symbolic algebra manipulation packages are necessary. The original presentation produced solutions for $M = 2$ and $M = 4$. The system of equations is overdetermined, in that, for example at $M = 2$ it contains seven nontrivial equations in the six unknowns a_0, b_0, a_1, b_1, a_2 and b_2.

Fenton (1986) solved the full equations numerically in the initial presentation of the method. The rather cumbersome set of nonlinear equations was solved by direct iteration and by least-squares methods. Although it worked well on smooth computationally-generated waves it has proved to be sensitive to noise in real data and does not calculate water particle velocities accurately. These problems were identified by Townsend (Townsend and Fenton 1996, Townsend,1997), who examined various versions of the method and placed the theory on a more sound footing. Subsequently his identification of the large advantage to be gained by using velocity measurements as well may justify the routine use of this approach in many field and laboratory studies (Townsend and Fenton, 1999). Considerable support for this is provided by the work of Sobey and Hughes (1998).

Use of velocity and pressure data: Townsend's initial attempts to improve the performance of the LPA concept as applied to pressure data used nonlinear least squares optimization methods. Difficulties were encountered with both convergence and sensitivity to some variables, particularly wave speed. However, he improved the governing polynomial representation. The horizontal and temporal variation as unscaled polynomials in $X = x - ct$ as shown above is convenient, but is not necessarily sound computationally. It would be better to use a set of functions which were orthogonal on the computational interval, to ensure that the equations were as linearly independent as possible. Chebyshev polynomials could be used, but Townsend adopted the artifice of scaling the X variation so that for a particular computational window it lay in the interval $[-1,+1]$. In this way, the set of monomials $\left\{X^j,\ j=0,1,\ldots,M\right\}$ is not an orthogonal set, but for M not large they approximate the oscillatory and orthogonal behavior of the first M Chebyshev polynomials.

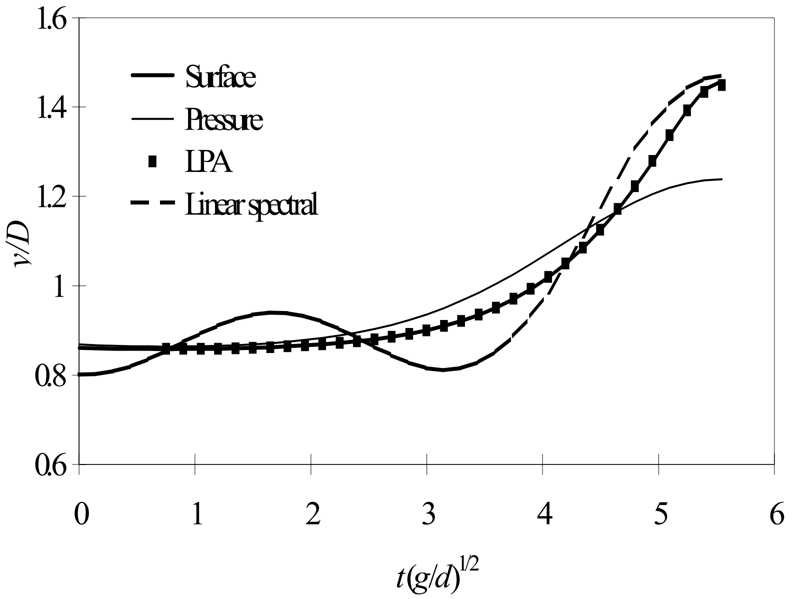

Figure 5-1. Actual and calculated free surface for a wave of $\lambda/d=10$, $H/d=0.6$.

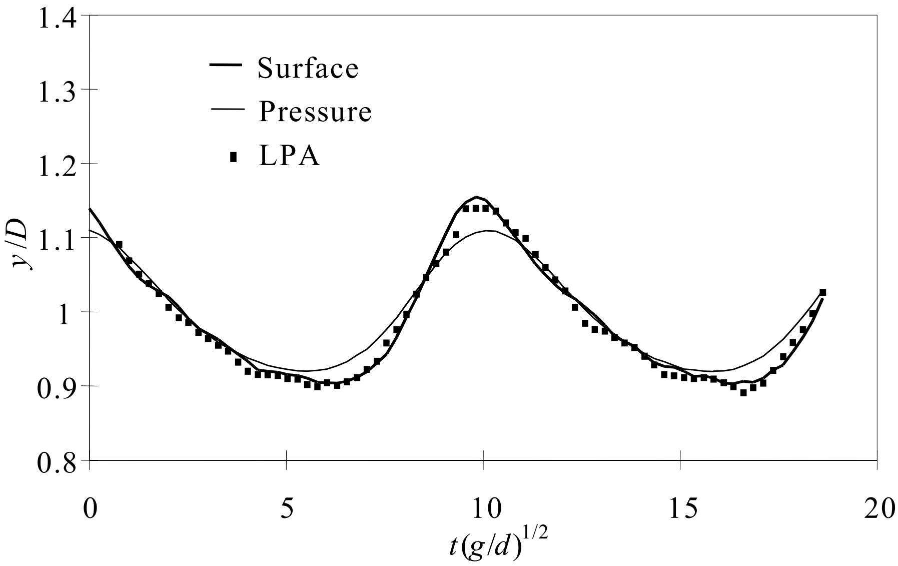

Figure 5-2. Comparison with laboratory wave $T\sqrt{g/d}=10.3$, $H/d\approx 0.17$. Measurements taken very close to floor.

As the indeterminate nature of the wave speed c and hence the horizontal velocity seemed to be the major problem it was decided to use the measured horizontal velocity as an additional input to a new LPA method. By writing a polynomial for the horizontal velocity similar to that for pressure, (5.8), a simpler set of equations with more useful input was obtained. Numerical solution could be reduced to finding the minimum of a single variable at each point, and the results were found to be rather more robust and more accurate, especially if the higher level of approximation $M = 6$ was used. It was recommended that, in view of the desirability of knowing the fluid velocities in determining the current, if any, and providing more useful input, in future the velocity be included as well as pressure in any field or laboratory studies where wave characteristics were to be inferred. Typical results of the method are shown below. Figure 5-1 shows the results for a high wave of intermediate length. The fine line shows the pressure signal used to generate the results; the velocity signal is not shown. The LPA method with $M = 6$, and with pressure and velocity provided, gives accurate results. The traditional linear spectral method has failed due to the imposition of a frequency cutoff; applying the next component would have led to a wildly erroneous solution.

In Figure (5-2) the results for a laboratory generated wave are shown, where the velocity and pressure measurements were taken close to the floor. The asymmetry of the laboratory wave is clear. Conventional linear spectral analysis gave very poor results for this, but the local polynomial approximation method seems quite robust and accurate. The method proved surprisingly accurate, even for waves as short as twice the water depth, generally held to be the limit where waves are considered to be so short that they do not feel the bottom and variation in the vertical is exponential.

5.2.4 Local Fourier approximation

Sobey and Hughes (1998) take this considerably further than the earlier presentation of Barker and Sobey (1996). Like Townsend for polynomial approximation, they have incorporated horizontal velocity measurements into the formulation. A powerful innovation was to use a combination of the dynamic and kinematic free surface boundary conditions which eliminated both temporal and spatial gradients of the free surface elevation from the surface equations. The problem formulation requires the least-squares solution of a number of nonlinear equations, which means that it is computationally intensive and ”probably not well-suited for routine analysis”, but they also noted ”it is possible to perform selective analysis of individual waves or large wave groups within the time series that may be of particular interest”. To test the method they applied it firstly to the smooth results from Fourier approximation methods for steady waves, and found excellent agreement. When applied to field data with ”very energetic wave and current conditions” they noted that ”credible solutions were obtained”. The methods of local approximation seem to offer considerable potential.

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