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Calculating hydrographs from stage records

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ABSTRACT: A method is developed for the routine analysis of river height records so that the corresponding discharges can be calculated. It is based on eliminating space derivatives between the long wave equations and by using an equation similar to the advection-diffusion approximation of river hydraulics to give an ordinary differential equation for discharge. An example shows that an explicit approximation may be as accurate. Either the differential equation or the explicit formula may also be used as an open boundary condition for numerical models of waterways.

KEYWORDS: Looped rating curve, hydrograph, stage, waterways.

Introduction

Almost universally, the routine measurement of the state of a river is that of the *stage*, the surface elevation at a gauging station. While this is an important quantity in determining the danger of flooding, another important quantity is the actual volume flow rate. The traditional way in which volume flow is inferred is for a rating curve to be derived, which is a relationship between the stage measured and the flow passing that point, obtained from measurements of corresponding stages and discharges over a period of time. This is then used in the future so that when stage is measured the corresponding discharge can be obtained from that relationship.

There are a number of factors that might cause the rating curve not to give the actual discharge. This paper addresses the problem of *unsteadiness*. Usually points on the rating curve are obtained for steady flow, whereas in a flood event the discharge will change with time as the flood wave passes, and the discharge will be different from that for a constant stage. On a plot of stage against discharge in an actual flood event, instead of following the rating curve, a looped trajectory is followed such that flows calculated from the rating curve may be in error.

In this paper, we obtain a differential equation for discharge at a station in terms of the stage history at that point. It can be solved using standard numerical techniques for ordinary differential equations, to give the actual discharge hydrograph. Another application of the differential equation obtained will be to computational models of rivers where there is no precise downstream control such as a weir or gate. The differential equation can be used as a “transparent” or open boundary condition, allowing computational flood waves to pass through the boundary without artificial reflections or other numerical effects.

Long wave equations

The flow of water and the propagation of long waves in waterways are described well by the long wave equations. Here we present them in the form where the dependent variables are the stage η and discharge Q:

$$\frac{\partial \eta}{\partial t}+\frac{1}{B}\frac{\partial Q}{\partial x}=\frac{q}{B}, \tag{1}$$

$$\frac{\partial Q}{\partial t}+\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial \eta}{\partial x}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+gAS_f-\beta\frac{Q^2B}{A^2}S_0+\frac{Q^2}{A}\frac{d\beta}{dx}=qu_q\,. \tag{2}$$

The equations in this form include the Boussinesq momentum coefficient β, and are as obtained by Fenton (1999), who developed a somewhat simpler derivation than usual by using Gauss' divergence theorem and showed that fewer assumptions are necessary than has been believed. In these equations x is distance down the channel, assumed straight for this work, t is time, g is gravitational acceleration, A is the cross-sectional area, B is width of the water surface, S_f is the friction slope, S_0 is the mean bed slope at a section, and q is the inflow per unit length which has a velocity of u_q before mixing. We use an empirical friction law for the friction slope S_f, in terms of a conveyance function K, so that we write

$$S_f=\frac{Q^2}{K^2}, \tag{3}$$

where the dependence of K on stage at a section may be determined empirically, or by a standard friction law.

Approximations to the long wave equations

We linearise the equations by considering disturbances about a uniform steady state to be small, which gives a single linear equation. It is convenient to replace temporarily the stage (surface elevation) by a local depth co-ordinate h such that $\eta=z_b+h$, where z_b is the elevation of the bed co-ordinate axis, such that $\partial\eta/\partial x=\partial z_b/\partial x+\partial h/\partial x=-S_0+\partial h/\partial x$. We consider the case where there is no inflow into the river $q=0$ and we assume that β is constant. Consider a steady uniform flow in the waterway of depth h_0, discharge Q_0, area A_0 and conveyance K_0 such that $S_0=Q_0^2/K_0^2$. We write $h=h_0+\varepsilon h_1+\ldots$ and $Q=Q_0+\varepsilon Q_1+\ldots$, where ε is a small quantity expressing the magnitude of the disturbance. Substituting these expansions into equations (1) and (2), using (3) and taking only first order terms in ε gives a pair of linear equations. After cross-differentiation and back-substitution between them, the equations can be reduced to a single equation, the Telegrapher's equation, well known in electrical engineering for describing transients on lines with losses:

$$2\alpha_0\left(\frac{\partial\phi}{\partial t}+c_0\frac{\partial\phi}{\partial x}\right)+\frac{\partial^2\phi}{\partial t^2}+2\beta U_0\frac{\partial^2\phi}{\partial t\partial x}+\left(\beta U_0^2-C_0^2\right)\frac{\partial^2\phi}{\partial x^2}=0\,, \tag{4}$$

where ϕ could stand for the actual depth or velocity, h or Q. The other symbols introduced here are: a parameter $\alpha_0=gA_0\sqrt{S_0}/K_0$, which combines the effects of slope and friction as shown, and increases with both. The quantity c_0 is a velocity such that

$$c_0 = U_0 + V_0 = \frac{\sqrt{S_0}}{B_0} \left. \frac{dK}{dh} \right|_0 , \tag{5}$$

which is simply the Kleitz-Seddon law for kinematic wave speed, which can be obtained rather more simply in the limited context of kinematic wave theory. We have introduced the symbol V_0 here as it makes subsequent expressions simpler to write. Other velocities appearing in the equation are – $C_0 = \sqrt{gA_0 / B_0}$, the traditional expression for the speed of long waves on still water; and $U_0 = Q_0 / A_0$, the mean fluid velocity in the stream.

An equation similar to (4) has been obtained by Deymie and by Lighthill and Whitham (1955) and several others, often for special cases. Surprisingly, little attention has been given to this formulation, which is a Telegrapher's equation.

A tradition in river hydraulics has been to eliminate higher time derivatives in favour of space derivatives, giving kinematic and diffusion theories. The latter was originally obtained by Hayami (see, for example, Henderson, 1966, #9.6), and involves a single time derivative. Here we proceed in the other direction by eliminating all but a single space derivative. We obtain this automatically by writing

$$\frac{\partial}{\partial x} = a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial^2}{\partial t^2} + a_3 \frac{\partial^3}{\partial t^3} + \ldots . \tag{6}$$

such that we assume that we can replace *x* differentiation with a series of space differentiations. We substitute expression (6) into equation (4) and equate powers of the $\partial / \partial t$ operator, to find a sequence of linear equations in the a_1 *etc*. These can be solved to give the equation which is an approximation to the Telegrapher's equation and hence to the Long Wave equations. We obtain the equation

$$\frac{\partial \phi}{\partial x} = -\frac{1}{c_0} \frac{\partial \phi}{\partial t} + \frac{D_0}{c_0^3} \frac{\partial^2 \phi}{\partial t^2} + \frac{G_0}{c_0^5} \frac{\partial^3 \phi}{\partial t^3} + \text{Higher order terms} , \tag{7}$$

where $D_0 = \dfrac{C_0^2 - \beta V_0^2 + (\beta - 1) c_0^2}{2\alpha_0}$, and $G_0 = -\dfrac{\left(C_0^2 - \beta V_0^2 + (\beta - 1) c_0^2\right)\left(C_0^2 + \beta U_0 V_0\right)}{2\alpha_0^2}$.

Equation (7) is an equation for the space derivative of discharge or area or depth at a point in terms of the time derivatives there, which is what we want. If we had proceeded in the other direction, of eliminating higher time derivatives, we would have obtained

$$\frac{\partial \phi}{\partial t} + c_0 \frac{\partial \phi}{\partial x} = D_0 \frac{\partial^2 \phi}{\partial x^2} + \ldots , \tag{8}$$

which is an advection-diffusion equation. This presentation is a generalisation of previous derivations, for we have included the momentum coefficient β . If there were no terms on the right of (8) it is the kinematic wave equation, whose solutions are simply waves which translate without change at a velocity of c_0 , as obtained by Lighthill and Whitham (1955).

An equation connecting stage and discharge at a point

Our task in this work is to eliminate spatial derivatives from the equations in favour of time derivatives so that we can take data measured at a point in a waterway, such as at a gauging station, and extract as much information as possible. If we eliminate the $\partial Q / \partial x$ term between the two equations (1) and (2), and using the friction law (3) we have

$$\frac{\partial Q}{\partial t} - 2\beta \frac{QB}{A} \frac{\partial \eta}{\partial t} + \left(gA - \beta \frac{Q^2 B}{A^2} \right) \frac{\partial \eta}{\partial x} + Q^2 \left(\frac{gA}{K^2} - \beta \frac{BS_0}{A^2} + \frac{1}{A} \frac{d\beta}{dx} \right) = q \left(u_q - 2\beta \frac{Q}{A} \right). \tag{9}$$

Such a procedure was adopted by Faye and Cherry (1980). To determine the discharge hydrograph from stage records the problem remains to eliminate the $\partial \eta / \partial x$ term, which would give us an ordinary differential equation for $Q(t)$, provided we knew the stage $\eta(t)$ at all times at a particular point, from which we could calculate the quantities A, B, and K at any time. Faye and Cherry used the kinematic wave equation to eliminate the space derivative. We, who thus far have made no approximations, eliminate the $\partial \eta / \partial x$ term by using the approximation (7) to the equations of motion equation, written for depth h. Substituting in (9) and with the identity $\partial \eta / \partial x = \partial h / \partial x - S_0$ we obtain

$$\frac{dQ}{dt} = \left(gA - \frac{\beta B}{A^2} Q^2 \right) \left(\frac{1}{c} \frac{d\eta}{dt} - \frac{D}{c^3} \frac{d^2 \eta}{dt^2} - \frac{G}{c^5} \frac{d^3 \eta}{dt^3} \right) + 2\beta \frac{B}{A} \frac{d\eta}{dt} Q + gAS_0 - \left(\frac{gA}{K^2} + \frac{1}{A} \frac{d\beta}{dx} \right) Q^2 . \tag{10}$$

We have replaced all partial derivatives with ordinary derivatives, as we are evaluating them at a fixed point. We have also dropped the subscripts 0 pertaining to the steady uniform flow about which we linearised, as there may be some gain in accuracy in using the actual local values of all the flow quantities. For a particular value of η, the geometric quantities A, B, S_0 and the conveyance K are known, as are c, D, and G. From the stage record we can calculate all the necessary time derivatives of η, so that (10) is an ordinary differential equation in $Q(t)$ which we can solve numerically.

Calculating the discharge hydrograph from a stage record

There are a number of approximations which could be introduced in solving the differential equation (10), such as neglecting the third derivative term. The computation of a third derivative from field data may well not be an accurate procedure, anyway. One could neglect momentum flux terms quadratic in Q, however this is not strictly necessary. Generally one could set $\beta = 1$, which is the sensible and common approximation used elsewhere in hydraulics, however one may not have to introduce it so readily, because usually at gauging stations the conveyance K is obtained from detailed measurements of the velocity distribution at a site. Unlike in most areas of channel hydraulics, one might be able to use a meaningful value of β. Naturally, it would be sensible to drop the term in $d\beta / dx$.

A considerable simplification can be had by neglecting the time derivative term dQ / dt in (10), which turns the differential equation into a transcendental equation for Q. As the right side of (10) is written it seems to be a quadratic equation in Q, however some of the coefficients actually depend upon Q and in general it is more complicated than just a quadratic. However, if one is neglecting the time derivative term, one may as well ignore the quadratic momentum flux terms as well, where the only place that Q appears is quadratically in the friction term, so that the solution is simply written

$$Q = K \sqrt{S_0 + \frac{1}{c} \frac{d\eta}{dt} - \frac{D}{c^3} \frac{d^2 \eta}{dt^2} - \frac{G}{c^5} \frac{d^3 \eta}{dt^3}} . \tag{11}$$

Ignoring the third and second derivative terms gives the well-known Jones formula. Further, ignoring even the first derivative term we have the simplest and the lowest approximation of all, $Q = K\sqrt{S_0}$, which is the method almost universally used which says that the discharge is always just that corresponding to the steady rating curve.

An example

We have a hierarchy of methods that could be used to calculate the flow hydrograph $Q(t)$ from a stage record $\eta(t)$, which we will now test in practice. We have solved the particular case of a stream of 10km length, of slope 0.001, which has a trapezoidal section 10m wide at the bottom with side slopes of 1:2, with a Manning's friction coefficient of 0.04. At a downstream regulator the depth of flow is 2m, while carrying a flow of $10\,\mathrm{m}^3/\mathrm{s}$. That flow is supposed linearly increased ten-fold to $100\,\mathrm{m}^3/\mathrm{s}$ over 30 mins and then reduced to the original flow. We solved the backwater curve problem and then solved the long wave equations in the channel over six hours. At a station halfway along the waterway we recorded the computed stages, which is the data one would normally have, as well as the computed discharges so that we could test the accuracy of this work, using some of the above-mentioned methods.

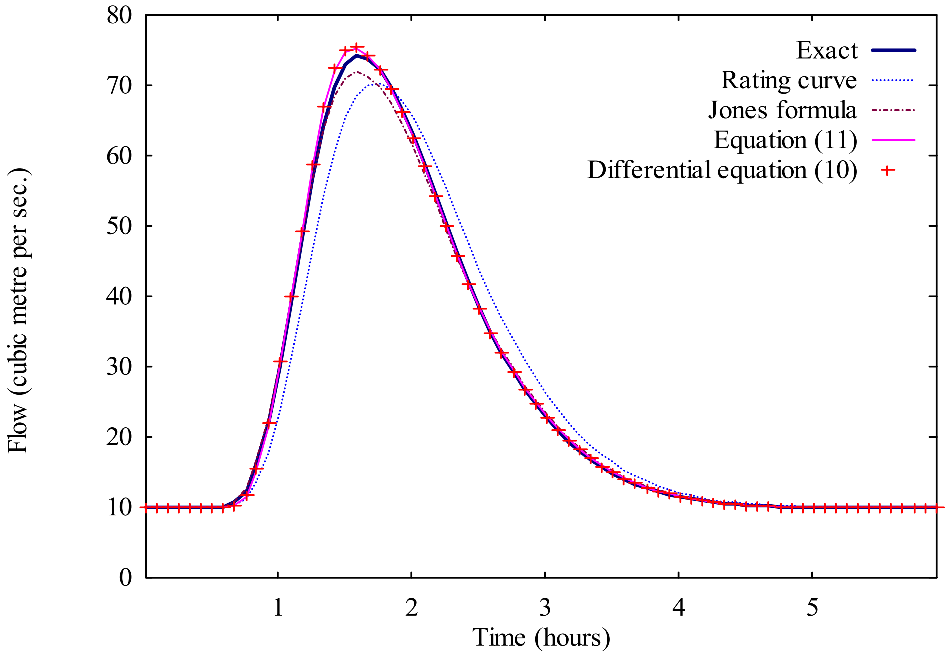

Figure 1. Actual and computed hydrographs from test example.

The thick curve ("Exact") in Figure 1 shows the discharge computed in the simulation part of the study. Other curves are approximations to the discharge. Results obtained by simply using the rating curve formula $Q = K\sqrt{S_0}$ are shown, and these underestimate the peak. The next approximation is the Jones Formula (Henderson, 1966, p393) which is simply the first two terms under the square root sign in (11). Next, equation (11) is used, but only with the second derivative term included, which is the next significant approximation beyond present practice, and finally results from numerical solution of the differential equation (10) are given. It can be seen that the results from the present work are rather more accurate, but that the explicit approximation (11) is just as accurate as the solution of the differential equation.

An application to computational hydraulics

A problem encountered in computational models of waterways is that of providing an open boundary condition where the waterway extends for some distance further than the desired computational domain. At such a boundary, for subcritical flow, it is necessary to provide some sort of relationship between the two dependent variables, such as discharge Q and stage η. Often it has simply been assumed that flow is uniform there, and the relationship provided by one of the well-known friction laws. Alternatively, the fiction is adopted that the channel is artificially extended by some distance. Equation (10) (or approximations based on it, such as dropping the third derivative term and/or using equation (11)) provide a solution to this problem, as it is a relationship between the two variables, discharge and stage, which would exist if disturbances were propagating along a waterway, oblivious to any computational boundary through which they might pass.

Conclusions

We have derived a differential equation for the discharge at a gauging station in terms of the time-record of the stage there. Several approximations to this are possible. An example has shown the equation and approximations to it have promise. It might also be useful as an open boundary condition in numerical models.

References

Faye, R. E., and Cherry, R. N. (1980) Channel and dynamic flow characteristics of the Chatthoochee River, Buford Dam to Georgia Highway 141, U.S. Geological Survey, Water Supply Paper 2063.

Fenton, J. D. (1999) On the long wave equations, manuscript submitted for publication.

Henderson, F. M. (1966) *Open Channel Flow*, Macmillan, New York.

Lighthill, M. J. and Whitham, G. B. (1955) On kinematic waves, I, *Proc. R. Soc. Lond. A*, **229**, 281-316.