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The Cnoidal Theory of Water Waves

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Introduction

Throughout coastal and ocean engineering the convenient model of a steadily-progressing periodic wave train is used to give fluid velocities, pressures and surface elevations caused by waves, even in situations where the wave is being slowly modified by effects of viscosity, current, topography and wind or where the wave propagates past a structure with little effect on the wave itself. In these situations the waves do seem to show a surprising coherence of form, and they can be modelled by assuming that they are propagating steadily without change, giving rise to the so-called steady wave problem, which can be uniquely specified and solved in terms of three physical length scales only: water depth, wave length and wave height. In practice, the knowledge of the detailed flow structure under the wave is so important that it is usually considered necessary to solve accurately this otherwise idealised model.

The main theories and methods for the steady wave problem which have been used are: Stokes theory, an explicit theory based on an assumption that the waves are not very steep and which is best suited to waves in deeper water; cnoidal theory, an explicit theory for waves in shallower water; and Fourier approximation methods which are capable of high accuracy but which solve the problem numerically and require computationally-expensive matrix techniques. A review and comparison of the methods is given in Sobey, Goodwin, Thieke & Westberg (1987) and Fenton (1990). For relatively simple solution methods that are explicit in nature, Stokes and cnoidal theories have important and complementary roles to play, and indeed it has relatively recently been shown that they are more accurate than has been realised (Fenton 1990).

This Chapter describes cnoidal theory and its application to practical problems. It has probably not been applied as often as it might. One reason is the unfamiliarity of the Jacobian elliptic functions and integrals and perceived difficulties in dealing with them. One possibility might be too that in the long wave limit for which the cnoidal theory is meant to apply, almost all expressions for elliptic functions and integrals given in standard texts are very slowly convergent, for example, those in Abramowitz & Stegun (1965). Both of these factors need not be a disincentive; relatively recently some remarkable formulae have been given (Fenton & Gardiner-Garden 1982) which are simple, short, and converge most quickly in the limit corresponding to cnoidal waves. These will be presented below.

It may be, however, that the author has inadvertently provided further reasons for not preferring cnoidal theory. In Fenton (1979) he presented a fifth-order cnoidal theory which was both apparently complicated, requiring the presentation of many coefficients as unattractive floating point numbers, and also gave poor results for fluid velocities under high waves. In a later work (Fenton 1990), however, the author showed that instead of fluid velocities being expressed as expansions in wave height, if the original spirit of cnoidal theory were retained and they be written as series in shallowness, then the results are considerably more accurate. Also in that work it was shown that, in the spirit of Iwagaki (1968), the series can be considerably shortened and simplified by a good approximation.

There have been many presentations of cnoidal theory, most with essentially the same level of approximation, and with relatively little to distinguish the essential common nature of the different approaches. However, there has been such a plethora of different notations, expansion parameters, definitions of wave speed, and so on, that the practitioner could be excused for thinking that the whole field was very complicated. The aim of this article is to review developments in cnoidal theory and to present the most modern theory for practical use, together with a number of practical aids to implementation. My hope is that this surprisingly simple and accurate theory becomes accessible to practitioners and regains its rightful place in the study of long waves.

Initially a history of cnoidal theory is given, and various contributions are described and reviewed. Then, the theory which can be used to generate high-order solutions is outlined and theoretical results from that theory are presented. This chapter contains the first full presentation of those results in terms of rational numbers, as previous versions have used some floating point numbers. Two forms of the theory are presented: the first is a full third-order theory, the second is a fifth-order theory in which a coherent approximation is introduced which, it is suggested, is accurate for most cases where cnoidal theory is used. It is suggested that this be termed the ”Iwagaki Approximation”. Next a detailed procedure for the application of the cnoidal theory is presented, allowing for cases where wavelength or period is specified.

Some new simplifications are introduced here. A number of practical tools and hints for the application of the theory are presented, including a numerical check on the coefficients used in this paper, a simple test to check that the series are correct as programmed by the user, some simple approximations are presented for the elliptic functions and integrals used, and techniques for convergence enhancement of the series are described. A numerical cnoidal theory developed by the author is then presented, which is a numerical method based on cnoidal theory. Finally, the accuracies of the methods are examined and appropriate limits are suggested.

Background

There have been many books and articles written on the propagation of surface gravity waves. The simplest theory is conventional long-wave theory, which assumes that pressure at every point is equal to the hydrostatic head at that point, and which gives the result that any finite amplitude disturbance must steepen until the assumptions of the theory break down. Unsettling evidence that this is not the case was provided by the publication in 1845 by John Scott Russell of his observations on the ”great solitary wave of translation” generated by a canal barge and seeming to travel some distance without modification. This was derided by Airy (”We are not disposed to recognize this wave as deserving the epithets ’great’ or ’primary’ ...”, (Rayleigh 1876)) who believed that it was nothing new and could be explained by long-wave theory. This is one aspect of the *long-wave paradox*, later resolved by Ursell (1953), who showed the importance of a parameter that incorporates the height and length of disturbance and the water depth in determining the behaviour of waves. The value of the parameter determines whether they are true long waves and show the steepening behaviour, or whether they are ”not-so-long” waves where pressure and velocity variation over the depth is more complicated, as is their behaviour. The cnoidal theory fits into this latter category.

Boussinesq (1871) and Rayleigh (1876) introduced an expansion based on the waves being long relative to the water depth. They showed that a steady wave of translation with finite amplitude could be obtained without making the linearising assumption, and that the waves were inherently nonlinear in nature. The solutions they obtained assumed that the water far from the wave was undisturbed, so that the solution was a solitary wave, theoretically of infinite length. Cnoidal theory obtained its name in 1895 when Korteweg & de Vries (1895) obtained their eponymous equation for the propagation of waves over a flat bed, using a similar approximation to Boussinesq and Rayleigh. However they obtained periodic solutions which they termed ”cnoidal” because the surface elevation is proportional to the square of the Jacobian elliptic function $\mathrm{cn}()$. The cnoidal solution shows the familiar long flat troughs and narrow crests of real waves in shallow water. In the limit of infinite wavelength, it describes a solitary wave.

Since Korteweg and de Vries there have been a number of presentations of cnoidal theory. Keulegan & Patterson (1940), Keller (1948), and Benjamin & Lighthill (1954) have presented first-order theories. The latter is particularly interesting, in that it relates the wave dimensions to the volume flux, energy and momentum of a flow in a rectangular channel (or per unit width over a flat bed) and showed that waves of cnoidal form could approximate an undular hydraulic jump. Wiegel (1960, 1964) gave a detailed presentation of first-order theory with a view to practical application, including details of mathematical approximations to the elliptic integrals. A second-order cnoidal theory was presented in a formal manner by Laitone (1960, 1962) who provided a number of results, re-casting the series in terms of the wave height/depth. However, the second-order results are surprisingly inaccurate for high waves (see, for example, Le Méhauté, Divoky & Lin (1968)). The next approximation was obtained by Chappelear (1962), as one of a remarkable sequence of papers on nonlinear waves. He obtained the third-order solution, and expressed the results as series in a parameter directly proportional to shallowness: $(\mathrm{depth}/\mathrm{wavelength})^2$.

Iwagaki published his ”Hyperbolic theory” in 1967, with an English version appearing a year later, Iwagaki (1968). This was an interesting development, for it was an attempt to make the computation of the elliptic functions and integrals simpler by replacing all of them by their limiting behaviours in the limit of solitary waves, except for quantities related to wavelength. In this case, the cn function becomes the hyperbolic secant function sech, and other elliptic functions become other hyperbolic functions, giving rise to the name he proposed. This approach raises a number of interesting points, and further below we will discuss it in some detail.

Tsuchiya and Yasuda in 1974, with an English version in 1985 (Tsuchiya & Yasuda 1985), obtained a third-order solution with the introduction of another definition of wave celerity based on assumptions concerning the Bernoulli constant. Nishimura, Isobe & Horikawa (1977) devised procedures for generating high-order theories for both Stokes and cnoidal theories, making extensive use of recurrence relations. The authors concentrated on questions of the convergence of the series. They computed a 24th-order solution, however, few detailed formulae for application were given.

The author (Fenton 1979) produced a method in 1979 for the computer generation of high-order cnoidal solutions for periodic waves. It had been observed that second-order solutions for fluid velocity were quite inaccurate (Le Méhauté et al. 1968), and it was desired to produce more accurate results, as well as trying to make the method more readily available for practical application. Like Laitone results were expressed in terms of the relative wave height. The paper also raised some interesting points: how it is rather simpler to use the trough depth as the depth scale in presenting results, and that the effective expansion parameter is not simply the wave height but is actually the wave height divided by the elliptic parameter m. For the expansion parameter to be small and for the series results to be valid, the short wave limit is excluded. In this way the cnoidal theory breaks down in deep water (short waves) in a manner complementary to that in which Stokes theory breaks down in shallow water (long waves) (Fenton 1985). A solution was presented to fifth order in wave height, with a large number of numerical coefficients in floating point form. For moderate waves, results were good when compared with experiment, but for higher waves the velocity profile showed exaggerated oscillations and it was found that ninth-order results were worse. These results were unexpectedly poor.

Isobe, Nishimura & Horikawa (1982) continued the work of Nishimura et al. (1977) and presented a unified view of Stokes and cnoidal theories. They proposed a generalised double series expansion in terms of Ursell parameter and shallowness, the square of the ratio of water depth to wavelength. They also proposed a boundary between areas of application of Stokes and cnoidal theory of $\mathbf{U} = 25$, where $\mathbf{U}$ is the Ursell number, $H\lambda^2/d^3$.

In a review article in 1990, the author (Fenton 1990) considered cnoidal theory as well as Stokes theory and Fourier approximation methods such as the ”stream function method”. The approach to cnoidal theory in Fenton (1979) was re-examined and some useful advances made. It was found that if the series for velocity were expressed in terms of the shallowness rather than relative wave height, as done by Chappelear (1962), then results were very much better, and justified the use of cnoidal theory even for high waves. This would fit in with the fundamental approximation of the cnoidal theory being an expansion in shallowness. That review article also incorporated the fact that the wave theory does not determine the wave speed, and that neither Stokes' first nor second definitions of velocity are necessarily correct. In general it is necessary to incorporate the effects of current, as had been done using graphical means in Jonsson, Skougaard & Wang (1970) and Hedges (1978), and analytically for numerical wave theories in Rienecker & Fenton (1981) and for high-order Stokes theory in Fenton (1985).

We now present the theory and results. The theory is essentially that described in Fenton (1979) but with the advances made in Fenton (1990) incorporated plus some more contributions introduced in this chapter. A number of suggestions for practical use are made, and then the performance of the theory is compared with other methods. One of those is a new numerical version of cnoidal theory. In general, the theory as presented here is found to be surprisingly robust and accurate over a wide range of waves.

Cnoidal theory

The physical problem

Consider the wave as shown in Figure 1, with a stationary frame of reference (x, y), x in the direction of propagation of the waves and y vertically upwards with the origin on the flat bed. The waves travel in the x direction at speed c relative to this frame. It is this frame which is the usual one of interest for

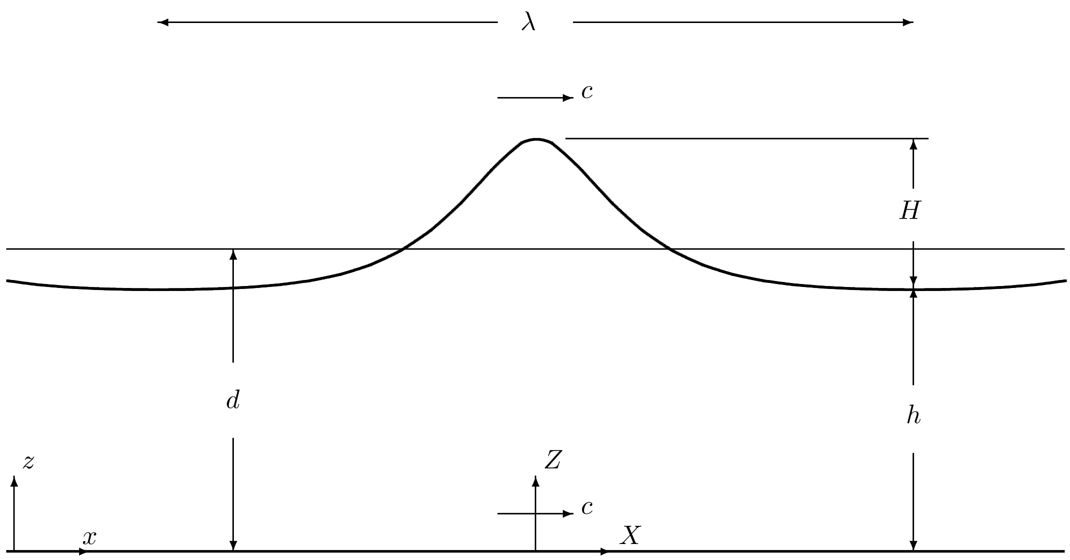

Figure 1. Wave train, showing important dimensions and coordinates

engineering and geophysical applications. Consider also a frame of reference (X, Y) moving with the waves at velocity c, such that $x = X + ct$, where t is time, and $y = Y$. The fluid velocity in the (x, y) frame is (u, v), and that in the (X, Y) frame is (U, V). The velocities are related by $u = U + c$ and $v = V$.

In the (X, Y) frame all fluid motion is steady, and consists of a flow in the negative X direction, roughly of the magnitude of the wave speed, underneath the stationary wave profile. The mean horizontal fluid velocity in this frame, for a constant value of Y over one wavelength λ is denoted by $-\bar{U}$. It is negative because the apparent flow is in the $-X$ direction. The velocities in this frame are usually not important, but they are used to obtain the solution rather more simply.

Equations of motion in a frame moving with the wave

We proceed to develop higher-order solutions for the problem where waves progress steadily without any change of form. Readers more interested in results than the details of the theory could proceed straight to the next section ”Presentation of theoretical results”.

It is easier to consider the equations of motion in the (X, Y) frame moving with the wave such that all motion in this frame is steady. If it is assumed that the water is incompressible and the flow two-dimensional, a stream function $\psi(X, Y)$ exists such that the velocity components (U, V) are given by

$$U = \frac{\partial\psi}{\partial Y} \quad \text{and} \quad V = -\frac{\partial\psi}{\partial X}, \tag{1}$$

and if the flow is irrotational, ψ satisfies Laplace's equation throughout the flow:

$$\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} = 0. \tag{2}$$

The boundary conditions are that the bottom $Y = 0$ is a streamline on which ψ is constant:

$$\psi(X, 0) = 0, \tag{3}$$

and that the free surface $Y = \eta(X)$ is also a streamline:

$$\psi(X, \eta(X)) = -Q, \tag{4}$$

where Q is the volume flux underneath the wave train per unit span. The negative sign is because the flow relative to the wave is in the negative X direction, such that relative to the water the waves will propagate in the positive x direction. The remaining boundary condition comes from Bernoulli's equation:

$$\frac{1}{2}\left(U^2+V^2\right)+gy+\frac{p}{\rho}=R, \tag{5}$$

in which g is gravitational acceleration, p is pressure, ρ is density and R is the Bernoulli constant for the flow, the energy per unit mass. If equation (5) is evaluated on the free surface $Y=\eta(X)$, on which pressure $p=0$, we obtain

$$\frac{1}{2}\left(\left(\frac{\partial\psi}{\partial X}\right)^2+\left(\frac{\partial\psi}{\partial Y}\right)^2\right)\Bigg|_{Y=\eta}+g\eta=R, \tag{6}$$

We assume a Taylor expansion for ψ about the bed of the form:

$$\psi=-\sin Y\frac{d}{dX}\,.\,f\left(X\right)=-Y\frac{df}{dX}+\frac{Y^3}{3!}\frac{d^3f}{dX^3}-\ldots, \tag{7}$$

as in Fenton (1979), where df/dX is the horizontal velocity on the bed. We have introduced the infinite differential operator $\sin Y\, d/dX$ as a convenient way of representing the infinite Taylor series, which has significance only as its power series expansion

$$\sin Y\frac{d}{dx}=Y\frac{d}{dx}-\frac{Y^3}{3!}\frac{d^3}{dx^3}+\frac{Y^5}{5!}\frac{d^5}{dx^5}-\ldots.$$

It can be shown that the velocity components anywhere in the fluid are

$$\begin{aligned} U &= \frac{\partial\psi}{\partial Y}=-\frac{\partial}{\partial Y}\sin Y\frac{d}{dX}\,.\,f\left(X\right)=-\cos Y\frac{d}{dX}\,.\,f'(X),\\ V &= -\frac{\partial\psi}{\partial X}=\frac{\partial}{\partial X}\sin Y\frac{d}{dX}\,.\,f\left(X\right)=\sin Y\frac{d}{dX}\,.\,f'(X), \end{aligned} \tag{8}$$

Further differentiation shows that the assumption of equation (7) satisfies the field equation (2) and the bottom boundary condition (3). The kinematic surface boundary condition (4) becomes

$$\sin\eta\frac{d}{dX}.\,f\left(X\right)=Q, \tag{9}$$

This equation is a nonlinear ordinary differential equation of infinite order for the local fluid depth η and $f'\left(X\right)$, the local fluid velocity on the bed, in terms of the horizontal coordinate X.

The remaining equation is the dynamic free surface condition, equation (6). Substituting equation (8) evaluated on the free surface we obtain:

$$\frac{1}{2}\left(\left(\cos\eta\frac{d}{dX}\cdot\frac{df}{dX}\right)^2+\left(\sin\eta\frac{d}{dX}\cdot\frac{df}{dX}\right)^2\right)+g\eta=R. \tag{10}$$

One of the squares of the infinite order operators can be eliminated by differentiating (9):

$$\sin\eta\frac{d}{dX}\cdot\frac{df}{dX}+\frac{d\eta}{dX}\cos\eta\frac{d}{dX}\cdot\frac{df}{dX}=\frac{dQ}{dX}=0,$$

as Q is constant along the channel, to give

$$\frac{1}{2}\left(1+\left(\frac{d\eta}{dX}\right)^2\right)\left(\cos\eta\frac{d}{dX}\cdot\frac{df}{dX}\right)^2+g\eta=R. \tag{11}$$

Equations (9) and (11) are two nonlinear ordinary differential equations in the unknowns $\eta(X)$ and $f'(X)$, the horizontal velocity on the bed. They are of infinite order, and will have to be approximated in some way. It is possible that they could be solved as differential equations, but that would require an infinite number of boundary conditions. In this and subsequent sections we use two methods, one using power series solution methods, the traditional way, and another using a numerical spectral approach based on assuming series of known functions.

Series solution

The equations have the trivial solution of uniform flow with constant depth: $\eta = h$ and $f'(X) = U$. We proceed to a series expansion about the state of a uniform critical flow. We will assume that all variation in X is relatively slow and can be expressed in terms of a scaled dimensionless variable $\alpha X/h$, where α is a small quantity which expresses the relative slowness of variation in the X direction, and h is the minimum or trough depth of fluid. At this stage, while solving the equations, it is more convenient to write them in terms of dimensionless variables. We let the scaled horizontal variable be $\theta = \alpha X/h$. Writing $\eta_* = \eta/h$ and $f_* = f/Q$, equation (9) can be written

$$\frac{1}{\alpha}\sin\eta_*\alpha\frac{d}{d\theta}\,.\,f_* - 1 = 0. \tag{12}$$

The dynamic boundary condition (11) can be written in terms of these quantities as

$$\frac{1}{2}\left(1+\alpha^2\left(\frac{d\eta_*}{d\theta}\right)^2\right)\left(\cos\eta_*\alpha\frac{d}{d\theta}\,.\,f_*'(\theta)\right)^2 + g_*\eta_* = R_*, \tag{13}$$

where $g_* = gh^3/Q^2$ is a dimensionless gravity number (actually the inverse square of the Froude number) and $R_* = Rh^2/Q^2$ is a dimensionless Bernoulli constant.

The form of equations (12) and (13) suggests that we use α^2 as the expansion parameter. We write the series expansions

$$\eta_* = 1 + \sum_{j=1}^{N}\alpha^{2j}Y_j(\theta), \tag{14}$$

$$f_*' = 1 + \sum_{j=1}^{N}\alpha^{2j}F_j(\theta), \tag{15}$$

$$g_* = 1 + \sum_{j=1}^{N}\alpha^{2j}g_j, \tag{16}$$

$$R_* = \frac{3}{2} + \sum_{j=1}^{N}\alpha^{2j}r_j, \tag{17}$$

where N is the order of solution required. Now, these are substituted into equations (12) and (13). Grouping all the terms in α^0, α^2, α^4, ..., and requiring that the coefficient equation of each be satisfied identically, a hierarchy of equations is obtained which can be solved sequentially. At α^0 the equations are satisfied identically; at α^2 we obtain

$$F_1 + Y_1 = 0,$$
$$F_1 + Y_1 + g_1 - r_1 = 0,$$

with solution $Y_1 = -F_1$ and $g_1 = r_1$. At the next order α^4 we obtain

$$F_2 + Y_2 + F_1Y_1 - \frac{1}{6}F_1'' = 0,$$
$$F_2 + Y_2 + g_2 - r_2 - \frac{1}{2}F_1'' + \frac{1}{2}F_1^2 + g_1Y_1 = 0,$$

and by subtracting one from the other, and using information from the previous order, we obtain

$$\frac{1}{3}F_1'' - \frac{3}{2}F_1^2 + r_2 - g_2 + g_1F_1 = 0. \tag{18}$$

This is a differential equation of second order which is nonlinear because of the F_1^2 term. The usual way of integrating such an equation (for example, #3.3.3of Shen 1993) is to write the F_1'' term as

$d(F_1'^2/2)/dF_1$, integrate the equation with respect to F_1, from which the solution for F_1 in terms of $\mathrm{cn}^2(\theta|m)$, a Jacobian elliptic function (see, for example, #16 of Abramowitz & Stegun 1965), can be obtained. This is a rather complicated procedure. Here we prefer a rather simpler approach to solve the nonlinear differential equation by presuming knowledge of the nature of the solution. We write

$$F_1 = A_1\,\mathrm{cn}^2(\theta|m), \tag{19}$$

where A_1 is independent of θ, and m is the parameter of the elliptic function. Using the properties as set out in Abramowitz & Stegun (1965), (#16.9 and #16.16), reproduced as equations (40) and (41) below, it can be shown that

$$\frac{d^2}{d\theta^2}\,\mathrm{cn}^2\,(\theta|m) = 2 - 2\,m + (8\,m-4)\,\mathrm{cn}^2\,(\theta|m) - 6\,m\,\mathrm{cn}^4\,(\theta|m)\,. \tag{20}$$

Substituting into equation (18) and collecting coefficients of powers of $\mathrm{cn}^2\,(\theta|m)$ we obtain:

$$A_1 = -\frac{4}{3}m, \quad g_1 = \frac{4}{3}\,(1-2m)\,, \quad r_2 - g_2 = \frac{8}{9}m\,(1-m)\,,$$

such that the first-order cnoidal solution is

$$\begin{aligned}
\eta_* &= 1 + \alpha^2\frac{4}{3}m\,\mathrm{cn}^2\,(\theta|m)\,,\\
f_*' &= 1 - \alpha^2\frac{4}{3}m\,\mathrm{cn}^2\,(\theta|m)\,,\\
g_* &= 1 + \alpha^2\frac{4}{3}\,(1-2m)\,,\\
R_* &= \frac{3}{2} + \alpha^2\frac{8}{9}m\,(1-m)\,.
\end{aligned} \tag{21}$$

These solutions should have been shown with order symbols $O\left(\alpha^4\right)$ after them, showing that the neglected terms are at least of the order of α^4. As this is obvious anyway, we choose not to do that here or elsewhere in this work, where the order of neglected terms is almost everywhere obvious.

The procedure described here can be repeated at all orders of α^2, and at each higher order a differential equation is obtained which is linear in the unknowns, and with increasingly complicated and lengthy terms involving the already-known lower orders of solution. The procedure has been described in some detail in Fenton (1979). At each order j, the solution for F_j and Y_j involves polynomials in $\mathrm{cn}^2\,(\theta|m)$ of degree j. With increasing complexity, the operations quickly become too lengthy for hand calculation and it is necessary to use computer software. In the 1979 paper floating point arithmetic and a conventional language (FORTRAN) was used, however now, at the time of writing of this chapter, it is much easier to use modern software which can perform mathematical manipulations. In the preparation of this article the author used the symbolic manipulation software MAPLE.

After the operations have been completed, the solutions which are to hand are power series in α^2, up to the order desired, for η_*, f_*', g_* and R_*. It is convenient to obtain the series for $Q/\sqrt{gh^3}$ by taking the power series of $g_*^{-1/2}$ and the series for R/gh by taking the power series of R_*/g_*, and the series for $\psi/\sqrt{gh^3}$ by taking the power series of $\psi_* \times Q/\sqrt{gh^3}$, where ψ_* is evaluated from

$$\psi_* = -\frac{1}{\alpha}\sin\eta_*\alpha\frac{d}{d\theta}\cdot f_*. \tag{22}$$

Expressions for velocity components follow by differentiation.

So far, all series have been in terms of α^2. It is simpler to express the series in terms of δ, where

$$\delta = 4\alpha^2/3, \tag{23}$$

as suggested by the results of equation (21). Physical solutions could be presented in terms of these power series, and they do reflect the nature of the theory, that the essential nature of the approximation is that the waves be long (α small). However the majority of presentations have converted to expansions in terms of $\varepsilon = H/h$, the ratio of wave height to trough depth. This can be done by expressing a series for ε in terms of δ or α^2 by evaluating $\eta_* - 1$ with $\mathrm{cn} = 1$. The series can then be reverted to express δ or α^2 in terms of ε, which can then be substituted.

The parameter m is determined by the geometry of the wave. As the function $\mathrm{cn}\,(\theta|m)$ has a real period of $4K(m)$, where K is the complete elliptic integral of the first kind, (Abramowitz & Stegun 1965), it is easily shown that $\mathrm{cn}^2\,(\theta|m)$ has a period of $2K(m)$, and as the wave has a wavelength of λ the elementary geometric relation holds:

$$\alpha\frac{\lambda}{h} = 2K(m). \tag{24}$$

The mean depth d is known in physical problems, but it has not entered the calculations yet. The ratio d/h can be obtained from the series for $\eta_* = \eta/h$, by replacing each cn^{2j} by I_j where I_j is the mean value of $\mathrm{cn}^{2j}\,(\theta|m)$:

$$I_j = \overline{\mathrm{cn}^{2j}\,(\theta|m)}, \tag{25}$$

then, (#5.13 of Gradshteyn & Ryzhik 1965): $I_0 = 1$, $I_1 = (-1 + m + e(m))/m$, where $e(m) = E(m)/K(m)$, and $E(m)$ is the complete elliptic integral of the second kind, and the other values may be computed from

$$I_{j+2} = \left(\frac{2j+2}{2j+3}\right)\left(2 - \frac{1}{m}\right) I_{j+1} + \left(\frac{2j+1}{2j+3}\right)\left(\frac{1}{m} - 1\right) I_j, \quad \text{for all } j. \tag{26}$$

This allows all quantities to be calculated with d as the non-dimensionalising depth scale. Similarly the mean fluid speed: $\bar{U}/\sqrt{gh}$, which is related to wave speed, can be obtained from the series for horizontal velocity $U/\sqrt{gh}$, by replacing each cn^{2j} by I_j.

Presentation of theoretical results

Two sets of results are presented here. For the first time a complete solution is given in terms of rational numbers, whereas in Fenton (1990) at least some floating-point numbers were used. Firstly, a full solution is presented to third-order, which is a reasonable limit for space reasons. Next, a fifth-order solution is presented, but in which the approximation is made of setting the parameter m to 1 wherever it explicitly occurs in the coefficients of the series expansions. This makes feasible the presentation of the theory to two higher orders. Here we present the solutions; the application and use of these theoretical results will be described in the subsequent section.

Features of solutions

Although the underlying method relies on an expansion in shallowness, it is often convenient to present results in terms of expansions in wave height. It was found in Fenton (1979) that the best parameter for this was the wave height relative to the trough depth, H/h, which we denote by ε. If the mean depth had been used, to give a series in H/d, many more terms would be involved, because, as equation (A.8) shows, the expression for h/d involves a double polynomial in m and $e = E/K$ of degree n at order n, such that, for example, equation (A.1) for η/h is a triple series in ε/m, m and cn^2, but the corresponding expression for η/d would be a quadruple series in terms of e as well. It is, of course, a simple matter to evaluate η/h from the results given and then to obtain η/d by multiplying by h/d.

The expression of the series as power series in ε/m rather than ε was suggested in Fenton (1979), when it was observed that as m could be less than 1 it was better to monitor the magnitude of ε/m than to have a power series in ε with coefficients which are polynomials in $1/m$, which could become large without it being obvious.

The author has experimented with presenting all the series in terms of α^2, which relates much more closely to the theory being based on an expansion in shallowness, however for all the quantities of cnoidal theory but one, series in ε/m gave more rapid convergence and better accuracy. The only exception is a very notable one, however, and that is the series for the velocity components. The author in Fenton (1979) presented results for fluid velocity which fluctuated wildly and were not accurate for high waves, and this has given cnoidal theory something of a bad name. However, in Fenton (1990) the series were expressed in terms of α^2 (actually $\delta = 4\alpha^2/3$). Much better results were obtained, and were found to be surprisingly accurate even for high waves, and that approach has been retained here.

In the presentation of results, the order of neglected terms such as $O\left((\varepsilon/m)^6\right)$ has not been shown, as it is obvious throughout.

A. Third-order solution

Here the full solution to third order is presented. This will be more applicable to shorter and not-so-high waves, where the parameter m might be less than, say, 0.96.

The symbol cn is used to denote $\mathrm{cn}(\alpha X/h|m) = \mathrm{cn}(\alpha(x-ct)/h|m)$. Equation numbers are shown prefixed with A. Subsequently below, in Table 1 a number is presented corresponding to each equation number. This is meant to provide a check if anybody uses these expressions, to indicate whether a typographical or transcription error might have been made: if the series expression is evaluated with all mathematical symbols on the right set to 1 (a meaningless operation in the context of the theory), then the result should be the number shown in Table 1.

Surface elevation

$$\begin{aligned}\frac{\eta}{h} &= 1+\left(\frac{\varepsilon}{m}\right) m\,\mathrm{cn}^2+\left(\frac{\varepsilon}{m}\right)^2\left(-\frac{3}{4}m^2\,\mathrm{cn}^2+\frac{3}{4}m^2\,\mathrm{cn}^4\right)\\ &\quad+\left(\frac{\varepsilon}{m}\right)^3\left(\left(-\frac{61}{80}m^2+\frac{111}{80}m^3\right)\mathrm{cn}^2+\left(\frac{61}{80}m^2-\frac{53}{20}m^3\right)\mathrm{cn}^4+\frac{101}{80}m^3\,\mathrm{cn}^6\right)\end{aligned} \tag{A.1}$$

Coefficient α

$$\alpha=\sqrt{\frac{3\varepsilon}{4m}}\left(1+\left(\frac{\varepsilon}{m}\right)\left(\frac{1}{4}-\frac{7}{8}m\right)+\left(\frac{\varepsilon}{m}\right)^2\left(\frac{1}{32}-\frac{11}{32}m+\frac{111}{128}m^2\right)\right) \tag{A.2}$$

Horizontal fluid velocity in the frame of the wave

$$\begin{aligned}&\frac{U}{\sqrt{gh}}=-1+\delta\left(\frac{1}{2}-m+m\,\mathrm{cn}^2\right)\\
&+\delta^2\begin{pmatrix}-\frac{19}{40}+\frac{79}{40}m-\frac{79}{40}m^2+\mathrm{cn}^2\left(-\frac{3}{2}m+3m^2\right)-m^2\,\mathrm{cn}^4\\ +\left(\frac{Y}{h}\right)^2\left(-\frac{3}{4}m+\frac{3}{4}m^2+\mathrm{cn}^2\left(\frac{3}{2}m-3m^2\right)+\frac{9}{4}m^2\,\mathrm{cn}^4\right)\end{pmatrix}\\
&+\delta^3\begin{pmatrix}\frac{55}{112}-\frac{3471}{1120}m+\frac{7113}{1120}m^2-\frac{2371}{560}m^3+\mathrm{cn}^2\left(\frac{71}{40}m-\frac{339}{40}m^2+\frac{339}{40}m^3\right)\\ +\mathrm{cn}^4\left(\frac{27}{10}m^2-\frac{27}{5}m^3\right)+\frac{6}{5}m^3\,\mathrm{cn}^6\\ +\left(\frac{Y}{h}\right)^2\begin{pmatrix}\frac{9}{8}m-\frac{27}{8}m^2+\frac{9}{4}m^3+\mathrm{cn}^2\left(-\frac{9}{4}m+\frac{27}{2}m^2-\frac{27}{2}m^3\right)\\ +\mathrm{cn}^4\left(-\frac{75}{8}m^2+\frac{75}{4}m^3\right)-\frac{15}{2}m^3\,\mathrm{cn}^6\end{pmatrix}\\ +\left(\frac{Y}{h}\right)^4\begin{pmatrix}-\frac{3}{16}m+\frac{9}{16}m^2-\frac{3}{8}m^3+\mathrm{cn}^2\left(\frac{3}{8}m-\frac{51}{16}m^2+\frac{51}{16}m^3\right)\\ +\mathrm{cn}^4\left(\frac{45}{16}m^2-\frac{45}{8}m^3\right)+\frac{45}{16}m^3\,\mathrm{cn}^6\end{pmatrix}\end{pmatrix}\end{aligned} \tag{A.3.1}$$

The leading term -1 should not cause concern, for if the wave is considered to be travelling in the positive x direction, then relative to the wave the fluid is flowing under the wave in the negative x direction with velocities of the order of the wave speed.

Vertical fluid velocity This can be obtained from equation (A.3.1) by using the mass conservation equation $\partial U/\partial X + \partial V/\partial Y = 0$, and the result from equation (40) that $d(\mathrm{cn}(\theta|m))/d\theta = -\mathrm{sn}(\theta|m)\,\mathrm{dn}(\theta|m)$, with the result that each term in (A.3.1) containing $(Y/h)^i\,\mathrm{cn}^j\,(\alpha X/h|m)$, for

$j > 0$, is replaced by $\alpha \, \mathrm{sn}() \, \mathrm{dn}() \left(\frac{j}{i+1} \right) \times (Y/h)^{i+1} \, \mathrm{cn}^{j-1}()$. Hence if we write equation (A.3.1) as

$$\frac{U}{\sqrt{gh}} = -1 + \sum_{i=1}^{5} \delta^i \sum_{j=0}^{i-1} \left(\frac{Y}{h} \right)^{2j} \sum_{k=0}^{i} \mathrm{cn}^{2k}() \, \Phi_{ijk},$$

where each coefficient Φ_{ijk} is a polynomial of degree i in the parameter m, the vertical velocity component follows:

$$\frac{V}{\sqrt{gh}} = 2\alpha \; \mathrm{cn}() \, \mathrm{sn}() \, \mathrm{dn}() \sum_{i=1}^{5} \delta^i \sum_{j=0}^{i-1} \left(\frac{Y}{h} \right)^{2j+1} \sum_{k=1}^{i} \mathrm{cn}^{2(k-1)}() \frac{k}{2j+1} \Phi_{ijk}. \tag{A.3.2}$$

Discharge

$$\begin{aligned} \frac{Q}{\sqrt{gh^3}} &= 1 + \left(\frac{\varepsilon}{m} \right) \left(-\frac{1}{2} + m \right) + \left(\frac{\varepsilon}{m} \right)^2 \left(\frac{9}{40} - \frac{7}{20} m - \frac{1}{40} m^2 \right) \\ &\quad + \left(\frac{\varepsilon}{m} \right)^3 \left(-\frac{11}{140} + \frac{69}{1120} m + \frac{11}{224} m^2 + \frac{3}{140} m^3 \right) \end{aligned} \tag{A.4}$$

Bernoulli constant

$$\begin{aligned} \frac{R}{gh} &= \frac{3}{2} + \left(\frac{\varepsilon}{m} \right) \left(-\frac{1}{2} + m \right) + \left(\frac{\varepsilon}{m} \right)^2 \left(\frac{7}{20} - \frac{7}{20} m - \frac{1}{40} m^2 \right) \\ &\quad + \left(\frac{\varepsilon}{m} \right)^3 \left(-\frac{107}{560} + \frac{25}{224} m + \frac{13}{1120} m^2 + \frac{13}{280} m^3 \right) \end{aligned} \tag{A.5}$$

Mean fluid speed in frame of wave

$$\begin{aligned} \frac{\bar{U}}{\sqrt{gh}} &= 1 + \left(\frac{\varepsilon}{m} \right) \left(\frac{1}{2} - e \right) + \left(\frac{\varepsilon}{m} \right)^2 \left(-\frac{13}{120} - \frac{1}{60} m - \frac{1}{40} m^2 + \left(\frac{1}{3} + \frac{1}{12} m \right) e \right) \\ &\quad + \left(\frac{\varepsilon}{m} \right)^3 \left(-\frac{361}{2100} + \frac{1899}{5600} m - \frac{2689}{16800} m^2 + \frac{13}{280} m^3 + \left(\frac{7}{75} - \frac{103}{300} m + \frac{131}{600} m^2 \right) e \right) \end{aligned} \tag{A.6}$$

Wavelength in terms of H/d

$$\begin{aligned} \frac{\lambda}{d} &= 4K(m) \left(3 \frac{H}{md} \right)^{-1/2} \left(1 + \left(\frac{H}{md} \right) \left(\frac{5}{4} - \frac{5}{8} m - \frac{3}{2} e \right) \right. \\ &\quad \left. + \left(\frac{H}{md} \right)^2 \left(-\frac{15}{32} + \frac{15}{32} m - \frac{21}{128} m^2 + \left(\frac{1}{8} - \frac{1}{16} m \right) e + \frac{3}{8} e^2 \right) \right) \end{aligned} \tag{A.7}$$

Trough depth in terms of H/d

$$\begin{aligned} \frac{h}{d} = 1 \quad & + \left(\frac{H}{md} \right) (1 - m - e) + \left(\frac{H}{md} \right)^2 \left(-\frac{1}{2} + \frac{1}{2} m + \left(\frac{1}{2} - \frac{1}{4} m \right) e \right) \\ & + \left(\frac{H}{md} \right)^3 \left(\frac{133}{200} - \frac{399}{400} m + \frac{133}{400} m^2 + \left(-\frac{233}{200} + \frac{233}{200} m - \frac{1}{25} m^2 \right) e + \left(\frac{1}{2} - \frac{1}{4} m \right) e^2 \right) \end{aligned} \tag{A.8}$$

B. Fifth-order solution with Iwagaki approximation

A simplification which can be made is suggested by the fact that for waves which are not low and/or short, the values of the parameter m used in practice are very close to unity indeed. This suggests the simplification that, in all the formulae, wherever m appears as a coefficient, it be replaced by $m = 1$, which results in much shorter formulae. In honor of the originator of this approach (Iwagaki 1968), we suggest that this be termed the "Iwagaki approximation". Here, this is implemented (but in a manner different from Iwagaki's original suggestion) that wherever m appears as the argument of an elliptic integral or function, such as the elliptic functions $\mathrm{cn}(\theta|m)$, $\mathrm{sn}(\theta|m)$ and $\mathrm{dn}(\theta|m)$, and the elliptic integrals $K(m)$, $E(m)$ and their ratio $e(m)$, the approximation is not made, as the quantities can be evaluated by methods which do not need to make the approximation.

This theory will be applicable for longer waves, where $m \geq 0.96$. Iwagaki (1968) observed that in many applications of cnoidal theory m can be set to 1 with no loss of practical accuracy. He presented results to second order and termed the resulting waves "hyperbolic waves" because the Jacobian elliptic functions approach hyperbolic functions in that limit. In Fenton (1990) theoretical results to fifth order were presented with this approximation, and it was shown that it was accurate for longer and higher waves. The present author, however, prefers not to use the term "hyperbolic waves" as in this work we will present a number of useful approximations to the elliptic functions which have a wider range of validity than merely replacing $\mathrm{cn}()$ by the hyperbolic function $\mathrm{sech}()$. The version of the theory which we present is a simple modification of the full theory: that wherever m appears explicitly as a coefficient, not as an argument of an elliptic integral or function, it is replaced by 1, but is retained in all elliptic integrals and functions.

The use of the Iwagaki approximation for typical values of m in the cnoidal theory is rather more accurate than the conventional approximations on which the theory is based, namely the neglect of higher powers of the wave height or the shallowness. For example, $m = 0.9997$ for a wave of height 40% of the depth and a length 15 times the depth; in this case the error introduced by neglecting the difference between m^6 and 1^6 (0.002) in first-order terms is less than the neglect of sixth-order terms not included in the theory ($0.4^6 = 0.004$).

All the results presented here agree with those presented in Fenton (1990) (where some coefficients were presented as floating point numbers), except for two typographical errors in that work: in the equivalent of equation (B.7) the term $3H/d$ should have appeared with a negative exponent, and in the equivalent of (B.8) a third-order coefficient ($-e/25$) was shown with the sign reversed, (Poulin & Jonsson 1994).

Surface elevation

$$\begin{aligned}\frac{\eta}{h} &= 1+\varepsilon\,\mathrm{cn}^2+\varepsilon^2\left(-\frac{3}{4}\,\mathrm{cn}^2+\frac{3}{4}\,\mathrm{cn}^4\right)+\varepsilon^3\left(\frac{5}{8}\,\mathrm{cn}^2-\frac{151}{80}\,\mathrm{cn}^4+\frac{101}{80}\,\mathrm{cn}^6\right)\\ &\quad+\varepsilon^4\left(-\frac{8209}{6000}\,\mathrm{cn}^2+\frac{11641}{3000}\,\mathrm{cn}^4-\frac{112393}{24000}\,\mathrm{cn}^6+\frac{17367}{8000}\,\mathrm{cn}^8\right)\\ &\quad+\varepsilon^5\left(\frac{364671}{196000}\,\mathrm{cn}^2-\frac{2920931}{392000}\,\mathrm{cn}^4+\frac{2001361}{156800}\,\mathrm{cn}^6-\frac{17906339}{1568000}\,\mathrm{cn}^8+\frac{1331817}{313600}\,\mathrm{cn}^{10}\right)\end{aligned} \tag{B.1}$$

Coefficient α

$$\alpha=\sqrt{\frac{3\varepsilon}{4}}\left(1-\frac{5\,\varepsilon}{8}+\frac{71\,\varepsilon^2}{128}-\frac{100627\,\varepsilon^3}{179200}+\frac{16259737\,\varepsilon^4}{28672000}\right) \tag{B.2}$$

Horizontal fluid velocity in the frame of the wave:

$$
\begin{aligned}
\frac{U}{\sqrt{gh}} \;=\; & -1+\delta\left(-\frac{1}{2}+\mathrm{cn}^2\right)+\delta^2\left(-\frac{19}{40}+\frac{3}{2}\,\mathrm{cn}^2-\mathrm{cn}^4+\left(\frac{Y}{h}\right)^2\left(-\frac{3}{2}\,\mathrm{cn}^2+\frac{9}{4}\,\mathrm{cn}^4\right)\right) \\
& +\delta^3\left(\begin{array}{l}
-\frac{55}{112}+\frac{71}{40}\,\mathrm{cn}^2-\frac{27}{10}\,\mathrm{cn}^4+\frac{6}{5}\,\mathrm{cn}^6+\left(\frac{Y}{h}\right)^2\left(-\frac{9}{4}\,\mathrm{cn}^2+\frac{75}{8}\,\mathrm{cn}^4-\frac{15}{2}\,\mathrm{cn}^6\right) \\
+\left(\frac{Y}{h}\right)^4\left(\frac{3}{8}\,\mathrm{cn}^2-\frac{45}{16}\,\mathrm{cn}^4+\frac{45}{16}\,\mathrm{cn}^6\right)
\end{array}\right) \\
& +\delta^4\left(\begin{array}{l}
-\frac{11813}{22400}+\frac{53327}{42000}\,\mathrm{cn}^2-\frac{13109}{3000}\,\mathrm{cn}^4+\frac{1763}{375}\,\mathrm{cn}^6-\frac{197}{125}\,\mathrm{cn}^8 \\
+\left(\frac{Y}{h}\right)^2\left(-\frac{213}{80}\,\mathrm{cn}^2+\frac{3231}{160}\,\mathrm{cn}^4-\frac{729}{20}\,\mathrm{cn}^6+\frac{189}{10}\,\mathrm{cn}^8\right) \\
+\left(\frac{Y}{h}\right)^4\left(\frac{9}{16}\,\mathrm{cn}^2-\frac{327}{32}\,\mathrm{cn}^4+\frac{915}{32}\,\mathrm{cn}^6-\frac{315}{16}\,\mathrm{cn}^8\right) \\
+\left(\frac{Y}{h}\right)^6\left(-\frac{3}{80}\,\mathrm{cn}^2+\frac{189}{160}\,\mathrm{cn}^4-\frac{63}{16}\,\mathrm{cn}^6+\frac{189}{64}\,\mathrm{cn}^8\right)
\end{array}\right) \\
& +\delta^5\left(\begin{array}{l}
-\frac{57159}{98560}-\frac{144821}{156800}\,\mathrm{cn}^2-\frac{1131733}{294000}\,\mathrm{cn}^4+\frac{757991}{73500}\,\mathrm{cn}^6-\frac{298481}{36750}\,\mathrm{cn}^8+\frac{13438}{6125}\,\mathrm{cn}^{10} \\
+\left(\frac{Y}{h}\right)^2\left(-\frac{53327}{28000}\,\mathrm{cn}^2+\frac{1628189}{56000}\,\mathrm{cn}^4-\frac{192481}{2000}\,\mathrm{cn}^6+\frac{11187}{100}\,\mathrm{cn}^8-\frac{5319}{125}\,\mathrm{cn}^{10}\right) \\
+\left(\frac{Y}{h}\right)^4\left(\frac{213}{320}\,\mathrm{cn}^2-\frac{13563}{640}\,\mathrm{cn}^4+\frac{68643}{640}\,\mathrm{cn}^6-\frac{5481}{32}\,\mathrm{cn}^8+\frac{1701}{20}\,\mathrm{cn}^{10}\right) \\
+\left(\frac{Y}{h}\right)^6\left(-\frac{9}{160}\,\mathrm{cn}^2+\frac{267}{64}\,\mathrm{cn}^4-\frac{987}{32}\,\mathrm{cn}^6+\frac{7875}{128}\,\mathrm{cn}^8-\frac{567}{16}\,\mathrm{cn}^{10}\right) \\
+\left(\frac{Y}{h}\right)^8\left(\frac{9}{4480}\,\mathrm{cn}^2-\frac{459}{1792}\,\mathrm{cn}^4+\frac{567}{256}\,\mathrm{cn}^6-\frac{1215}{256}\,\mathrm{cn}^8+\frac{729}{256}\,\mathrm{cn}^{10}\right)
\end{array}\right)
\end{aligned}
\tag{B.3.1}
$$

Vertical fluid velocity: In the same way as above, each term in (B.3.1) containing $(Y/h)^i\,\mathrm{cn}^j()$, for $j>0$, is replaced by $\alpha\,\mathrm{sn}()\,\mathrm{dn}()\left(\frac{j}{i+1}\right)(Y/h)^{i+1}\,\mathrm{cn}^{j-1}()$. Hence the expression for $V/\sqrt{gh}$ can be written

$$
\frac{V}{\sqrt{gh}}=2\alpha\;\mathrm{cn}()\,\mathrm{sn}()\,\mathrm{dn}()\sum_{i=1}^{5}\delta^i\sum_{j=0}^{i-1}\left(\frac{Y}{h}\right)^{2j+1}\sum_{k=1}^{i}\mathrm{cn}^{2(k-1)}()\frac{k}{2j+1}\Phi_{ijk},
\tag{B.3.2}
$$

where the coefficients Φ_{ijk} can be extracted from equation (B.3.1), or from Table III of Fenton (1990).

Discharge

$$
\frac{Q}{\sqrt{gh^3}}=1+\frac{\varepsilon}{2}-\frac{3\,\varepsilon^2}{20}+\frac{3\,\varepsilon^3}{56}-\frac{309\;\varepsilon^4}{5600}+\frac{12237\,\varepsilon^5}{616000}
\tag{B.4}
$$

Bernoulli constant

$$
\frac{R}{gh}=\frac{3}{2}+\frac{\varepsilon}{2}-\frac{\varepsilon^2}{40}-\frac{3\,\varepsilon^3}{140}-\frac{3\;\varepsilon^4}{175}-\frac{2427\,\varepsilon^5}{154000}
\tag{B.5}
$$

Mean fluid speed in frame of wave

$$
\begin{aligned}
\frac{\bar{U}}{\sqrt{gh}}=1\quad & +\left(\frac{H}{h}\right)\left(\frac{1}{2}-e\right)+\left(\frac{H}{h}\right)^2\left(-\frac{3}{20}+\frac{5}{12}e\right)+\left(\frac{H}{h}\right)^3\left(\frac{3}{56}-\frac{19}{600}e\right) \\
& +\left(\frac{H}{h}\right)^4\left(-\frac{309}{5600}+\frac{3719}{21000}e\right)+\left(\frac{H}{h}\right)^5\left(\frac{12237}{616000}-\frac{997699}{8820000}e\right)
\end{aligned}
\tag{B.6}
$$

Wavelength in terms of H/d

$$
\frac{\lambda}{d}\;=\;4K(m)\left(3\frac{H}{d}\right)^{-1/2}\left(\begin{array}{l}
1+\left(\frac{H}{d}\right)\left(\frac{5}{8}-\frac{3}{2}e\right)+\left(\frac{H}{d}\right)^2\left(-\frac{21}{128}+\frac{1}{16}e+\frac{3}{8}e^2\right) \\
+\left(\frac{H}{d}\right)^3\left(\frac{20127}{179200}-\frac{409}{6400}e+\frac{7}{64}e^2+\frac{1}{16}e^3\right) \\
+\left(\frac{H}{d}\right)^4\left(-\frac{1575087}{28672000}+\frac{1086367}{1792000}e-\frac{2679}{25600}e^2+\frac{13}{128}e^3+\frac{3}{128}e^4\right)
\end{array}\right)
\tag{B.7}
$$

Trough depth in terms of H/d

$$\frac{h}{d}=1 \quad +\frac{H}{d}(-e)+\left(\frac{H}{d}\right)^2\frac{e}{4}+\left(\frac{H}{d}\right)^3\left(-\frac{1}{25}e+\frac{1}{4}e^2\right)+\left(\frac{H}{d}\right)^4\left(\frac{573}{2000}e-\frac{57}{400}e^2+\frac{1}{4}e^3\right)$$
$$+\left(\frac{H}{d}\right)^5\left(-\frac{302159}{1470000}e+\frac{1779}{2000}e^2-\frac{123}{400}e^3+\frac{1}{4}e^4\right) \tag{B.8}$$

Practical application of cnoidal theory

Here the procedure for application of the above results is outlined. Firstly, the problem of obtaining solutions in a frame through which the waves move will be outlined. We have not yet encountered this problem for the high-order cnoidal theory, as all operations were performed in a frame (X, Y) moving with the wave such that all fluid motion was steady.

The first step: solving for parameter m

In practical problems, usually the water depth d and the wave height H are known, and either the wave length λ or period τ is known. The problem is initially to solve for the parameter m. We now consider the different ways to do that whether the wave length or period is known.

Wavelength known

Either of the transcendental equations (A.7) for the full third-order solution or (B.7) for the fifth-order Iwagaki approximation can be used to solve for the parameter m. In the latter case one would of course check that the value of m so obtained was sufficiently close to unity that the Iwagaki approximation was justified. Both equations contain $K(m)$ and $e = e(m) = E(m)/K(m)$, for which convenient expressions are given below. The variation of $K(m)$ with m is very rapid in the limit as $m \to 1$, as it contains a singularity in that limit, hence, gradient methods such as the secant method for this might break down. The author prefers to use the bisection method, for which reference can be made to any introductory book on numerical methods, Conte & de Boor (1980) for example. This requires bracketing the solution, for which the author uses the range $m = 0.5$ to $m = 1 - 10^{-12}$, if 14 digit arithmetic is being used.

As an aside, here we develop an approximation for m in terms of the Ursell number which gives some insight into the nature of m. Consider equation (24):

$$\alpha\frac{\lambda}{h} = 2K(m).$$

If we introduce the first-order approximation from equation (A.2):

$$\alpha = \sqrt{\frac{3}{4}\frac{H}{mh}},$$

and as the lowest-order result from equation (A.8) is $h/d = 1$, we can write the lowest-order approximation to equation (24) as

$$\sqrt{\frac{3}{4}\frac{H}{md}}\frac{\lambda}{d} = 2K(m).$$

It is noteworthy that this can be written in terms of the Ursell parameter $\mathbf{U} = (H/d)/(d/\lambda)^2 = H\lambda^2/d^3$, giving

$$\sqrt{\frac{3}{16}\mathbf{U}} = \sqrt{m}K(m). \tag{27}$$

However, the limiting behaviour of K in the limit as $m \to 1$ is $K(m) \approx \frac{1}{2}\log\left(\frac{16}{1-m}\right)$ (#17.3.26 of Abramowitz & Stegun 1965), which shows strong singular behaviour in that limit and we can replace

$\sqrt{m}$ by 1 to give

$$\sqrt{m}K(m) \approx \frac{1}{2}\log\frac{16}{1-m}. \tag{28}$$

Substituting this into equation (27) and solving gives an explicit first-order approximation for m in the limit $m \to 1$:

$$m \approx 1 - 16\,e^{-\sqrt{3\mathbf{U}/4}}. \tag{29}$$

This has some theoretical as well as practical interest, in that we have shown that the parameter m is related to the Ursell number, and as such it might be interpreted as a measure of the relative importance of nonlinearity to dispersion, which is a common interpretation of the Ursell number. (Hedges 1995) has suggested that the boundary between the application of Stokes and cnoidal theory is $\mathbf{U} = 40$, in which case, equation (29) gives $m \approx 0.933$. This is an indication that, roughly speaking, m is always greater than 0.93 when cnoidal theory is used within its recommended limits.

The effects of current on wave period and fluid velocities

A steadily-progressing wave train is uniquely defined by three physical dimensions: the mean depth d, the wave crest-to-trough height H, and wavelength λ, such that it can be expressed in terms of two dimensionless quantities, usually H/d and λ/d for shallow waves. In many situations the wave period is known, rather than the wave length. In most marine situations waves travel on a finite current, and the wave speed and hence the measured wave period depends on the current, because waves travel faster with the current than against it. Most presentations of steady wave theory have used either one of two particular definitions of wave speed, such that (1) the wave moves such that the mean fluid speed at any point is the mean current observed, or, (2) that the depth-integrated mean fluid speed is the mean current observed. These are known as Stokes' first and second definitions of wave speed respectively. However, in general, the speed depends on the current, which cannot be predicted by theory, as it is determined by other topographic or oceanographic factors. What the theories do predict, however, is the speed of the waves relative to the current, and this is the quantity $\bar{U}$ introduced above.

The existence of a current has two main implications for the application of a steady wave theory. Firstly, the apparent period measured by an observer depends on the actual wave speed and hence on the current such that the apparent period is Doppler-shifted. This means that without explicit allowance for the current, if the period is known instead of the wave length it is not possible to solve the problem uniquely. This will have a relatively small effect, of the order of the ratio of fluid speed to wave speed. The second main effect of current is more important if fluid velocities are to be calculated, and this is the additive effect it has on the horizontal fluid velocities, which will be of the order of the current relative to wave-induced fluid velocities. To determine these velocities it is necessary to know the current. If the current is not known, then the problem is under-specified, and the error in fluid velocities thus computed will be of the order of the currents possible.

In the stationary frame of reference the time-mean horizontal fluid velocity at any point is denoted by $\bar{u}_1$, the mean current which a stationary meter would measure. It can be shown that if the fluid flow is irrotational, on which the above theory has been based, that this is constant throughout the fluid. Relating the velocities in the two co-ordinate systems gives

$$\bar{u}_1 = c - \bar{U}. \tag{30}$$

If there is no current, $\bar{u}_1 = 0$, and hence $c = \bar{U}$, so that in this special case the wave speed is equal to $\bar{U}$, introduced above as the mean fluid speed in the frame of the wave. This is Stokes' first approximation to the wave speed, usually incorrectly referred to as his "first definition of wave speed", and is that relative to a frame in which the current is zero. Most wave theories have presented an expression for $\bar{U}$, obtained from its definition as a mean fluid speed, and it has often been referred to, incorrectly, as the "wave speed".

A second type of mean fluid speed is the *depth-integrated* mean speed of the fluid under the waves in the frame in which motion is steady. If Q is the volume flow rate per unit span underneath the waves in the (X, Y) frame, the depth-averaged mean fluid velocity is $-Q/d$, where d is the mean depth. In the physical (x, y) frame, the depth-averaged mean fluid velocity, the "mass-transport velocity", is $\bar{u}_2$, given by

$$\bar{u}_2 = c - Q/d. \tag{31}$$

If there is no mass transport, $\bar{u}_2 = 0$, then Stokes' second approximation to the wave speed is obtained: $c = Q/d$. This would be the condition in a closed wave tank in a laboratory.

In general, neither of Stokes' first or second approximations is the actual wave speed, and in fact the waves can travel at any speed. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed.

Wave period known and current at a point known

In many applications, instead of knowing the wavelength, it is the wave period and current which are known, in which case formulae based on equations (30) or (31) can be used. In this case it is simpler to present separate expansions for the quantities which appear in the equations.

Equation (30) can be shown to give

$$\bar{u}_1 + \bar{U} - \frac{\lambda}{\tau} = 0,$$

where τ is the wave period, as $c = \lambda/\tau$ by definition. We can substitute this and re-arrange the equation to give

$$\frac{\bar{u}_1}{\sqrt{gd}} + \frac{\bar{U}}{\sqrt{gh}}\left(\frac{h}{d}\right)^{1/2} - \frac{\lambda/d}{\tau\sqrt{g/d}} = 0. \tag{32}$$

In the case that water depth d, wave height H, gravitational acceleration g, period τ, and mean Eulerian current $\bar{u}_1$ are known, the quantities $\bar{u}_1/\sqrt{gd}$ and $\tau\sqrt{g/d}$ can be calculated. The dimensionless trough depth h/d and dimensionless wavelength λ/d are known as functions of the known wave height H/d and the as-yet-unknown m, as given by equations (A or B.7) and (A or B.8). The quantity $\bar{U}/\sqrt{gh}$ is given by equation (A.6 or B.6), which can be calculated also in terms of m and the known physical dimensions from

$$\varepsilon = \frac{H}{h} = \frac{H/d}{h/d}. \tag{33}$$

With these quantities substituted, equation (32) is now an equation in the single unknown m, and methods such as bisection can be applied to obtain a solution.

Equation (32) is simpler than that given by the author as equation (20) in Fenton (1990), where he did not realise that the series for the wavelength itself could be used so simply.

Wave period known and mean current over the depth known

In the other case, where the depth-integrated mean current $\bar{u}_2$ is known, the equation to solve for m is

$$\frac{\bar{u}_2}{\sqrt{gd}} + \frac{Q}{\sqrt{gh^3}}\left(\frac{h}{d}\right)^{3/2} - \frac{\lambda/d}{\tau\sqrt{g/d}} = 0, \tag{34}$$

where the procedure is the same as before but the dimensionless discharge $Q/\sqrt{gh^3}$, known as a function of ε and m from (A.4) and (B.4), appears instead of the mean fluid speed $\bar{U}/\sqrt{gh}$. This is also a simpler formulation than the author's equation (25) in Fenton (1990).

Wave period known, current not known

In this case, the problem is not uniquely defined, and an assumption will have to be made for the current, and one of the above two approaches adopted. It will have to be recognised that any horizontal fluid velocities calculated have an error of the magnitude of the real current relative to the assumed current.

An alternative approach

Poulin & Jonsson (1994) have expressed products of two series in equations (32) and (34) as single power series. Thus, they provided a power series for $\bar{U}/\sqrt{gd}$ and one for $Q/\sqrt{gd^3}$ in terms of the known H/d. Hence, in equation (32), if one were to work to the full fifth-order accuracy of the current theory (the series was presented to fourth order only in Poulin & Jonsson (1994)), then the series for $\bar{U}/\sqrt{gd}$ contains 21 terms, compared with the procedure adopted here, of evaluating the product of two series, that for $\bar{U}/\sqrt{gh}$ with a total of 11 terms in the series and that for h/d with 12 terms, a total of 23 terms. (Similarly they expressed a term in $h/d/\alpha$ as a single power series, which has now been superseded by the author's realisation above that the expression is simply related to the wavelength).

Equivalently considering equation (34), the series for $Q/\sqrt{gd^3}$ in Poulin & Jonsson (1994) (which is actually wrong at third and fourth orders as presented therein), would contain 21 terms at fifth order, compared with 6 terms for $Q/\sqrt{gh^3}$ plus 12 terms for h/d, a total of 18 terms using the present approach.

The current author, who originated the formulation of equations (32) and (34), deliberately chose the sequential evaluation of series (not "simultaneous" or "coupled" as stated in Poulin & Jonsson (1994)) rather than combining the series, as to him it seemed that the necessity of providing more series expansions as part of the theory was not justified.

Application of the theory

Having solved for m iteratively, the cnoidal theory can now be applied.

Trough depth h: Equation (A or B.8) can be used to calculate h/d. This will probably already have been calculated as part of the converged solution process for m.

Wavelength λ: This follows easily from equation (A or B.7), and also will probably already have been calculated.

Dimensionless wave height $\varepsilon = H/h$: Equation (33).

Coefficient α: Equation (A or B.2). This is used as an argument of the elliptic functions in all quantities which vary with position and is used to calculate δ.

Shallowness parameter δ: It has been found by the author (Fenton 1990) that it is more accurate to present results for fluid velocity in terms of α rather than $\varepsilon = H/h$, and it is more convenient to present the results in terms of δ, rather than in terms of α, where

$$\delta = \frac{4}{3}\alpha^2. \tag{35}$$

Mean fluid speed in frame moving with wave $\bar{U}$: Equation (A or B.6) is used to calculate $\bar{U}/\sqrt{gh}$.

Discharge Q: Equation (A or B.4) is used to calculate $Q/\sqrt{gh^3}$.

Wave speed c: follows from equation (30) if the current at a point is known: $c = \bar{u}_1 + \bar{U}$, or from equation (31) if the depth-integrated mean current is known: $c = \bar{u}_2 + Q/d$.

Surface elevation: For a particular point and time (x, t) the elliptic function $\mathrm{cn}(\alpha\,(x-ct)\,/h|m)$ can be computed using the approximation in Table 2 and equation (A or B.1) used.

Fluid velocity components (u, v): Fluid velocities in the physical (x, y) frame are given by

$$u(x, y, t) = c + U(x - ct, y), \tag{36}$$

where $U(X, Y)$ is given by equation (A or B.3). These equations can be written

$$\frac{u(x, y, t)}{\sqrt{gh}} = \frac{c}{\sqrt{gh}} - 1 + \sum_{i=1}^{5} \delta^i \sum_{j=0}^{i-1} \left(\frac{y}{h}\right)^{2j} \sum_{k=0}^{i} \mathrm{cn}^{2k}(\alpha(x - ct)/h|m)\, \Phi_{ijk}, \tag{37}$$

where the coefficients Φ_{ijk} can be extracted from equation (A.3.1), where each is a polynomial of degree i in the parameter m, or in the Iwagaki approximation where they are rational numbers, from equation (B.3.1), or from Table III of Fenton (1990). The vertical velocity components follow, using the mass conservation equation, differentiating with respect to x and integrating with respect to y to give:

$$\frac{v(x, y, t)}{\sqrt{gh}} = 2\alpha\ \mathrm{cn}()\,\mathrm{sn}()\,\mathrm{dn}() \sum_{i=1}^{5} \delta^i \sum_{j=0}^{i-1} \left(\frac{y}{h}\right)^{2j+1} \sum_{k=1}^{i} \mathrm{cn}^{2(k-1)}(\alpha(x - ct)/h|m) \frac{k}{2j+1} \Phi_{ijk}, \tag{38}$$

It will be seen below that this theory predicts velocities accurately over a wide range of wave conditions.

Derivatives of fluid velocity: In some applications it is necessary to know the spatial and time derivatives of the velocity. These follow from differentiation of equations (37) and (38) and the use of elementary properties of elliptic functions, and application of the mass conservation and irrotationality equations:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2\alpha\sqrt{\frac{g}{h}}\,\mathrm{cn}()\,\mathrm{sn}()\,\mathrm{dn}() \sum_{i=1}^{5} \delta^i \sum_{j=0}^{i-1} \left(\frac{y}{h}\right)^{2j} \sum_{k=1}^{i} \mathrm{cn}^{2(k-1)}(\alpha(x - ct)/h|m)\, k\, \Phi_{ijk}, \\
\frac{\partial u}{\partial y} &= 2\sqrt{\frac{g}{h}} \sum_{i=1}^{5} \delta^i \sum_{j=1}^{i-1} \left(\frac{y}{h}\right)^{2j-1} \sum_{k=0}^{i} \mathrm{cn}^{2k}(\alpha(x - ct)/h|m)\, j\Phi_{ijk}, \\
\frac{\partial u}{\partial t} &= -c\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial t} = -c\frac{\partial v}{\partial x}, \\
\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}.
\end{aligned}$$

Bernoulli constant R: Equation (A or B.5) is used to calculate R/gh.

Fluid pressure p: By applying Bernoulli's theorem in the frame in which motion is steady, equation (5) can be used to give an expression for the fluid pressure at a point:

$$\frac{p(x, y, t)}{\rho} = R - gy - \frac{1}{2}\left[(u - c)^2 + v^2\right].$$

Practical tools and hints for application

Here we provide some methods and results which may make the application of cnoidal theory somewhat more accessible.

Numerical check for coefficients

In the presentation of high-order series results it is very easy to make errors, whether the author preparing the work for publication or a reader transcribing the results for application. To provide a check on this, Table 1 provides a list of numbers, one for each of the equations (A.1-8) and (B.1-8) which have been obtained by evaluating each of the expressions with all mathematical symbols set to 1. This is a meaningless operation physically, and the fact that the numbers from the full third-order theory and fifth order theory disagree does not imply that something is wrong. If a user checks their own calculations and does not obtain the values shown here, an error has been made by someone, either the author or themselves, and checking should be carried out. As a possible extra check, it should be mentioned that the fifth-order Iwagaki approximation as presented in (Fenton 1990) is believed to be correct as printed but for two sign errors: the exponent $-1/2$ of $3H/d$ in equation (19) (*cf.* B.7 above) and in the coefficient $-e/25$ in the third-order term of equation (21), (*cf.* B.8 above).

Table 1. Values of expressions evaluated with all symbols set to 1

Quantity	Third-order full solution		Fifth-order Iwagaki approximation	
	Equation	Check value	Equation	Check value
$\frac{\eta}{h}$	(A.1)	2	(B.1)	2
α	(A.2)	$\frac{119}{256}\sqrt{3} \approx 0.80513$	(B.2)	$\frac{26815417}{57344000}\sqrt{3} \approx 0.80995$
$\frac{U}{\sqrt{gh}}$	(A.3)	$\frac{33}{560} \approx 0.05893$	(B.3)	$-\frac{4572863}{2464000} \approx -1.85587$
$\frac{Q}{\sqrt{gh^3}}$	(A.4)	$\frac{393}{280} \approx 1.40357$	(B.4)	$\frac{842847}{616000} \approx 1.36826$
$\frac{R}{gh}$	(A.5)	$\frac{547}{280} \approx 1.95357$	(B.5)	$\frac{295783}{154000} \approx 1.92067$
$\frac{\bar{U}}{\sqrt{gh}}$	(A.6)	$\frac{138}{175} \approx 0.78857$	(B.6)	$\frac{158576387}{194040000} \approx 0.81724$
$\frac{\lambda}{d}$	(A.7)	$\frac{17}{32}\sqrt{3} \approx 0.92015$	(B.7)	$\frac{6826061}{4300800}\sqrt{3} \approx 2.74904$
$\frac{h}{d}$	(A.8)	$\frac{23}{50} = 0.46$	(B.8)	$\frac{2176261}{1470000} \approx 1.48045$

Numerical Richardson test for the series results

There is one method which is very simple to apply which can test whether or not a series solution to a problem is correct, and if not, at which order of accuracy it is wrong. It gives a simple answer as to whether all the series used in the computation are correct, but it does not reveal where any errors might be. The method, in the context of this work might prove helpful to a practitioner having written a program based on the theoretical results above who might want to check the accuracy of the series as programmed.

The method, proposed in Fenton (1985), is based on Richardson extrapolation to the limit. It can be used almost anywhere, but a simple test for some of the most important quantities presented above would be to calculate the pressure at an arbitrary point on the free surface, where the method would test whether or not all of the expressions were correct: the elevation of the surface from (A or B.1), the coefficient α from (A or B.2), the velocity components from (A or B.3.1) and (A or B.3.2), the mean fluid speed from (A or B.6) *and* the Bernoulli constant from (A or B.5)!

If one has a series approximation to a quantity which should be zero, such as pressure, or testing the series by evaluating an identity such as $H/h - (H/d)/(h/d)$, then evaluating it will not give zero in general, but a finite error, such as non-zero pressure at a point on the surface. We denote this error by Δ, and suppose it to be a function only of an expansion parameter ϵ (probably H/d in the present context), for all other quantities given numerical values. For example we might take a wave of length $\lambda/d = 20$ and calculate the pressure on the surface at $(x - ct)/h = 0.5$. Now, if we assume that the error is proportional to the nth power of ϵ, then we can write, where ϵ is the expansion quantity, whether ε, δ, or H/d:

$$\Delta = a\epsilon^n,$$

where a is independent of ϵ. Now if we evaluate the error numerically for two different values of ε: ϵ_1 and ϵ_2, to give Δ_1 and Δ_2, then we can eliminate a such that

$$\frac{\Delta_1}{\Delta_2} = \left(\frac{\epsilon_1}{\epsilon_2}\right)^n,$$

and this can be solved to give

$$n = \frac{\log(\Delta_1/\Delta_2)}{\log(\epsilon_1/\epsilon_2)}, \tag{39}$$

thus giving a numerical estimate of the error. The error of this expression can be shown to be $O(\epsilon)$, so to provide convincing evidence of the order of the theory it is necessary to use a small value of ϵ. In practice it is very reassuring to obtain a value of $n = 5.98$, for example, providing strong evidence that all series that have gone into the calculation are correct to fifth order.

Formulae and methods for elliptic integrals and functions

Elementary properties of elliptic functions and integrals

For elementary properties, reference can be made to Abramowitz & Stegun (1965), Gradshteyn & Ryzhik (1965) and Spanier & Oldham (1987), for example. Those sources contain a number of approximations, but the expressions given usually do not have the same remarkable accuracy as those given in Fenton & Gardiner-Garden (1982) for the limit required for cnoidal theory of waves, $m \to 1$. If a reader were interested enough to investigate the theory, Eagle (1958) contains a refreshing different approach to the subject which inspired the work described below, originally obtained in Fenton & Gardiner-Garden (1982).

Approximations to functions and integrals

One perceived practical problem with the application of cnoidal theory has been that the theory makes use of Jacobian elliptic functions and integrals, seen as being difficult to calculate. This has some justification, as conventional formulae such as in Abramowitz & Stegun (1965) are very poorly convergent, if convergent at all, in the limit $m \to 1$, precisely the limit in which cnoidal theory is most appropriate. However, alternative formulae can be obtained which are most accurate and remarkably quickly convergent in the limit of $m \to 1$. This has been done in (Fenton & Gardiner-Garden 1982), which provide a number of useful expressions for both elliptic functions and integrals. The formulae are dramatically convergent, even for values of m not in the $m \to 1$ limit. Convenient approximations to these formulae can be obtained and are given here in Table 2. For values of m likely to be encountered using cnoidal wave theory the formulae are probably accurate to machine accuracy. It is remarkable that even for $m = 1/2$, the simple approximations given in the table are accurate to five significant figures. For the case $m < 1/2$, when cnoidal theory becomes less valid, conventional approximations could be used, for which reference can be made to Fenton & Gardiner-Garden (1982) or to standard references. However, cnoidal theory should probably be avoided in this case.

Derivatives

$$\begin{aligned} \frac{\partial}{\partial\theta}\,\mathrm{cn}(\theta|m) &= -\,\mathrm{sn}(\theta|m)\,\mathrm{dn}(\theta|m), \\ \frac{\partial}{\partial\theta}\,\mathrm{sn}(\theta|m) &= \mathrm{cn}(\theta|m)\,\mathrm{dn}(\theta|m), \\ \frac{\partial}{\partial\theta}\,\mathrm{dn}(\theta|m) &= -m\,\mathrm{sn}(\theta|m)\,\mathrm{cn}(\theta|m) \end{aligned} \tag{40}$$

Relations between squares

$$\begin{aligned} \mathrm{dn}^2(\theta|m) &= 1 - m\left(1 - \mathrm{cn}^2(\theta|m)\right), \\ \mathrm{sn}^2(\theta|m) &= 1 - \mathrm{cn}^2(\theta|m) \end{aligned} \tag{41}$$

Fourier series for cn^2

Although not necessary for the application of the above theory, there is an apparently little-known Fourier series for the cn^2 function which might prove useful in certain applications. It is presented here, partly because in some fundamental references – (#911.01 of Byrd & Friedman 1954), (#8.146,

Table 2. Approximations for elliptic functions and integrals in the case most appropriate for cnoidal theory, $m \geq 1/2$.

Elliptic integrals

Complete elliptic integral of the first kind $K(m)$

$$K(m) \approx \frac{2}{\left(1+m^{1/4}\right)^2} \log \frac{2\left(1+m^{1/4}\right)}{1-m^{1/4}}$$

Complementary elliptic integral of the first kind $K'(m)$

$$K'(m) \approx \frac{2\pi}{\left(1+m^{1/4}\right)^2}$$

Complete elliptic integral of the second kind $E(m)$

$$E(m) = K(m)\, e(m), \text{ where}$$

$$e(m) \approx \frac{2-m}{3} + \frac{\pi}{2KK'} + 2\left(\frac{\pi}{K'}\right)^2 \left(-\frac{1}{24} + \frac{q_1^2}{\left(1-q_1^2\right)^2}\right),$$

where $q_1(m)$ is the complementary nome $q_1 = e^{-\pi K/K'}$.

Jacobian elliptic functions

$$\operatorname{sn}(z|m) \approx m^{-1/4} \frac{\sinh w - q_1^2 \sinh 3w}{\cosh w + q_1^2 \cosh 3w},$$

$$\operatorname{cn}(z|m) \approx \frac{1}{2}\left(\frac{m_1}{m\, q_1}\right)^{1/4} \frac{1-2q_1 \cosh 2w}{\cosh w + q_1^2 \cosh 3w},$$

$$\operatorname{dn}(z|m) \approx \frac{1}{2}\left(\frac{m_1}{q_1}\right)^{1/4} \frac{1+2q_1 \cosh 2w}{\cosh w + q_1^2 \cosh 3w},$$

in which $w = \pi z/2K'$.

26 of Gradshteyn & Ryzhik 1965) – an incorrect expression (an odd function) is given. The correct expression is given in #2.23 of(Oberhettinger 1973) as a Fourier series for $\operatorname{dn}^2()$ which can be used to convert to a Fourier Series for $\operatorname{cn}^2()$, which can be written

$$\operatorname{cn}^2(\theta|m) = 1 + \frac{e-1}{m} + \frac{\pi^2}{mK^2} \sum_{j=1}^{\infty} \frac{j}{\sinh\left(j\pi K'/K\right)} \cos\left(\frac{j\pi\theta}{K}\right). \tag{42}$$

For typical shallow water waves, $m \to 1$, and $K \to \infty$, such that the series would be slowly convergent, as would be expected for a wave which is so non-sinusoidal as a long wave with its long trough and short crest. The series could be re-cast to give a complementary rapidly-convergent formula which would involve a series of hyperbolic functions.

Convergence enhancement of series

The series above have been presented to third order for the full theory and to fifth order for the Iwagaki approximation. There are several techniques available for obtaining more accurate results by taking the series results and attempting to extract more information from the series than is apparently there.

The Shanks transform

One simple way of doing this is to use Shanks transforms, which are delightfully introduced in Shanks (1955), and used in the context of water wave theory to enhance the convergence of series in Fenton (1972). They take the first few terms of a series and attempt to mimic the behaviour of the series as if it had an infinite number of terms. The method takes three successive terms in a sequence (such as the first, second and third order solutions for a wave property), and extrapolates the behaviour of the sequence to infinity, hopefully mimicking the behaviour of the series if there were an infinite number of terms. There is little theoretical justification for the procedure, but it can work surprisingly well. It is easily implemented: if the last three terms in a sequence of n terms are S_{n-2}, S_{n-1}, and S_n, an estimate of the value of S_∞ is given by

$$S_\infty \approx S_n - \frac{\left(S_n - S_{n-1}\right)^2}{\left(S_n - S_{n-1}\right) - \left(S_{n-1} - S_{n-2}\right)}. \tag{43}$$

This is not the form which is usually presented, but it is that which is most suitable for computations, when in the possible case that the sums have nearly converged and both numerator and denominator of the second term on the right go to zero the result is less liable to round-off error. The transform does indeed possess some remarkable properties. For example, it gives the exact sum to infinity for geometric series, which can be verified by substituting $S_n = \sum_{j=0}^{n} r^j$, then equation (43) gives $1/(1-r)$, the exact result for the sum to infinity.

The transform is simply applied and can be used in many areas of numerical computations. It gives surprisingly good results, but its theoretical justification is limited and sometimes it does not work well.

Padé approximants

A form of approximation of the series which has more justification is that of Padé approximation, where a rational function of the expansion variable is chosen such as to match the series expansion as much as possible, Baker (1975), which was introduced to water wave theory by Schwartz (1974). The calculations for Padé approximants are not as trivial as for Shanks transforms, however the properties are usually more powerful. The $[i, j]$ Padé approximation is defined to be the rational function $p(\epsilon)/q(\epsilon)$, where $p(\epsilon)$ is a polynomial of degree $\leq m$ and $q(\epsilon)$ is a polynomial of degree $\leq n$, such that the series expansion of $p(\epsilon)/q(\epsilon)$ has maximal initial agreement with the series expansion of the function. In normal cases, the series expansion agrees through the term of degree $m+n$, and it is this way that the coefficients in the two polynomials are computed. An example is $(1+x/2)/(1-x/2)$ as the $[1, 1]$ approximation to e^x, which for small values of x is more accurate than the equivalent series with quadratic terms $1+x+x^2/2$. Another example is where the function $1+x+x^2$ has as its $[1, 1]$ approximant the function $1/(1-x)$, and this too has ascertained that the first three terms of the series look like a geometric series.

Use of convergence acceleration procedures in cnoidal theory

The author has tested the use of both Shanks transforms and Padé approximants in applying the cnoidal theory described in this work. As Padé approximation is considered more powerful, some attention was given to that, however, a limitation became quickly obvious, when at the first step in application, solving equation (B.7) for the wavelength, approximating the quartic in H/d in the large brackets by a $[2, 2]$ Padé approximant, with a quadratic in numerator and denominator, the latter passed through zero for an intermediate value of H/d, such that in the vicinity of that point very wildly varying results were obtained. The author considered that this was sufficiently dangerous that generally $[2, 2]$ or $[3, 2]$ Padé approximants could not be recommended for the approximation of fifth-order cnoidal theory. When he examined Padé approximants with a linear function in the denominator, it was found that, given an n-term series, the $[n-1, 1]$ Padé approximant is, in fact, exactly equal to the Shanks transform of the last three sums in the series, as given in the equations above. As the Shanks transform is more simply implemented, we will refer to the series convergence acceleration using this method by that name.

In practice, obtaining solutions for given values of wavelength and wave height, the use of the Shanks transforms everywhere gave better results than just using the raw series in the case of global wave quantities such as α, Q, *etc.* which are independent of position, and it is recommended that for both third-order theory and the Iwagaki approximation that Shanks transforms be used to improve the accuracy of all series computations for those quantities. They are of course trivially implemented, given say, three numbers for the third, fourth and fifth solutions.

For the surface elevation and the fluid velocity components, however, because they are functions of position, then depending on that position the series could show rather irregular behaviour, and it was found that the Shanks transform results could also be irregular. As in Fenton (1990), it is then recommended that for quantities which are functions of position, that no attempts be made to improve the accuracy by numerical transforming of the results, but that for all other quantities, characteristic of the wave as a whole, the Shanks transform be applied to all numerical evaluations of series. This procedure was adopted for all the results shown further below.

A numerical cnoidal theory

Introduction

In Fenton (1990) the accuracies of various theories were examined by comparison with experimental results and with results from high-order numerical methods. It was found that fifth-order Stokes and cnoidal theory were of acceptable engineering accuracy almost everywhere within the range of validity of each. For long waves which are very high however, even the high-order cnoidal theory presented above becomes inaccurate. In such cases the most accurate method is to use a numerical method. The usual method, suggested by the basic form of the Stokes solution, is to use a Fourier series which is capable of accurately approximating any periodic quantity, provided the coefficients in that series can be found. A reasonable procedure, then, instead of assuming perturbation expansions for the coefficients in the series as is done in Stokes theory, is to calculate the coefficients numerically by solving the full nonlinear equations. This approach began with Chappelear (1961), and has been often but inappropriately known as ”stream function theory” (Dean 1965). Further developments include those of Rienecker & Fenton (1981). A comparison of the various methods has been given in Sobey et al. (1987), where the conclusion was drawn that there was little to choose between them. A more recent development has been the simpler method and computer program given in Fenton (1988).

This Fourier approach breaks down in the limit of very long waves, when the spectrum of coefficients becomes broad-banded and many terms have to be taken, as the Fourier approximation has to approximate both the short rapidly-varying crest region and the long trough where very little changes. More of a problem is that it is difficult to get the method to converge to the solution desired, (Dalrymple & Solana 1986).

A new approach was suggested in Fenton (1995), which describes a numerical cnoidal theory, which is to cnoidal theory what the various Fourier approximation methods are to Stokes theory, in that it solves the problem numerically by assuming series of cnoidal-type functions, but rather than solving them by analytical power series methods as above, the coefficients in the equations are found numerically and there is no essential mathematical approximation introduced. The method will be described here briefly.

Theory

A spectral approach is used, in which all functions of x are approximated by polynomials of degree N in terms of the square of the Jacobian elliptic function $\mathrm{cn}^2(\theta|m)$ for the surface elevation and bottom velocity of the form suggested by conventional cnoidal theory:

$$\eta_* = 1 + \sum_{j=1}^{N} Y_j \, \mathrm{cn}^{2j}(\theta|m), \quad (44)$$

$$f'_* = F_0 + \sum_{j=1}^{N} F_j \, \mathrm{cn}^{2j}(\theta|m), \quad (45)$$

where the Y_j and F_j are numerical coefficients for a particular wave. Note that the N here is not the order of approximation but the number of terms in the series. Conventional cnoidal theory expresses the coefficients as expansions in terms of the parameter α which is related to the shallowness (depth/wavelength)^2, equations (14) and (15), and produces a hierarchy of equations and solutions based on series expansions in terms of α, which is required to be small. In this work there is no attempt to solve the equations by making expansions in terms of physical quantities. The surface velocity components are then given by

$$\begin{aligned} u_{*s} &= \frac{u_s h}{Q} = -\cos\alpha\eta_* \frac{d}{d\theta} \cdot f'_*, \\ v_{*s} &= \frac{v_s h}{Q} = \sin\alpha\eta_* \frac{d}{d\theta} \cdot f'_*. \end{aligned} \quad (46)$$

On substituting these into the nonlinear surface boundary conditions, equations (12) and (13) we have two nonlinear algebraic equations valid for all values of θ. The equations include the following unknowns: α, m, g_*, R_*, plus a total of N values of the Y_j for $i = 1 \ldots N$, and $N+1$ values of the F_j for $i = 0 \ldots N$, making a total of $2N+5$ unknowns. For the boundary points at which both boundary conditions are to be satisfied we choose $M+1$ points equally spaced in the vertical between crest and trough such that:

$$\mathrm{cn}^2\left(\theta_i|m\right) = 1 - i/M, \quad \text{for} \quad i = 0 \ldots M, \tag{47}$$

where $i = 0$ corresponds to the crest and $i = M$ to the trough. This has the effect of clustering points near the wave crest, where variation is more rapid and the conditions at each point will be relatively different from each other. If we had spaced uniformly in the horizontal, in the long trough where conditions vary little the equations obtained would be similar to each other and the system would be poorly conditioned. We now have a total of $2M+2$ equations but so far, none of the overall wave parameters have been introduced. It is known that the steady wave problem is uniquely defined by two dimensionless quantities: the wavelength λ/d and the wave height H/d. In many practical problems the wave period is known, but Fenton (1995) considered only those where the dimensionless wavelength λ/d is known. It can be shown that λ/d is related to α using the expression (24) which we term the Wavelength Equation:

$$\alpha \frac{\lambda}{d}\frac{d}{h} - 2K(m) = 0, \tag{48}$$

where $K(m)$ is the complete elliptic integral of the first kind, and where the equation has introduced another unknown d/h, the ratio of mean to trough depth.

The equation for this ratio is obtained by taking the mean of equation (44) over one wavelength or half a wavelength from crest to trough:

$$\frac{d}{h} = 1 + \sum_{j=1}^{N} Y_j \, \overline{\mathrm{cn}^{2j}\left(\theta|m\right)}. \tag{49}$$

The mean values of the powers of the cn function over a wavelength can be computed from the recurrence relations (26) for the I_j such that equation (49) can be written

$$1 + \sum_{j=1}^{N} Y_j \, I_j - \frac{d}{h} = 0, \tag{50}$$

thereby providing one more equation, the Mean Depth Equation.

Finally, another equation which can be used is that for the wave height:

$$\frac{H}{h} = \frac{\eta_0}{h} - \frac{\eta_M}{h}, \tag{51}$$

which, on substitution of equation (44) at $x = x_0 = 0$ where $\mathrm{cn}(0|m) = 1$ and, because $\mathrm{cn}(\alpha x_M|m) = 0$ from equation (47), gives

$$\frac{H}{d}\frac{d}{h} - \sum_{j=1}^{N} Y_j = 0, \tag{52}$$

the Wave Height Equation.

We write the system of equations as

$$\mathbf{e}\left(z\right) = \left\{e_i\left(\mathbf{z}\right),\ i = 1 \ldots 2M+5\right\} = 0, \tag{53}$$

where e_i is the equation with reference number i, the $2M+2$ equations described above plus the three equations (48), (50), and (52), and where the variables which are used are the $2N+5$ unknowns described above plus d/h:

$$\mathbf{z} = \left\{z_j,\ j = 1 \ldots 2N+6\right\}, \tag{54}$$

Whereas the parameter m has been used in cnoidal theory, it has the unpleasant property that it has a singularity in the limit as $m \to 1$, which corresponds to the long wave limit, and as we will be using gradient methods to solve the nonlinear equations this might make solution more difficult. It is more convenient to use the ratio of the complete elliptic integrals as the actual unknown, which we choose to be the first:

$$z_1 = \frac{K(m)}{K(1-m)}. \tag{55}$$

The solution of the system of nonlinear equations follows that in Fenton (1988), using Newton's method in a number of dimensions, where it is simpler to obtain the derivatives by numerical differentiation.

As the number of equations and variables can never be the same ($2M+5$ can never equal $2N+6$ for integer M and N), we have to solve this equation as a generalised inverse problem. Fortunately this can be done very conveniently by the Singular Value Decomposition method (for example #2.6 of Press, Teukolsky, Vetterling & Flannery 1992) so that if there are more equations than unknowns, $M > N$, the method obtains the least squares solution to the overdetermined system of equations. In practice this was found to give a certain rugged robustness to the method, despite the equations being rather poorly conditioned.

The set of functions $\left\{\mathrm{cn}^{2j}(\theta|m), j = 0 \ldots N\right\}$ used to describe spatial variation in the horizontal do not form an orthogonal set, and they all tend to look like one another, which result, although apparently an esoteric mathematical property, has the important effect that the system of equations is not particularly well-conditioned, and numerical solutions show certain irregularities and a relatively slow convergence with the number of terms taken in the series. It was difficult to obtain solutions for $N > 10$. The Fourier methods, however, using the robustly orthogonal trigonometric functions, do not seem to have these problems. Fortunately, however, in the case of the numerical cnoidal theory, good results could be obtained with few terms.

For initial conditions in the iteration process, it was obvious to choose the fifth order Iwagaki theory presented in equations (B.1-B.8). The first step is to compute an approximate value of m and hence z_1 using the analytical expression for wavelength in terms of m from equation (B.7), combined with the bisection method of finding the root of a single transcendental equation. After that the rest of the fifth order expressions presented above can be used.

Accuracy of the methods

In this section we examine the applicability of the full third-order cnoidal theory, the fifth-order Iwagaki approximation and the numerical cnoidal theory by considering several high waves and showing results for the surface profile and possibly more importantly, for the velocity profile under the crest.

The region of possible waves and the validity of theories

The range over which periodic solutions for waves can occur is given in Figure 2, which shows limits to the existence of waves determined by computational studies. The highest waves possible, $H = H_m$, are shown by the thick line, which is the approximation to the results of Williams (1981), presented as equation (32) in Fenton (1990) :

$$\frac{H_m}{d} = \frac{0.141063\,\frac{\lambda}{d} + 0.0095721\left(\frac{\lambda}{d}\right)^2 + 0.0077829\left(\frac{\lambda}{d}\right)^3}{1 + 0.0788340\,\frac{\lambda}{d} + 0.0317567\left(\frac{\lambda}{d}\right)^2 + 0.0093407\left(\frac{\lambda}{d}\right)^3}. \tag{56}$$

Nelson (1987,) has shown from many experiments in laboratories and the field, that the maximum wave height achievable in practice is actually only $H_m/d = 0.55$. Further evidence for this conclusion is provided by the results of Le Méhauté et al. (1968), whose maximum wave height tested was $H/d = 0.548$, described as "just below breaking". It seems that there may be enough instabilities at work that real waves propagating over a flat bed cannot approach the theoretical limit given by equation (56). This is fundamental for the application of the present theories. If indeed the highest waves do have a height

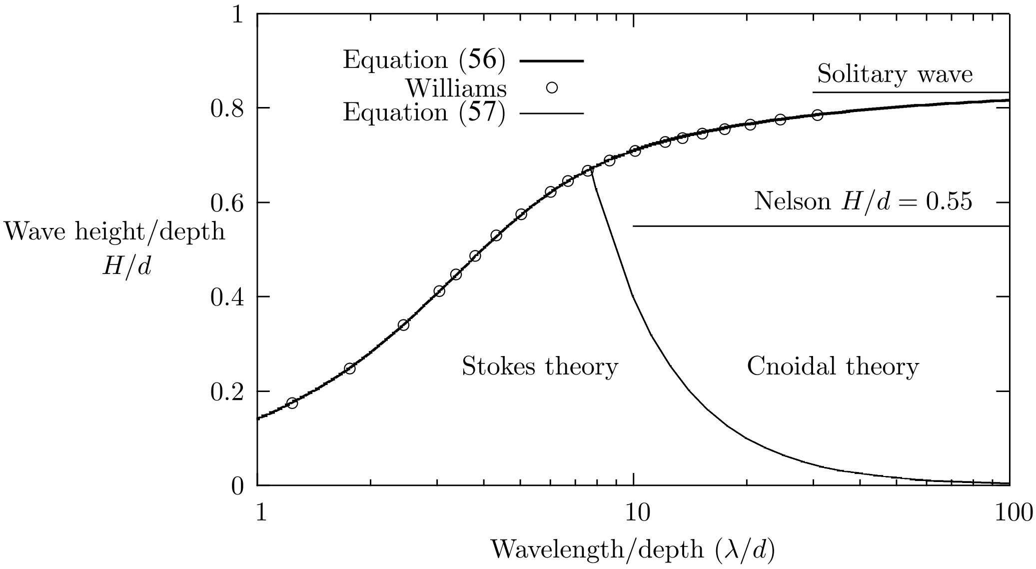

Figure 2. The region of possible steady wave, showing the theoretical highest waves (Williams), the highest long waves in the field (Nelson) with cases reported on here, and Hedges' proposed demarcation line between regions of applicability of Stokes and cnoidal theories.

to depth ratio of only 0.55, it seems that both fifth-order Stokes theory and fifth-order cnoidal theory are capable of giving accurate results over all possible waves (Fenton 1990).

In Fenton (1990) the author proposed a formula for the boundary between the use of Stokes theory and cnoidal theory. It has been pointed out by Hedges (1995), however, that a simpler criterion, and one agreeing more with the numerical evidence, is that cnoidal theory should be applied for

$$\mathbf{U} = \frac{H\lambda^2}{d^3} > 40, \tag{57}$$

while for $\mathbf{U} < 40$, for shorter waves, Stokes theory should be used. This line is plotted on Figure 2, and it shows an interesting and important property for small waves, that cnoidal theory should not be used below a certain wave height, even for very long waves! This was explained in Fenton (1979), where it was shown that in the small amplitude limit, the waves tended to become sinusoidal and the parameter m became small, such that the *effective* expansion parameter ε/m became large, even if ε itself was not, and the series showed poor convergence.

Comparison of theories and numerical methods

Now we examine the accuracy of the various theories over the range of possible waves, considering $H/d = 0.55$ and increasing the wave length from 8 to 64, doubling each time. One with a height of 0.7, close to the theoretical maximum, will be considered.

Table 3. Wave trains for which results are presented here

H/d	λ/d	$\mathbf{U}$	m (3rd order)	m (5th order)
0.55	8	35.2	0.9168	0.8964
0.55	16	141	0.9983	0.9980
0.7	32	717	$1 - 0.14 \times 10^{-6}$	$1 - 0.24 \times 10^{-6}$
0.55	64	2250	$1 - 0.75 \times 10^{-13}$	$1 - 0.11 \times 10^{-12}$

These cases are summarised in Table 3, which shows the wave dimensions, the Ursell number, and the value of m obtained by solving equation (B.7).

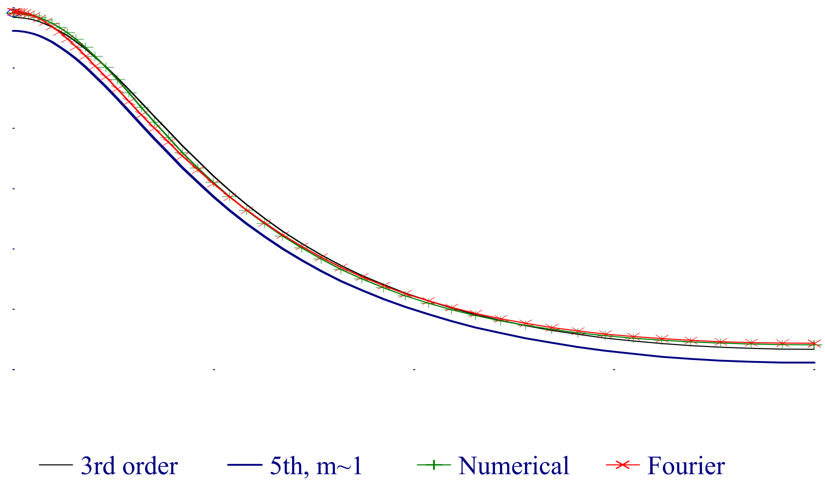

Figure 3. Surface profiles for $H/d = 0.55$, $\lambda/d = 8$.

Figure 3 shows the solution for the surface profiles obtained for a high wave of intermediate length, when conventional cnoidal theory has been considered not valid, and which falls outside Hedges' recommended boundary for cnoidal theory of $\mathbf{U} \geq 40$, as can be seen on Figure 2. Four curves are plotted, results from the full third-order method, the fifth-order Iwagaki approximation (shown as $m\sim 1$), the numerical cnoidal theory described above, and from the Fourier approximation method, which should be highly accurate in this relatively short wave limit. It can be seen that most results are almost indistinguishable at the scale of plotting, but that in this case of a relatively short wave, with $m \approx 0.9$, the Iwagaki approximation is not so accurate, as expected. Whereas conventional cnoidal theory should not be particularly accurate in this shorter wave limit, as it depends on the waves being long for its accuracy, there is nothing in the numerical cnoidal method which necessarily limits its accuracy to long waves. In fact, for the initial conditions for the numerical method only cnoidal theory was used, and it was not accurate enough for waves shorter than this example. If Stokes theory could be modified to provide the initial conditions, there is no reason why the numerical cnoidal method could not be used for considerably shorter waves.

Figure 4 shows the velocity profiles under the crest for the same wave. It is clear that the numerical cnoidal method and the Fourier method agree closely, and possibly strangely, that the Iwagaki approximation is accurate, even for this wave with $m \approx 0.9$. The third-order theory predicts the mean fluid speed under the wave poorly, but predicts the velocity variation in the vertical very well, so that the curve is displaced relative to the accurate results.

Figure 5 shows the results for a longer wave, of $\lambda/d = 16$. In this case, $m = 0.998$, and it is expected that the Iwagaki approximation would be accurate. It can be seen that even the third-order theory predicts the surface very accurately. For all subsequent cases studied, even for the higher wave with $H/d = 0.7$, the results for surface elevation were better even than this, and no more results for surface elevation will be presented here.

Figure 6 shows the velocity profiles under the crest. It is clear that the fifth-order Iwagaki theory is highly accurate for practical purposes, but that the third-order theory has a constant shift as before.

Figure 7 shows the behaviour of the numerical cnoidal method for very high and long waves, for a wave of length $\lambda/d = 32$ and a height of $H/d = 0.7$, close to the maximum theoretical height of $H_m/d = 0.737$, calculated from equation (56). There is evidence that no long wave in shallow water can exist at this height, and that a maximum of $H/d = 0.55$ is more likely (Nelson 1994). This wave is sufficiently long that the Fourier method is beginning to be tested considerably, yet it is capable of

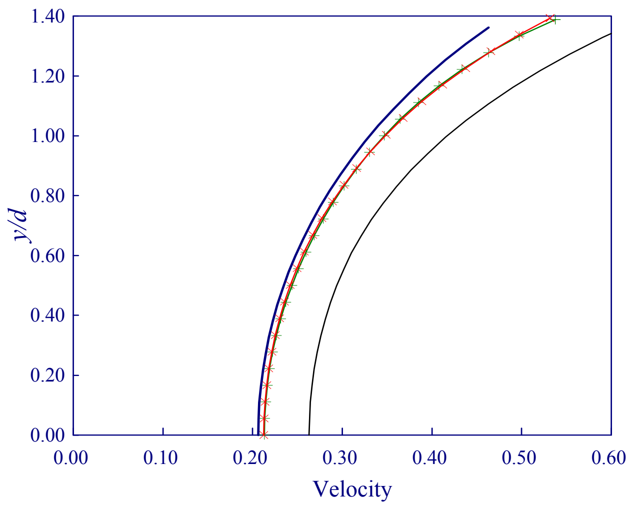

Figure 4. Velocity profiles under crest for the same wave as the previous figure; $U/\sqrt{gd}$ plotted.

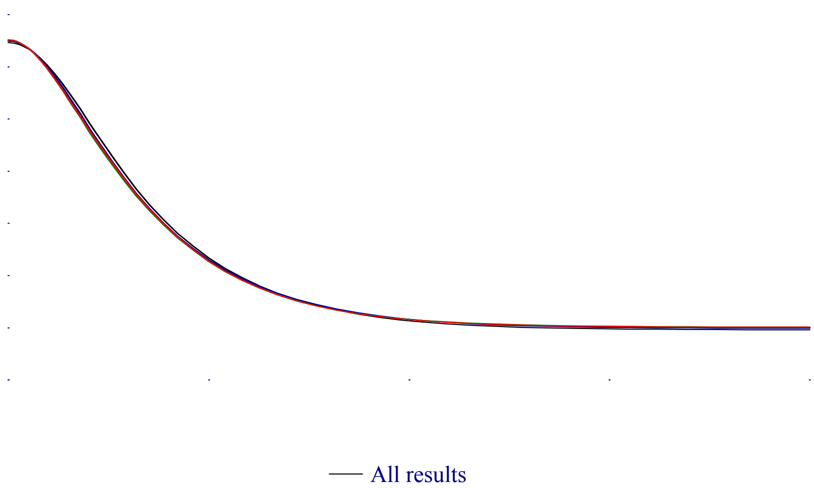

Figure 5. Surface profiles for $H/d = 0.55$, $\lambda/d = 16$.

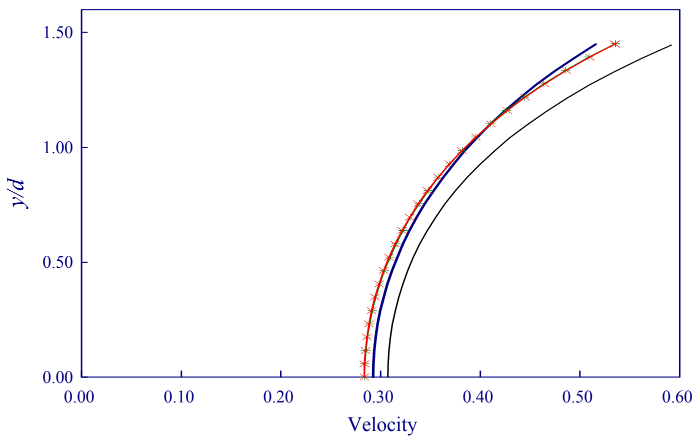

Figure 6. Velocity profiles under crest for $H/d = 0.55$, $\lambda/d = 16$.

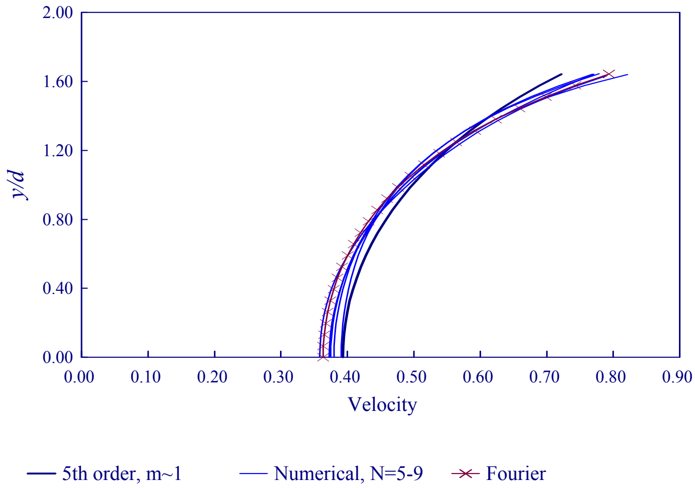

Figure 7. Velocity profiles under crest for high and long wave: $H/d = 0.7$, $\lambda/d = 32$.

giving results provided sufficient numbers of Fourier terms are taken and sufficient steps in wave height are taken. It can be seen that the present numerical cnoidal theory is also capable of high accuracy, as demonstrated by the close agreement between the two very different theories. It used much smaller computing resources, typically using 9-10 spectral terms with the solution of systems of 25 equations compared with the Fourier method with some 25 spectral terms and some 70 equations. However, it can be seen that there are some irregularities in the solution, and the results for different values of N do not agree to within plotting accuracy. Although the method shows difficulty with convergence, it does yield results of engineering accuracy. It is still remarkable, however, that such a demanding problem can be solved with so few ”spectral terms”.

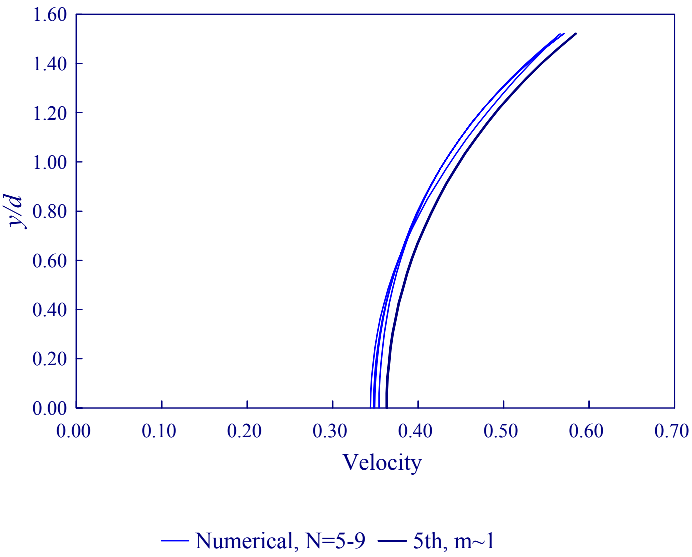

Figure 8. Velocity profile under crest for $H/d = 0.55$, $\lambda/d = 64$.

Figure 8 shows the velocity profile under the crest for a very long wave, $\lambda/d = 64$, with Nelson’s maximum height $H/d = 0.55$. The Fourier method took a large number of steps to converge that it was considered not worth while. The numerical cnoidal method performed quite well, although there was some variation between the solutions for $N = 5$ to $N = 9$. What is noteworthy, however, is that the fifth-order Iwagaki theory gave a good engineering accuracy solution to this problem.

Conclusions from computational results

The numerical cnoidal method has been shown to be accurate for waves longer than some eight times the water depth. It can treat very long waves rather more easily than Fourier methods can. As the theoretical highest waves are approached, however, the accuracy decreases to an approximate engineering accuracy. However there is strong evidence that these waves cannot be achieved in practice. Throughout, however, for waves with an Ursell number greater than 40, and apparently even for high waves, the fifth-order Iwagaki theory presented in this work gave satisfactory engineering solutions to the problems studied.

List of Symbols

Symbol	**Definition**
Roman symbols	
a	Constant in numerical test of order of accuracy
c	Wave speed
$\mathrm{cn}(\theta\|m)$	Elliptic function
$D(\epsilon)$	polynomial in denominator of Padé approximant
d	Mean water depth

$\mathrm{dn}(\theta|m)$ Elliptic function
$E(m)$ Elliptic integral of the second kind
E Kinematic free surface equation
$e(m)$ $= E(m)/K(m)$, Ratio of elliptic integrals
e_i Equation i in numerical cnoidal theory
$\mathbf{e}(\mathbf{z})$ Vector of errors in equations for numerical cnoidal theory
F_i Coefficients in expansion for f'_*
F_{ij} Coefficient in series for F_i
$f'(X)$ Velocity on bed
f_*' Dimensionless velocity on bed
g Gravitational acceleration
g_* $= gh^3/Q^2$, dimensionless number, inverse of square of Froude number
g_j Coefficients in expansion for g_*
H Wave height (crest to trough)
H_m Maximum wave height possible for a given wavelength
h Water depth under wave trough
$I(j)$ Mean value of $\mathrm{cn}^{2j}(\theta|m)$
i Integer used in sums *etc.*
j Integer used in sums *etc.*
$K(m)$ Elliptic integral of the first kind
$K'(m)$ $= K(1-m)$, Complementary elliptic integral
k Integer used in sums *etc.*
M Number of computational points in numerical cnoidal theory
m Parameter of elliptic functions and integrals
m_1 $= 1-m$, Complementary parameter
N Number of terms in series or polynomial in numerator of Padé approximant
n Order of errors or degree of polynomial or number of terms in series
p Pressure
Q Volume flux per unit span perpendicular to flow
q_1 $= \exp(-K/K')$, Complementary nome of elliptic functions
R Bernoulli constant (energy per unit mass)
R_* $= Rh^2/Q^2$, dimensionless energy per unit mass
S_n Sum to n terms of series
$\mathrm{sn}(\theta|m)$ Elliptic function
t Time
U Velocity component in X co-ordinate
$\bar{U}$ Mean value of fluid speed over a line of constant elevation
$\mathbf{U}$ $= H\lambda^2/d^3$, Ursell number
u Velocity component in x direction of frame fixed to bed
$\bar{u}_1$ Current at a point: mean value of u, averaged over time at a fixed point
$\bar{u}_2$ Depth-averaged current: mean value of u over depth, averaged over time
u_* Dimensionless velocity
u_{*s} Value of u_* on surface
V Velocity component in Y co-ordinate
v Velocity component in y co-ordinate
v_* Dimensionless velocity
v_{*s} Value of v_* on surface
w $= \pi z/2K'$, dummy variable
X $= x - ct$, horizontal co-ordinate in frame moving with wave crest
X_* $= X/h$
x Horizontal co-ordinate in frame fixed to bed
Y Vertical co-ordinate in frame moving with wave crest
Y_j Coefficients in expansion for η_*
Y_* $= Y/h$
y $= Y$, vertical co-ordinate in frame fixed to bed

z	Dummy argument used in elliptic function formulae
z_j	Variable j in numerical cnoidal theory, $z_1 = K(m)/K(1-m)$
$\mathbf{z}$	$= \{z_j,\ j = 1 \ldots 2M+6\}$, vector of variables
Greek symbols	
α	Coefficient of X/h in elliptic functions and expression of shallowness
Δ	Error in any equation
δ	$= \frac{4}{3}\alpha^2$, quantity used in series for velocity components
ϵ	general symbol for expansion quantity of series: ε, δ or H/d
ε	$= H/h$, dimensionless wave height
η	Water depth
η_*	$= \eta/h$, dimensionless water depth
θ	Argument of elliptic functions, often $\alpha X/h$ in this work
λ	Wavelength
ρ	Fluid density
τ	Wave period
Φ_{ijl}	Velocity coefficients in cnoidal theory
ψ	Stream function
ψ_*	$= \psi \times \sqrt{gh^3}/Q$, dimensionless stream function
Mathematical symbols	
$O()$	Order symbol: "neglected terms are at least of the order of"
$[i, j]$	Padé approximant with ith and jth degree polynomials in numerator and denominator

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