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Channel flow over curved boundaries and a new hydraulic theory

John D. Fenton
Department of Mechanical Engineering, Monash University
Clayton, Victoria, Australia 3168

Introduction

Dressler (1978) derived the St Venant long wave equations for two-dimensional irrotational flow over curved boundaries using orthogonal curvilinear coordinates based on the channel bottom. These equations are considerably more complicated than those in the cartesian system, but they do contain contributions to the pressure and the governing equations from the curvature of the bed, which does not occur in the cartesian formulation at the same level of approximation. Similarly, Chapman and Dressler (1984) obtained the shallow water equations for unsteady shallow groundwater flow with a free surface over a curved impermeable boundary. In both these cases, the flow was assumed to be irrotational, which is well-known to be justified in the case of seepage flow and for wave motion on otherwise quiescent fluid, but not so justified in the case of fast hydraulic flows such as over spillways.

In both formulations, the expressions obtained are complicated, and there have been few recorded attempts at solution, whether analytical or numerical. An exception to this is the work of Sivakumaran, Hosking, Tingsanchali and Yevjevich (SH&T, 1981; ST&H, 1983; and S&Y, 1987), who concentrated on the steady flow of fluid over spillways and curved beds. The authors drew the conclusions that Dressler's equations gave excellent agreement with experiment. For the steady flow cases studied, they concluded that his equations were simple to apply and described some important phenomena of the flow. Although the equations do describe some phenomena well, notably the pressures on a downstream spillway face, the level of approximation is still essentially of a conventional hydraulic nature with a centrifugal pressure correction

This paper shows that the scaling of variables introduced by Dressler has an inconsistency, such that the curvilinear equations governing fluid motion over curved beds are asymptotically only of the same level of approximation as the cartesian equations, and in most applications there would be little point in going to the extra complication of the curvilinear coordinates. Possibly more usefully, the paper, having noted some deficiencies in the hydraulic approximation in both curvilinear and cartesian co-ordinates, goes on to develop a general hydraulic theory in cartesian co-ordinates which incorporates a non-hydrostatic pressure distribution to first approximation by incorporating centrifugal effects. This can be applied to the flow of fluids over spillways or irregular topography, and seems to be able to describe the transition from sub-critical to super-critical flow as well as the existence of waves. As it does not suffer from some of the drawbacks of conventional theory it may have a number of uses in hydraulic practice.

Curvilinear co-ordinates

In this section devoted to curvilinear co-ordinates we suppose that the problem is two-dimensional, that there is no variation normal to the plane of flow. Let the distance along the channel bottom be denoted by s, and at any point on the bottom there is a local orthogonal curvilinear co-ordinate system (s, n), where n is perpendicular to s. The local radius of curvature of the bottom is r, such that in an elemental increment ds the bed turns through an angle $d\theta$, where θ is the slope angle of the bed, and $ds = r\, d\theta$, but it is more convenient to use curvature κ defined by $\kappa = 1/r = d\theta/ds$.

Now consider a typical horizontal length scale L, and d, a typical vertical length scale which, importantly, was assumed by Dressler to characterise not only the depth of flow, but also the vertical extent of variations in the bottom topography. Thus the appropriate scaling for $\sin\theta$ as used by Dressler is d/L:

$$\sin\theta(s) = \frac{d}{L}\, F(s/L), \tag{1}$$

where F is some function of the dimensionless arc length shown, of order of magnitude unity. Differentiating,

$$\cos\theta\, \frac{d\theta}{ds} = \frac{d}{L^2}\, F'(s/L). \tag{2}$$

However, $d\theta/ds = \kappa$, hence the dimensionless curvature is

$$\kappa d = \frac{d^2}{L^2} F'(s/L) \,/\, \cos\theta, \tag{3}$$

where $\cos\theta$, $F()$, and s/L are all of order unity. This shows that the dimensionless curvature is actually of the order of the shallowness parameter $\sigma = d^2/L^2$, and it will be legitimate to consider first-order approximations in curvature. The equations can be simplified considerably, and it can be shown that they are asymptotically of the same order as the St. Venant equations which it had been hoped to supplant.

Experimental test: flow over a mound in a channel

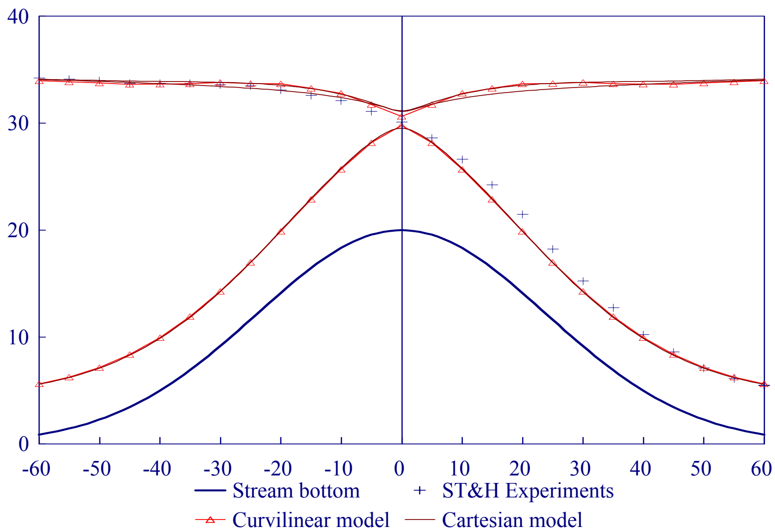

Figure 1. Flow of water over a Gaussian mound - experimental case of Figures 6(a) and 7(a) of Sivakumaran *et al.* (1983).

Sivakumaran *et al.* (1983) conducted interesting experiments on water flowing over a mound in a channel. Results are as shown in Figure 1. In the experimental situation water approached the mound sub-critically and passed over the crest and down the front face smoothly, becoming super-critical, as shown by the crosses in the figure. Results from the Dressler theory are shown by triangles and a solid line. It can be seen that according to the theory, flow is always symmetrical about the crest – if it starts subcritical it remains subcritical, dropping down to pass over the crest before going back to the same level as before; or, it remains super-critical, shooting over the hump and then back down again. The theory does not predict the transition from one state to the other which is a failure of the curvilinear formulation. The Dressler theory here, however, is a big improvement on conventional hydraulic theory, which would predict that the surface in either case develops a vertical tangent some time before critical flow is reached. Sivakumaran *et al.* solved a transcendental equation for the flow depth at each point. Here, we prefer to put it in the context of differential equation theory. It can be shown that their method is the equivalent of solving the differential equation

$$\frac{dh}{ds} = \frac{\sin(\theta)\,(1-\kappa h) + \frac{q^2\kappa\kappa'\,(\ln(1-\kappa h)+\kappa h)}{g(1-\kappa h)^3(\ln(1-\kappa h))^3}}{\frac{-q^2\kappa^3(\ln(1-\kappa h)+1)}{g(1-\kappa h)^3(\ln(1-\kappa h))^3} - \cos(\theta)}, \tag{4}$$

where $h(s)$ is the n-co-ordinate of the free surface, the thickness of flow in a direction locally normal to the bed. If, however, one accepts the argument above, that the Dressler formulation implicitly contains the approximation that it is of order κh, then it is legitimate to take the first-order series expansion in this quantity of both numerator and denominator to give

$$\frac{dh}{ds} = \frac{\sin(\theta)\,(1-\kappa h) + \frac{1}{2}\frac{q^2}{gh}\kappa' + O(\kappa h)}{\frac{q^2}{gh^3}\left(1+\frac{1}{2}\kappa h\right) - \cos(\theta) + O(\kappa h)}. \tag{5}$$

When applied to the situation of Figure 1, the results agreed well, except in the vicinity of the crest. If it is recognised that the terms in $q^2/h^2 \times \kappa$ in equation (5) arise because they are essentially centrifugal terms, then it is reasonable to argue that as the velocity on the bed is actually zero, rather than the maximum value assumed in the Dressler irrotational theory, then the centrifugal contribution should be reduced. Accordingly, if one puts a factor of ω_0 in front of those terms to allow for real fluid effects, a more realistic model of the flow situation may be had. The equation becomes

$$\frac{dh}{ds} = \frac{\sin(\theta)\,(1-\kappa h) + \omega_0\frac{1}{2}\frac{q^2}{gh}\kappa' + O(\kappa h)}{\frac{q^2}{gh^3}\left(1+\omega_0\frac{1}{2}\kappa h\right) - \cos(\theta) + O(\kappa h)}. \tag{6}$$

Using a value of $\omega_0 = 3/4$ gave the results shown on Figure 1 by an unadorned line, and they are everywhere closely coincident with the results from the Dressler theory except at the crest. Reducing ω_0 slightly below the notional value of $3/4$ gave almost exact coincidence. One can conclude that this simpler theory, correct to $O(\kappa h)$, is just as accurate, and may reflect the fluid mechanics more accurately. It seems that there is nothing special about the curvilinear formulation, and that the good agreement found previously may be just due to the incorporation of centrifugal effects on the pressure.

A new theory for hydraulic flows

In this section we show that use of the integrated quantities, discharge Q and cross-sectional area A, enables the development of equations in cartesian co-ordinates for channels of arbitrary section and bottom topography. The flexibility of the formulation allows for the incorporation of dynamic effects in the pressure to at least the first order of approximation. It will be seen that this leads to a higher-order of hydraulic approximation.

The momentum equation for open channel flow can be written (Fenton, manuscript in preparation):

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + \frac{1}{\rho}\int_A \frac{\partial p}{\partial x} dA + gAS_f = 0, \tag{7}$$

where p is pressure, ρ is fluid density, and S_f is the friction slope, and where we use cartesian co-ordinates (x, y, z), with x horizontally along the channel and z vertically upwards. In fact, few essential approximations have been introduced in equation (7): the first term is exact, and the second is exact for a velocity distribution which is linear in vertical and transverse directions, and so will be a reasonable approximation to real flows, while requiring no information about the distribution of velocity over the section. The only major problem is the pressure term $\int_A \partial p/\partial x\, dA$. The conventional approximation in hydraulic engineering is that the pressure is given by the equivalent static head of fluid above each point. Here we attempt to go further than that approximation by allowing for centrifugal effects on the fluid pressure. We assume that the vertical pressure gradient is given by the usual gravitational component plus a centrifugal contribution proportional to the mean square fluid speed over the section. This effectively means that we have a modified local body force, which we assume constant over the whole section at any point along the channel:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g - \overline{\alpha}\frac{Q^2}{A^2}, \tag{8}$$

where $\overline{\alpha}$ is the mean value over the section of a quantity related to streamline curvature. It can plausibly be shown that $\bar{\alpha}$ is the mean value of $\kappa/\cos\theta$ over the section, where κ is the local streamline curvature and θ the angle of inclination.

Integrating equation (8) in the vertical between a point z and the free surface η, differentiating with respect to x and integrating across the channel, we obtain the contribution

$$\frac{1}{\rho}\int_A \frac{\partial p}{\partial x} dA = \frac{\partial \bar{\eta}}{\partial x}\left(gA + \overline{\alpha}\frac{Q^2}{A}\right) + A\bar{d}\left(\frac{Q^2}{A^2}\frac{\partial \overline{\alpha}}{\partial x} + 2\overline{\alpha}\frac{Q}{A^2}\frac{\partial Q}{\partial x} - 2\overline{\alpha}\frac{Q^2}{A^3}\frac{\partial A}{\partial x}\right), \tag{9}$$

where $\bar{\eta}$ is the mean elevation of the free surface across the channel and where $\bar{d}$ is the depth of the centroid of the section below the surface. Now, it can be shown that $\partial\bar{\eta}/\partial x = 1/B\,\partial A/\partial x + \overline{Z'}$,where $\overline{Z'}$ is the mean gradient of the bed, upwards positive, so that, incorporating all the terms in the full equation (7), we have the momentum equation:

$$\begin{aligned}\frac{\partial Q}{\partial t} + 2\frac{Q}{A}\left(1 + \overline{\alpha}\bar{d}\right)\frac{\partial Q}{\partial x} + \left(\frac{gA}{B} - \frac{Q^2}{A^2}\left(1 + 2\overline{\alpha}\bar{d} - \overline{\alpha}\frac{A}{B}\right)\right)\frac{\partial A}{\partial x}& \\ + gA\left(\overline{Z'} + S_f\right) + \overline{\alpha}\overline{Z'}\frac{Q^2}{A} + \bar{d}\frac{Q^2}{A}\frac{\partial \overline{\alpha}}{\partial x} &= 0. \end{aligned} \tag{10}$$

This is one of the two governing partial differential equations. The other is the mass-conservation equation $\partial A/\partial t + \partial Q/\partial x = 0$ – very much simpler than Dressler's equation (13.02) and with the feature that this one happens to be exact.

Now we consider only two-dimensional flows. Let the depth be H, and discharge per unit width q, such that $A = BH$, $Q = Bq$, where B is a constant, $\bar{\eta} = \eta = Z + H$ and $\bar{d} = H/2$. In this case, equation (10) becomes

$$\frac{\partial q}{\partial t} + \left(2\frac{q}{H} + q\overline{\alpha}H\right)\frac{\partial q}{\partial x} + \left(gH - \frac{q^2}{H^2}\right)\frac{\partial H}{\partial x} + gH\left(Z' + S_f\right) + \overline{\alpha}Z'\frac{q^2}{H} + \frac{1}{2}q^2\frac{\partial \overline{\alpha}}{\partial x} = 0. \tag{11}$$

Now we have to consider the terms $\overline{\alpha}$ and $\partial\overline{\alpha}/\partial x$. It can be shown that as $\bar{\alpha} = \overline{\kappa/\cos\theta}$, a plausible approximate value for $\bar{\alpha}$ is a weighted mean of contributions from the bed and the surface:

$$\bar{\alpha} \approx \omega_0\frac{Z''}{1 + Z'^2} + \omega_1\frac{\partial^2 H/\partial x^2}{1 + Z'^2}, \quad \text{and} \quad \frac{\partial \overline{\alpha}}{\partial x} \approx \omega_0\frac{Z'''}{1 + Z'^2} + \omega_1\frac{\partial^3 H/\partial x^3}{1 + Z'^2} \tag{12}$$

where the two factors ω_0 and ω_1 are such that half of ω_0 is due to its effect of the bed in determining the elevation of the surface, while half is associated with dynamic pressures due to slow moving flow near the bottom of the stream, so that it might have a value somewhat less than 1, while ω_1 might have a value of about $1/2$.

Now we will consider only steady flow, in which q is everywhere constant, and equation (11) becomes a third-order ordinary differential equation in x, which we now write in standard form as

$$\frac{\omega_1 q^2}{2}\frac{d^3H}{dx^3}+\omega_1 Z'\frac{q^2}{H}\frac{d^2H}{dx^2}+\quad\left(1+Z'^2\right)\quad\left(\left(gH-\frac{q^2}{H^2}\right)\frac{dH}{dx}+gH\left(Z'+S_f\right)\right)$$
$$+\omega_0 q^2\left(\frac{Z'''}{2}+\frac{Z'Z''}{H}\right)=0. \tag{13}$$

This equation could be used to describe surface profiles over spillways, backwater curves *etc.,* where dynamic pressure effects have been included to first order. The conventional hydraulic approximation is obtained by ignoring centrifugal effects altogether, $\omega_0=\omega_1=0$, to give the familiar expression (for example, Henderson, 1966, #4.4):

$$\left(gH-\frac{q^2}{H^2}\right)\frac{dH}{dx}+gH\left(Z'+S_f\right)=0. \tag{14}$$

Here we examine the nature of equation (13) by considering flow over a uniformly-sloping bed, such that $Z'=-S_0$, using the term from conventional hydraulics, and we ignore the square of the slope term. The equation becomes

$$\frac{\omega_1 q^2}{2}\frac{d^3H}{dx^3}-\omega_1 S_0\frac{q^2}{H}\frac{d^2H}{dx^2}+\left(gH-\frac{q^2}{H^2}\right)\frac{dH}{dx}+gH\left(S_f-S_0\right)=0. \tag{15}$$

This form, ignoring the higher derivatives of the bed, might find favour in open-channel hydraulics, whereas those terms would be retained for flow over structures. Now if we consider small perturbations about a uniform flow of depth D such that $H=D+h$, and if we assume the head loss formula can be written $S_f=S_0\left(D/H\right)^m$, where $m=10/3$ for Manning's Law and $m=3$ for Chézy, we linearise the equation about this uniform flow:

$$\frac{\omega_1}{2}q^2\frac{d^3h}{dx^3}-\omega_1 S_0\frac{q^2}{D}\frac{d^2h}{dx^2}+\left(gD-\frac{q^2}{D^2}\right)\frac{dh}{dx}-mS_0gh=0. \tag{16}$$

This is a third-order linear differential equation. Standard methods can be used to show that there are three independent solutions of the form

$$h=C_1e^{\lambda_1 x/D}+C_2e^{\lambda_2 x/D}+C_3e^{\lambda_3 x/D}, \tag{17}$$

where the constants C_1 *etc.* will be given by boundary conditions, and the λ_1 *etc.* are solutions of the cubic equation corresponding to equation (16). Unfortunately the general expressions are very lengthy indeed, but by making an expansion in bed slope such that the neglected terms are $O\left(S_0^2\right)$, of the order of the square of the slope, which should be a very good approximation, values for the exponents are

$$\lambda_1=\frac{S_0\,m}{1-F^2}+O\left(S_0^2\right),\quad \lambda_{2,3}=\pm 2\,\frac{\sqrt{F^2-1}}{F}+\frac{S_0\,\left(F^2-1+m/2\right)}{F^2-1}+O\left(S_0^2\right). \tag{18}$$

These solutions are interesting: for *sub-critical flow*, $F<1$, λ_1 is positive, showing exponential growth, but at a slow rate, proportional to the slope, while λ_2 and λ_3 show oscillatory behaviour (the first part is imaginary) but with a slow exponential decay given by the second part. This corresponds, presumably, to what one sees in an undular hydraulic jump. The oscillatory behaviour cannot be predicted by conventional hydraulic theory. For *super-critical flow*, $F>1$, λ_1 is negative, showing slow exponential decay, but the solution will be dominated by λ_2 or λ_3, as in one of those cases the first part is real and positive, corresponding to exponential growth at a finite rate. This will make it necessary usually to solve the full differential equation (13) as a boundary value problem, providing a value of surface elevation and its derivative at one end of computations and, say, an elevation value at the other end. If computations were performed as a pure initial value problem with values of the elevation and its first two derivatives, then parasitic solutions showing exponential growth may well be generated with the computations becoming meaningless.

Results

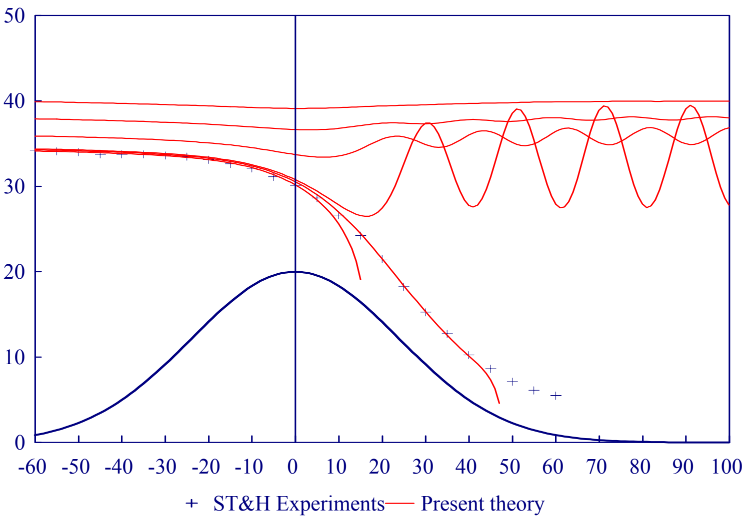

Figure 2. Results from the present theory for the same case as Figure 1.

Figure 2 shows the same experimental situation as Figure 1, but this time with results from the theory developed here, equation (13). The differential equation was solved by an enhanced Runge-Kutta method, re-casting the third-order equation as three first-order equations and with values of $H(-100)$, $H'(-100) = 0$ and $H''(-100) = 0$ provided. The curve for $H(-100) = 34.26$, corresponding to the experimental conditions is the lowest curve, which dives into the bed just past the crest. This is not surprising, because as described in the previous section, solving the problem with only initial conditions should always allow the possibility of the parasitic exponential-increasing solution, which is certainly the case here. To solve the differential equation with a downstream boundary condition is a rather demanding problem which will not be addressed here. The conventional shooting method, satisfying a downstream condition iteratively is difficult here because of the strength of the parasitic solution. This is shown by the second curve, which follows the experimental points very closely, and which was for $H(-100) = 34.395969$, every digit of which was necessary to carry the solution as far as shown, before it too dived into the bed. What is clear, however, from both these solutions, is that the differential equation can describe the flow over rapidly varying topography very accurately, and seems to give excellent agreement with experiment. It makes the transition from sub-critical to super-critical flow and generally behaves just as the physical flow did. It is unfortunate that the parasitic solution destroyed the solution, but provided a means can be found of incorporating a downstream boundary condition the method seems to be capable of obtaining solutions where conventional hydraulics cannot. It seems that the simple incorporation of centrifugal pressures provides the key.

The other curves shown on the figure are for initial depths of 34.5, 36, 38 and 40cm, and it can be seen that in each case a good description of a plausible hydraulic flow was obtained. For flows remaining only marginally sub-critical downstream wave trains (undular bores) were displayed, as commonly observed in real streams downstream of obstacles. For gradually increasing inundation, the wave train decreased in amplitude, until for 40cm the flow passed over the obstacle in an almost symmetric manner.

Conclusions

A new level of hydraulic theory has been attempted, where centrifugal effects on the pressure distribution have been incorporated into the long wave equations for an arbitrary channel. For the case of steady flow, the equations have been solved analytically for the case of a near-uniform flow, and seem to show characteristics of hydraulic flows which conventional theory cannot, including the existence of standing waves. Numerical solutions were obtained for flow over an obstacle in a channel. Good agreement with experiment was observed, and the transition from sub-critical to super-critical flow can be simulated. However, parasitic solutions may be generated, and the problem of obtaining solutions with downstream boundary conditions is yet to be solved. Overall, the new theory presented here seems capable of considerable use in describing real hydraulic problems, and may prove to be of benefit in research and practice.

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