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Exploiting periodicity in boundary integral equations

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1. Introduction

This paper shows how the inherent periodicity in boundaries of two-dimensional domains may be exploited in numerical computations. The boundary of a region is inherently periodic, for if it is circumnavigated in any particular direction, as a second or any other traverse is made, both the geometry and any variables on the boundary are the same as in the first, and so on. It is possible to exploit this periodicity of the problem to give methods for interpolation and associated differentiation which are capable of greater accuracy than traditional methods, yet are much simpler to implement.

In practice, some effort has to be made at corners to ensure sufficient piecewise continuity. Provided this is done, methods based on Fourier interpolation and approximation can be used with greater accuracy than low-degree polynomial methods which are routinely used. Apparently paradoxically, the Fourier methods are almost always easier to implement. They can be applied to relatively simple operations such as interpolation of the boundary of the body for graphical representation, or for differentiation around the body, where convenient explicit formulae can be obtained.

A particularly powerful application of the Fourier methods is in the numerical approximation of boundary integral equations. Traditionally low-order polynomial approximation methods have been used, which at lowest order are not complicated. Neither are they particularly accurate. For increasing accuracy, higher-order methods are used, and these become very complicated indeed for example Brebbia (1984) and Brebbia & Dominguez (1989), and the numerical approximation of the boundary and the field variable becomes a demanding task analytically.

In particular this paper demonstrates the application of the exploitation of boundary periodicity by considering the solution of potential problems in two dimensions using the Cauchy integral theorem. It is shown that the singularity of the kernel can be eliminated, such that the integrand is everywhere continuous. Although the boundary integral equation methods described here are for potential problems in two dimensions only, they are capable of extension to many other problems and into three dimensions.

2. Fourier approximation of a two-dimensional region

Consider a simply-connected domain in two dimensions. Suppose the complex coordinate is known for each of the N points $z_j = x_j + \mathrm{i} y_j$, for $j = 0, 1, \cdots, N-1$, where $\mathrm{i} = \sqrt{-1}$. Consider the discrete Fourier transform of the points:

$$Z_m = \frac{1}{N} \sum_{j=0}^{N-1} z_j \, \mathrm{e}^{-\mathrm{i}2\pi mj/N} = D(z_j\,; m), \tag{1}$$

which is a sequence of the complex Fourier coefficients Z_m, for $m = -N/2, \ldots, +N/2$. The inverse discrete Fourier transform is

$$z(j) = \sum_{m=-N/2}^{+N/2}{}'' Z_m\, \mathrm{e}^{+\mathrm{i}2\pi mj/N}, \tag{2}$$

where the sum Σ'' is interpreted in a trapezoidal rule sense, such that a value of $1/2$ multiplies the end contributions at $\pm N/2$.

The usual manner in which this is interpreted is that evaluating for integer j, the complex coordinate z_j is recovered, which is the inverse discrete transform, denoted by the symbol D^{-1}:

$$z_j = D^{-1}(Z_m\,; j). \tag{3}$$

In the approach of this work it is suggested that it is not necessary to restrict j to having integer values in Equation (2). If, instead, j is interpreted as varying continuously from 0 to N in a circumnavigation of the boundary, then Equation (2) can be interpreted for arbitrary j as the Fourier series which interpolates the points z_j. Hence, it could be used for purposes such as accurately plotting the boundary of a region defined by a finite number of points and many others.

For this approach to be accurate it will be necessary for the z_j to be sufficiently smooth in j that the finite spectrum Z_m for $m = -N/2, \ldots, +N/2$ for $\pm N/2$ will have decayed sufficiently quickly that as the ends of the spectrum are approached the coefficients will be so small that truncation of the series at $\pm N/2$ will be accurate. This is treated in the next section.

To obtain expressions for numerical values of derivatives, the interpolating function Equation (2) can be differentiated to give:

$$z'_j = \frac{\mathrm{i}2\pi}{N} \sum_{m=-N/2}^{+N/2}{}'' m\, Z_m\, \mathrm{e}^{+\mathrm{i}2\pi mj/N} = \frac{\mathrm{i}2\pi}{N} D^{-1}(mZ_m\,; j). \tag{4}$$

In this way, the z'_j may be computed by taking the discrete Fourier transform of the points z_j, multiplying each coefficient by m and inverting. If fast Fourier transform programs are available this can be done in $O(N \log N)$ operations.

If N is not large, and if FFT programs are not available it would be quite in order to write simple programs to evaluate the series (1), (2) and (4) directly as written. However, it is possible to obtain an expression for the derivatives z'_j in terms of the z_j by substituting equation (1) written in terms of n instead of j, which gives, after changing the order of summation:

$$z'_j = \frac{\mathrm{i}2\pi}{N^2} \sum_{n=0}^{N-1} z_n \sum_{m=-N/2}^{+N/2}{}'' m\, \mathrm{e}^{+\mathrm{i}2\pi m(j-n)/N}. \tag{5}$$

This can be written as

$$z'_j = \frac{\mathrm{i}2\pi}{N^2} \sum_{n=0}^{N-1} z_n\; d(j-n), \tag{6}$$

such that the equation is a simple finite difference type of expression for the derivative, in terms of a weighted sum of all N point values of the co-ordinates z_j. We expect the accuracy of this expression to be rather higher than conventional finite difference expressions, as it should have the accuracy of the underlying Fourier approximation. The coefficients $d(j-n)$ are given by the finite sum

$$d(j-n) = \sum_{m=-N/2}^{+N/2}{}'' m\, \mathrm{e}^{+\mathrm{i}2\pi m(j-n)/N}, \tag{7}$$

and it is possible to obtain an analytical expression for these coefficients, as follows.

Consider the geometric series

$$S = \sum_{m=-N/2}^{+N/2}{}'' r^m \quad \text{where} \quad r = e^{i2\pi(j-n)/N}. \tag{8}$$

Differentiating with respect to r gives

$$\frac{dS}{dr} = \sum_{m=-N/2}^{+N/2}{}'' m\, r^{m-1} = \frac{1}{r} \sum_{m=-N/2}^{+N/2}{}'' m\, r^m, \tag{9}$$

with the result

$$d(j-n) = r\,\frac{dS}{dr}, \tag{10}$$

using equation (7). An explicit expression can be obtained using the properties of geometric series. Considering equation (8) and following the standard procedure for evaluating geometric series of multiplying both sides of the equation by r and subtracting gives

$$S = \frac{r^{-N/2} + r^{-N/2+1} - r^{N/2} - r^{N/2+1}}{2(1-r)}. \tag{11}$$

This expression is able to be differentiated, and after some manipulations the result is obtained that

$$\frac{dS}{dr} = \frac{N}{2}(-1)^{j-n}\frac{r+1}{r(r-1)}. \tag{12}$$

If $j = n$ such that $r = 1$, it can be shown, after some manipulations using a limiting procedure, that the result is zero. Substituting this result and equation (12) into equation (10) and using equation (8), it can be shown after further manipulations that

$$d(j-n) = \begin{cases} -i\,\frac{N}{2}(-1)^{j-n}\,\cot\frac{\pi(j-n)}{N} & \text{for} \quad j \neq n, \\ 0 \quad \text{for} \quad j = n. \end{cases} \tag{13}$$

Substituting these into Equation (6) gives

$$z'_j = \frac{\pi}{N}\sum_{n=0,\ n\neq j}^{N-1} z_n\,(-1)^{j-n}\cot\frac{\pi(j-n)}{N}, \tag{14}$$

for $j = 0, \ldots, N-1$, which is easily programmed. The computational cost to evaluate all N of the z'_j is $O(N^2)$, but the accuracy is that of the rest of this work, which is Fourier accuracy: if the function is periodic, has continuous derivatives, and the kth derivative is piecewise continuous, then the error of approximation by the finite Fourier series is (#6.5 Conte & de Boor 1980):

$$\text{Error} \ \leq\ \frac{\text{Constant}}{N^k}. \tag{15}$$

For functions which are of low degrees of continuity, where k might be 2, 3 or 4 say, the accuracy will be comparable to traditional low-level polynomial approximation of the integrals, however if high degrees of continuity exist, the method should be very accurate indeed. Throughout this work, the accuracy attained is this "Fourier" accuracy.

It will be seen later just why the quantity z'_j is important in the numerical approximation of boundary integral equations. However, a possible simple practical problem is the computation of the slope of the boundary at a point, in which it is trivially shown that

$$\frac{dy}{dx}(j) = \Im\left(z'_j\right)/\Re\left(z'_j\right), \tag{16}$$

where $\Re()$ and $\Im()$ denote the real and imaginary parts respectively.

3. Distribution of computational points

The expressions for the interpolating Fourier series and for the derivatives around the boundary, have been expressed above in terms of the co-ordinates of the computational points z_j, and it has been shown that the accuracy of those expressions depends on how continuous the latter are in terms of the co-ordinate j. To do this it is necessary to spend some effort in ensuring piecewise continuity across corners in the boundary, something which conventional methods which use local approximation do not have to consider.

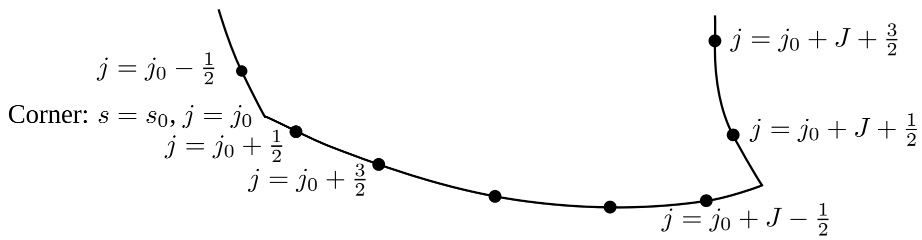

Figure 1. Arc of boundary showing point numbering

Consider an arc of a boundary that has a corner at each end, as shown in Figure 1. Let there be some parameter such as arc length s which uniquely specifies position on the arc. If ds/dj is continuous along the arc *and across the corners at its ends*, then so is z_j', and the problem is to distribute the computational points such that ds/dj is continuous across the ends. It seems that the easiest way of doing this is to choose the points such that ds/dj and possibly several higher derivatives go to zero at the ends. If this is done for all arcs of the boundary, there will be a high degree of continuity at each corner. This was implemented and found to work very well. Here this is done using s as the point specifying parameter, but any monotonically increasing quantity could be used. In fact we will see that even if no special spacing is used, the accuracy is still surprisingly high for integral equations, but here we attempt to produce a system capable of exploiting fully the potential accuracy of the method.

We suppose that in terms of s the arc has a total length of L, and that there are J computational points on the arc. If we were to put computational points at the corners of the arc, we would have to introduce a special combined treatment for different types of boundary condition when approximating boundary integral equations. It is simpler to place computational points away from the corner such that j increases by 1/2 between the corner and the first computational point, by 1 between each of the computational points, and by 1/2 between the last point and the corner. In this way the parameter j increases by an integer quantity J from the initial value of j_0 at the first corner, and the corner points are equidistant in terms of the variable j between computational points. Now, to obtain a degree of continuity across the corner we assume a form for ds/dj of the form

$$\frac{ds}{dj} = D(j - j_0)^p (j_0 + J - j)^q, \tag{17}$$

where D is a constant for the arc considered, and p and q are numbers (probably integers) such that at the first corner $j = j_0$ the first p derivatives of s with respect to j are zero, and at the other corner the first q derivatives are zero. In practice it is likely that it will be possible to use $p = q$ everywhere unless there are some unusual combinations of arcs or boundary conditions. It is simpler to assume that q is an integer, such that the series obtained by expanding $(j_0 + J - j)^q$ by the binomial theorem terminates. Integrating that and evaluating the constant D by requiring that when $s = s_0 + L$, $j = j_0 + J$ gives the spacing formula for the arc length s_j of point j:

$$\frac{s_j - s_0}{L} = \frac{S((j - j_0)/J, p, q)}{S(1, p, q)}, \tag{18}$$

where

$$S(t,p,q)=t^p\left(\frac{t}{p+1}-\frac{qt^2}{p+2}+\frac{q(q-1)t^3}{2!\,(p+3)}+\cdots+\frac{(-1)^q t^{q+1}}{p+q+1}\right). \qquad (19)$$

The degree of continuity in terms of p and q is arbitrary. The dependence of results on those parameters will be examined below.

Now that the arc lengths s_j corresponding to computational points have been calculated, the complex co-ordinates of the points may be assigned. For a regular boundary, with simple explicit formulae for the arcs such as straight lines, circular or elliptical arcs *etc.*, the values of the z_j' may be calculated from the expression

$$\frac{dz(j)}{dj}=\frac{dz(j)}{ds}\frac{ds}{dj}, \qquad (20)$$

where $dz(j)/ds=\cos\alpha+i\ \sin\alpha$, where α is the angle the tangent makes with the x axis, and where ds/dj may be calculated using equation (17) and the result that

$$D=\frac{L}{J^{p+q+1}\,S(1,p,q)}. \qquad (21)$$

However, for irregular boundaries it should be more convenient to compute the coefficients numerically by the method described in the previous section.

4. Boundary integral equation solution of potential problems in two dimensions

4.1 Theory

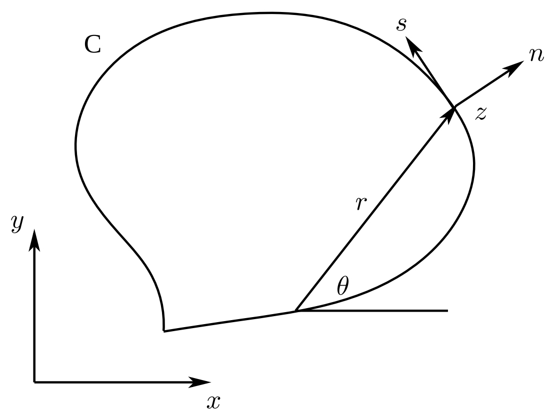

Figure 2. Typical computational domain showing important points and coordinates

Consider a two-dimensional region such as that shown in Figure 2 in which a scalar potential function ϕ exists and satisfies Laplace's equation, $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$. A typical boundary value problem is where the value of ϕ or its normal derivative $\partial\phi/\partial n$ or a combination of the two is known at all points on the closed boundary C. The usual way of doing this is to set up and solve the integral equation:

$$\alpha\phi\left(z_m\right)=\int_{\mathrm{C}}\left(\frac{\phi}{r}\frac{\partial r}{\partial n}-\frac{\partial\phi}{\partial n}\log\ r\right)ds, \qquad (22)$$

where s is an arc-length co-ordinate around the contour, α is the interior angle at the point m, which if the boundary turns continuously is π, r is the distance of a general point from m, and at the general point a normal and tangential co-ordinate system (s,n) exists. Equation (22) is the form which has received most attention in the literature. Numerical approximation of the integral near the singularity at m is demanding, and a great deal of effort has to be given to the details of computation schemes. For example, the second order scheme described in Brebbia & Dominguez (1989) approximates the integrands allowing quadratic variation and a great deal of complicated mathematics has to be worked through and presented. For higher orders of approximation the effort would be prohibitive.

It is the aim of this paper to develop a formulation of the problem in which the integrands are not singular, and more importantly to exploit the fact that the problem is periodic as one traces around the boundary. In this way, apparently paradoxically, numerical schemes can be developed which have the advantages of being both simple and accurate.

As ϕ is an harmonic function, another function ψ exists, and the complex function $w = \phi + i\psi$, where $i = \sqrt{-1}$. Cauchy's theorem states that w satisfies

$$\oint w(z)\, dz = 0, \tag{23}$$

where the path of integration is around the contour C.

In this work, an approach is adopted in which the singularity is subtracted. This can be easily done, for if $w(z)$ is analytic within and on C, then so is the function $(w(z) - w(z_m))/(z - z_m)$, and Equation (23) can be written with this as integrand:

$$\oint \frac{w(z) - w(z_m)}{z - z_m}\, dz = 0, \tag{24}$$

where the integrand is everywhere continuous, even at $z = z_m$, and its numerical approximation should be simpler and potentially more accurate.

4.2 Numerical scheme with boundary periodicity

The continuous co-ordinate j is introduced here, which is 0 at some reference point on the boundary, and after a circumnavigation of the boundary, takes on a value N, which is assumed to be an integer. The integral in Equation (24) can be written

$$\int_0^N \frac{w(z(j)) - w(z_m)}{z(j) - z_m} \frac{dz}{dj}\, dj = 0. \tag{25}$$

Now the numerical approximation is introduced to transform the integral equation into an algebraic one in terms of point values. To do this, the integral in Equation (25) is replaced by the trapezoidal rule approximation:

$$\sum_{j=0}^{N-1} \frac{w(z_j) - w(z_m)}{z_j - z_m} z_j' = 0, \tag{26}$$

where $z_j = z(j)$ and $z_j' = dz(j)/dj$, but where, after the differentiation,j now takes on only integer values. It might be thought undesirable that the usually low-accuracy trapezoidal rule has been used, however, where the integrand is periodic, as it is here, the trapezoidal rule is capable of very high accuracy indeed, related to the accuracy of interpolation as expressed by Equation (15).

It is interesting that Equation (26), which required little effort in its derivation and which has remarkably simple weights for the $w(z_j)$, is a numerical approximation to the original integral which is very much more accurate than conventional approximations which assume particular variations of the unknown (ϕ or w) and of the boundary, whether constant, linear or quadratic over an "element", and which may use high-order Gauss formulae for the integrals. Such calculations are long and tedious, as are the formulae obtained Brebbia & Dominguez (1989). The present formulation seems to be rather simpler, and has no need for the concept of boundary elements, as it uses the points on the boundary as mere interpolation points.

In the form of Equation (26), the expression is not yet useful, as the case at the singularity $j = m$ has to be addressed. It is easily shown that in this limit, the integrand and hence the summand becomes $dw(m)/dm$, and extracting this term from the sum gives the expression in terms of a "punctured sum" $j \neq m$:

$$\frac{dw}{dm}(m) + \sum_{j=0,\ j\neq m}^{N-1} \frac{w_j - w_m}{z_j - z_m}\, z_j' = 0, \tag{27}$$

for $m = 0,\ 1,\ 2, \dots,\ N-1$, and where the obvious notation $w_j = w(j)$ *etc.* has been introduced. The notation $dw(m)/dm$ means differentiation with respect to the continuous variable$m$, evaluated at integer value$m$. It is convenient here to introduce the symbol Ω_{mj} for the geometrical coefficients:

$$\Omega_{mj} = \alpha_{mj} + i\beta_{mj} = \frac{z_j'}{z_j - z_m}, \tag{28}$$

with the real geometrical coefficients α_{mj} and β_{mj} thus defined. One is free to use either the real or imaginary part of the integral equation and of the sum, Equation (27). Here the two parts are extracted, to give

$$\frac{d\phi}{dm}(m) + \sum_{j=0,\ j\neq m}^{N-1} \left[\alpha_{mj}(\phi_j - \phi_m) - \beta_{mj}(\psi_j - \psi_m)\right] = 0, \tag{29}$$

$$\frac{d\psi}{dm}(m) + \sum_{j=0,\ j\neq m}^{N-1} \left[\alpha_{mj}(\psi_j - \psi_m) + \beta_{mj}(\phi_j - \phi_m)\right] = 0. \tag{30}$$

Provided either $d\phi/dm$ or $d\psi/dm$ is known at every point on the boundary, one of these equations can be used at each of the N computational points, written in terms of the $2N$ values of ϕ_j and ψ_j . If N of these are known, specified as boundary conditions, then there are enough linear algebraic equations and it is possible to solve for all the remaining unknowns.

4.3 Boundary conditions

There are two common boundary conditions:

4.3.1 Dirichlet conditions

These are when ϕ or ψ is specified along an arc, a term used here to denote part of the boundary. If the computational point spacing is known, such that $dz(m)/dm$ can be calculated, then for example, $d\phi(m)/dm$ in Equation (29) can be calculated from

$$\frac{d\phi}{dm}(m) = \frac{\partial\phi}{\partial s}\frac{ds}{dm} = \frac{\partial\phi}{\partial s}\left|\frac{dz}{dm}\right|, \tag{31}$$

where s is a boundary arc-length variable with arbitrary origin, and where ϕ is known along an arc such that $\partial\phi/\partial s$ can be calculated. A common occurrence is where ϕ or ψ is constant along a boundary, in which case the corresponding $d\phi/dm$ or $d\psi/dm$ is zero, and the implementation is particularly simple.

4.3.2 Neumann conditions

These occur where the normal derivative $\partial\phi/\partial n$ is specified along an arc such that $\partial\phi/\partial n = f(s)$, where $f(s)$ is known, then using one of the Cauchy-Riemann equations gives $\partial\psi/\partial s = f(s)$ and integrating gives $\psi = F(s) + C$, where C is an arbitrary constant of integration. For computational purposes, this is written $\psi_m = \Psi(m) + C_n$, where $\Psi(m)$ is a known function of m, and C_n is a constant on this, the nth Neumann boundary arc. It can be shown that Neumann boundaries can be treated by methods similar to those described above for Dirichlet conditions.

4.4 Set-up and solution of equations

When the z_j' have been calculated, the coefficients α_{mj} and β_{mj} are known, and can be used in equations (29) or (30), one for each point at which an unknown exists. There is a certain amount of good fortune here, for if the magnitudes of the coefficients are examined, it may be seen that the system of equations is such that the equation written for point m is dominated by the coefficient of the unknown at that point. This suggests that the iterative procedure of Gauss-Seidel is a possibility, which was found to work well. It was found that a lot of programming detail could be avoided if the step of assembling into a matrix was by-passed. In this case, the equations (29) and (30) may simply be written in terms of the appropriate dominant unknown on the left side and sequentially evaluated. It was found that large numbers of points could be used if the coefficients of each equation were generated at the time the equation was evaluated, as it is only necessary to store the information from one equation at a time. The implementation of the scheme in this equation-by-equation iterative form was particularly simple.

4.5 Post-processing: calculation of values

After a solution has been obtained it is usually necessary to be able to calculate values of ϕ or w at an arbitrary point in the plane, z_m say, which is neither a computational point nor even a point on the boundary. In this case, it is no longer necessary to take the limiting procedure, and (26) can be written as

$$\sum_{j=0}^{N-1}(w_j - w_n)\Omega_{nj} = 0, \tag{32}$$

where the Ω_{nj} are given by equation (28). Provided $w(z)$ is sufficiently continuous, this is an excellent approximation to the original integral equation (24). Separating the components gives the value of w at z_n:

$$w_n = \frac{\sum_{j=0}^{N-1} w_j\ \Omega_{nj}}{\sum_{j=0}^{N-1} \Omega_{nj}}, \tag{33}$$

from which it is simple to extract either the real or the imaginary part.

4.6 Results

The test problem which was chosen for comparing results from the present work with traditional polynomial approximation methods was that of the St-Venant torsion of a square bar (Brebbia & Dominguez 1989). The problem can be stated in mathematical terms: $\nabla^2\phi = 0$ within a rectangle of x dimension $2a$ and y dimension $2b$. It is possible to exploit symmetry so that we can just consider a quarter of the rectangle, and in this work we consider a square with $a = b = 1$, such that $0 \le x \le 1$ and $0 \le y \le 1$, with the boundary conditions: $\phi = 0$ on the two sides $y = 0,\ 0 \le x \le 1$ and $x = 0,\ 0 \le y \le 1$, while $\partial\phi/\partial n = y$ on $x = 1,\ 0 \le y \le 1$, and $\partial\phi/\partial n = -x$ on $y = 1$,$0 \le x \le 1$. This problem has an analytical solution which can be used for comparison purposes.

Figure 3 shows the results obtained by using three conventional boundary element computer programs (Brebbia & Dominguez 1989) which use a hierarchy of approximations based on polynomial approximation, successively assuming that quantities are (1) constant over an element, (2) vary linearly across an element and (3) vary quadratically across two elements.

Results from the present method are shown for comparison. The parameter p refers to the degree of continuity across corners of the square, and throughout q was equal to p. The case $p = 0$ means that the zeroth derivative was continuous, giving equally-spaced points. It is clear that the present method is capable of rather greater accuracy than the polynomial approximation method using boundary elements. For few computational points, $N = 8$, the traditional method was more accurate - reflecting the fact that an 8-term Fourier series approximation to a square is not a very good approximation. For larger values of N, however, the present work was capable of considerably greater accuracy, especially for

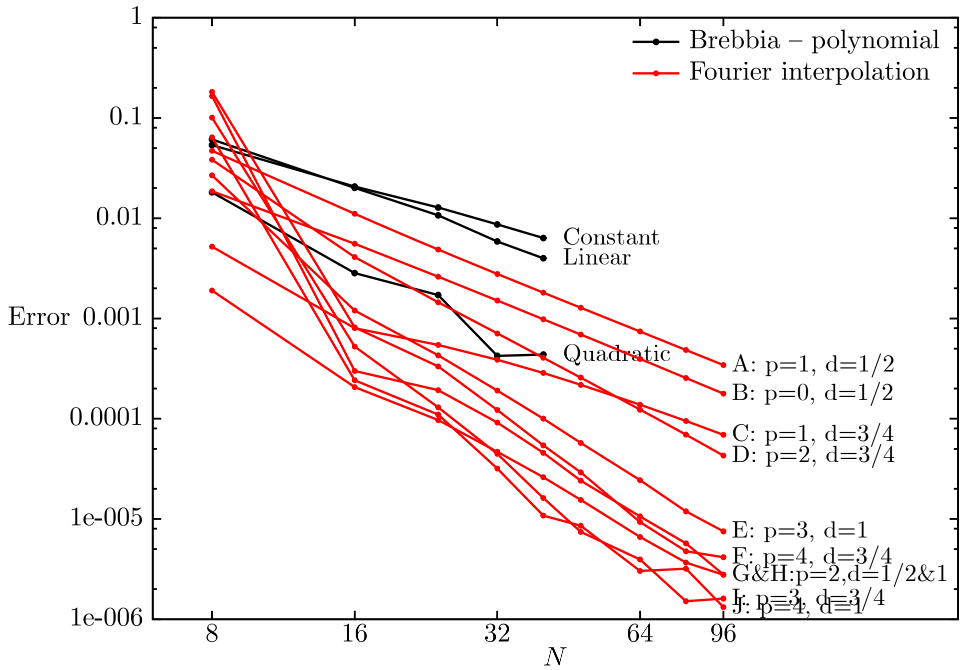

Figure 3. Comparison of accuracy of the present method with boundary element programs

large numbers of points.

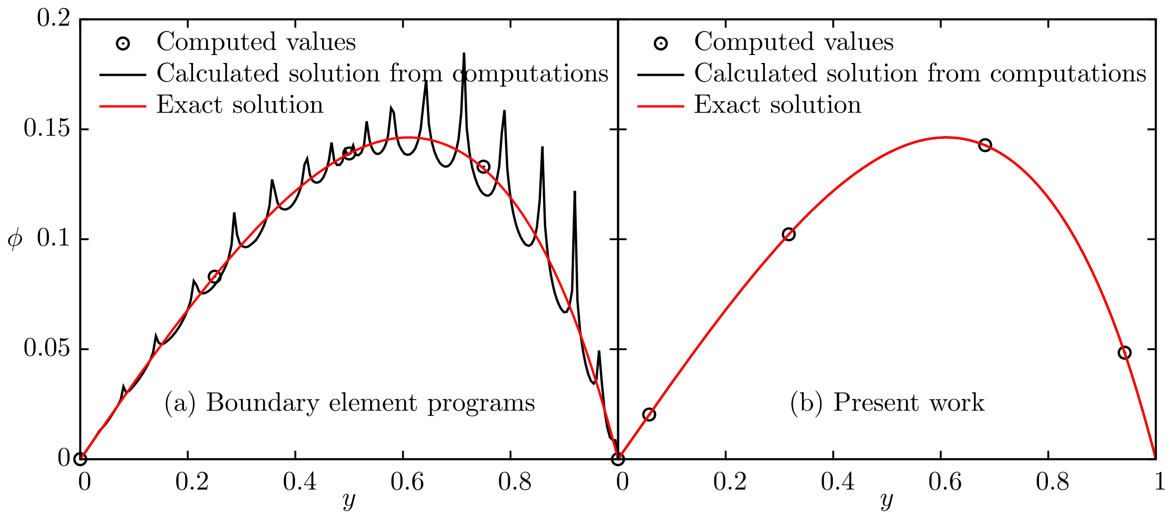

Figure 4. Distribution of ϕ along one face showing the results at computational points plus the distribution computed from those points, compared with the exact solution

It is not until the details of the field solution are computed, that other advantages of the present method become apparent. In Figure 4(a) are the computational results from the three traditional boundary element methods for the distribution of ϕ along the face of the square: $x = 1,\ 0 \le y \le 1$, using a total of 16 computational points. It can be seen that the computed point values agree closely with the exact solution. When, however, the value of ϕ is computed at points along the boundary the results are very poor indeed. It seems that the singular nature of the integrals used in that method give large fluctuations as computational singularities are traversed around the boundary (there are 16 of the little peaks, equal to the number of points). In Figure 4(b) results from the present work are shown, using Equation (33). The accuracy of that expression, obtained via an implicit singularity subtraction, is clearly very high indeed, and there seems to be no numerical ill-conditioning. The method did fail if a computational point ever coincided to machine accuracy with one of the points used in the solution, however, provided it was not coincident there was no apparent ill-conditioning at all, a rather surprising result. For example, as close as 10^{-11} to any computational point, the computed solution was still accurate to the full machine accuracy, some 14 figures. The implications of this are important, for if field values cannot be computed at an arbitrary point by the traditional method to acceptable accuracy, then achieving accuracy for the computational points themselves is somewhat in vain. For example, any contour plot of the field would be very ragged. Using the present approach, however, enables contour plots of high accuracy to be produced.

5. Conclusions

Some useful formulae have been obtained for the numerical description of boundaries of simply-connected two-dimensional regions. These have been shown to be able to simply obtained yet are capable of high accuracy. In particular they are useful when approximating boundary integral equations, and results for a potential problem show that the methods proposed are considerably superior to traditional boundary element methods. The extension to three-dimensional bodies can be trivially obtained, as it would simply be necessary to supplement the counting-type variable j in one dimension with another, such that the body would be expressed in terms of the two periodic variables, something like latitude and longitude. While multiply-connected bodies in two dimensions can be easily treated by the present methods by breaking up the boundary description into two or more separate Fourier series, it is not yet clear how practically useful that would be for three-dimensional bodies which are not simply connected.

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7. Biographical sketch

John Fenton obtained a Diploma of Civil Engineering from Bendigo Technical College, a Bachelor of Engineering and Master of Engineering Science from the Civil Engineering Department at Melbourne University, and a Doctor of Philosophy in the Department of Applied Mathematics and Theoretical Physics at Cambridge University. His research has been in the areas of maritime engineering, hydraulic engineering, and computational fluid mechanics. He is currently an A.R.C. Senior Research Fellow in the Department of Mechanical Engineering at Monash University.