9. Subsea control systems - on the nature of wave propagation in long hydraulic transmission lines

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Abstract

Two models governing the propagation of signals in hydraulic control lines are considered. One is a linear model starting with two-dimensional viscous equations of motion, and the other a nonlinear model with a nonlinear friction term. It is shown that the speed of propagation of disturbances is not the commonly accepted expression, but is modified by friction, and in fact, depends on the nature of the disturbance. In most practical cases this effect is small, but it is desirable to be aware of the effect. It is shown how both linear and nonlinear model are related, and then a computational scheme is developed which is based on the method of characteristics and high-order approximation by splines. It is found to be robust and accurate and capable of treating problems with rapidly-varying transients.

1 Introduction

Subsea control systems are characterised by very long hydraulic control lines, 10-15 km in length are not uncommon. Responses of hydraulic actuators which operate production control valves on a subsea christmas tree are directly affected by transmission characteristics of the connecting lines. One of the regulatory requirements for a subsea control system is to determine, at the design stage, the time needed to close all production control valves under the scenario that all hydraulic valve actuators are vented to the surface at the platform level. Thus, we need to have an exact model of the connecting line. As the lumped parameter approximation of a very long transmission line requires long computation time, distributed parameter models are investigated and described in this paper.

We examine the equations for compressible flow in rigid pipes to obtain a greater understanding of the nature of the propagation of waves and with a view to developing computational schemes. We start with a linear two-dimensional model, which is converted to a one dimensional model which we investigate in the time domain. The results contain some novel features: the speed of propagation of disturbances depends on the amount of friction, and the waves are dispersive such that distortion of the wave form may be important. In the majority of cases such effects are small, however it is important to have a detailed understanding of the real processes at work and to know when the effects may be finite. Then the case of nonlinear friction is considered, for which time domain methods are necessary. It is shown how the linear and nonlinear cases are related. Then a numerical method is developed for both linear and nonlinear cases. The method is based on a characteristic formulation, and it is shown how the formulation reduces to simple processes of interpolation, for which the use of spline and exponential spline approximation is suggested, giving a fast method of high accuracy and for which most programming details may be relegated to subroutines. When applied to the propagation of rapidly-varying transients the method is seen to be robust and accurate and able to describe well the rapid variation across the shocks.

The proposed model may be extended to include compliant lines. The line model is easily interfaced with lumped parameter models (eg. bondgraph or block diagrams) or other components in the control system.

2 Nomenclature

Latin symbols

a	Radius of line
$C_\pm$	Characteristics of gradients $u \pm c_0$
C	Actual propagation speed
c_0	Mean speed of sound in the line
d	Diameter of line
D	Coefficient matrix
E	Coefficient matrix
$F_{0,L}$	Function relating p, u & t at ends
G	Negative of the pressure gradient
$H_{1,2}$	Coefficients
i	$\sqrt{-1}$, also used as integer
k	Wavenumber
L	Length of line
p, $\bar{p}$	Pressure and its mean across line
$p_\pm$	Value of p at $x = x_\pm$
Q	Volume flux along line
r	Radial coordinate
t	Time
u	x velocity component and mean
$\bar{u}$	Mean of u over section
$u_\pm$	Value of u at $x = x_\pm$
u	Vector of variables
U	Coefficient vector

Latin symbols (continued)

v	y velocity component and mean
$\bar{v}$	Mean of v over section
$v_\pm$	Value of v at $x = x_\pm$
x	Coordinate along line
x_i	Coordinate of point i
$x_\pm$	$x_i - (u \pm c_0)\Delta$

Greek symbols

α	Friction coefficient / Decay constant
β	Dimensionless quantity (eqn. 7)
Δ	Computational time step
Λ	Parameter introduced for characteristics
λ	Dimensionless nonlinear friction coefficient
μ	Coefficient of time in exponent of e
ν_0	Mean kinematic viscosity of the fluid
θ	Dimensionless frictional parameter
ρ_0	Mean fluid density
ω	Angular frequency of waves

Subscripts

R,I	Real and imaginary parts
+/-	Associated with $C_\pm$ characteristics
$0/L$	Mean value in line, or value at left/right end

Other symbols

$O(\Delta^2)$	Landau order symbol "at least of order of"

3 Equations of motion

We consider two different models of an hydraulic line. The first will be based on linearised equations in two dimensions for the motion of a viscous incompressible fluid, the second will be a one-dimensional nonlinear model. We will examine them and relate the two.

3.1 TWO-DIMENSIONAL LINEAR EQUATIONS FOR A VISCOUS COMPRESSIBLE FLUID

Consider the two-dimensional equations of motion of a viscous compressible fluid in cylindrical coordinates:

$$\frac{\partial p}{\partial t} + \rho_0 c_0^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} \right) = 0, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} = \nu_0 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \tag{1.2}$$

where (u, v) are the velocity components in the (x, r) directions, along and transverse to the pipe respectively, p is the pressure, t is time, ρ_0 is the average fluid density, c_0 is the mean speed of sound in the fluid transmission line, and ν_0 is the average kinematic viscosity of the fluid. These equations form one of eight models considered in [1] and [2], from the full Navier-Stokes equation to a simple one-dimensional incompressible linear model, which after testing it was identified as being not too complex but gave very good results in the frequency domain to model the phenomena in long lines.

Now we convert to a single space dimension by integrating over the cross sectional area. For the first terms in equation (1.1) this is trivial, for the terms in radial velocity we have

$$\frac{\partial v}{\partial r} + \frac{v}{r} = \frac{1}{r} \frac{\partial}{\partial r}(vr), \tag{2}$$

and multiplying by $2\pi r$ and integrating from $r = 0$ to $r = a$ where $v = 0$ we obtain zero contribution, leaving for the mass-conservation equation in integrated form

$$\frac{\partial \bar{p}}{\partial t} + \rho_0 c_0^2 \frac{\partial \bar{u}}{\partial x} = 0, \tag{3}$$

where the overbars denote mean quantities over the area of the line. Now considering the momentum equation (1.2), the derivative terms on the right side are

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \tag{4}$$

and multiplying by $2\pi r$ and integrating gives

$$2\pi a \left. \frac{\partial u}{\partial r} \right|_{r=a}, \tag{5}$$

in terms of the velocity gradient at the pipe wall. Hence the mean momentum equation becomes

$$\frac{\partial \bar{u}}{\partial t} + \frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial x} = \frac{2\nu_0}{a} \left. \frac{\partial u}{\partial r} \right|_{r=a}, \tag{6}$$

and if we assume that the velocity gradient at the wall is given by a linear relation in terms of the mean velocity over the section by

$$\left. \frac{\partial u}{\partial r} \right|_{r=a} = -\frac{\beta}{a} \bar{u}, \tag{7}$$

where β is a dimensionless quantity, then the momentum equation becomes

$$\frac{\partial \bar{u}}{\partial t}+\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}+\frac{2\beta\nu_0}{a^2}\bar{u}=0. \tag{8}$$

In the case of steady incompressible Poiseuille flow in a tube, [3, p.181] shows that

$$\left.\frac{\partial u}{\partial r}\right|_{r=a}=-\frac{Ga}{2\rho_0\nu_0}, \tag{9}$$

where G is the negative of the pressure gradient, and also that the volume flux down the tube is

$$Q=\frac{\pi G a^4}{8\rho_0\nu_0}, \tag{10}$$

such that $\bar{u}=Ga^2/8\rho_0\nu_0$, and we can write

$$\left.\frac{\partial u}{\partial r}\right|_{r=a}=-\frac{4}{a}\bar{u}. \tag{11}$$

Comparing with equation (7), we have, for steady incompressible flow,

$$\beta=4. \tag{12}$$

Now if we introduce the symbol

$$\alpha=\frac{2\beta\nu_0}{a^2}=\frac{8\beta\nu_0}{d^2}, \tag{13}$$

where d is the pipe diameter, then, suppressing the overbars for mean quantities across the section, equations (3) and (8) become:

$$\frac{\partial p}{\partial t}+\rho_0 c_0^2\frac{\partial u}{\partial x}=0, \tag{14.1}$$

$$\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\alpha u=0, \tag{14.2}$$

where p is mean pressure across the section, u is the mean velocity in the direction of the pipe, and α is the friction coefficient defined in equation (13). These equations are linear, and provide a framework for examining the behaviour of the line.

3.2 ONE-DIMENSIONAL NONLINEAR EQUATIONS

These equations include nonlinear terms, as well as a nonlinear empirical friction term. They can be written:

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\rho_0 c_0^2\frac{\partial u}{\partial x}=0, \tag{15.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\frac{\lambda}{2d}u|u|=0, \tag{15.2}$$

where λ is a dimensionless coefficient of friction, and where the p and the u are mean quantities across the section.

4 The Telegrapher's equation

Initially in this work we are concerned about examining the nature of the propagation of disturbances in hydraulic lines, and most information will be able to be obtained by using the linear model, equations (14.1) and (14.2). One way of examining the equations is to obtain a single higher-order partial differential equation in a single unknown. This is easily done as the system is linear. We consider equation (14.1) differentiated with respect to t:

$$\frac{\partial^2 p}{\partial t^2}+\rho_0 c_0^2\frac{\partial^2 u}{\partial x\partial t}=0, \tag{16}$$

and equation (14.2) differentiated with respect to x:

$$\frac{\partial^2 u}{\partial x\partial t}+\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2}+\alpha\frac{\partial u}{\partial x}=0. \tag{17}$$

Eliminating $\partial^2 u/\partial x\partial t$ between these equations and eliminating $\partial u/\partial x$ using equation (14.1) gives a single partial differential equation in terms of p:

$$\alpha\frac{\partial p}{\partial t}+\frac{\partial^2 p}{\partial t^2}-c_0^2\frac{\partial^2 p}{\partial x^2}=0. \tag{18}$$

This is one form of the Telegrapher's equation, well known in Electrical Engineering, for which a massive literature exists in studying the effects on the propagation of alternating current on lines with losses. Note that for no friction, $\alpha=0$, we recover the wave equation, with solutions those of waves travelling up and down the pipe at speed c_0.

To examine the behaviour of solutions we assume a solution, periodic in x, of the form

$$p=\exp(ikx+\mu t), \tag{19}$$

such that the solution has a wavelength of $2\pi/k$. Substituting into equation (18) to determine the behaviour in time gives

$$\mu^2+\alpha\mu+k^2c_0^2=0, \tag{20}$$

with solutions

$$\mu_\pm=-\frac{\alpha}{2}\pm\sqrt{\frac{\alpha^2}{4}-k^2c_0^2}\,. \tag{21}$$

In fact, if we introduce the angular frequency of the waves,

$$\omega=kc_0, \tag{22}$$

and extract a factor of -1 from under the square root, the expression can be written

$$\mu_\pm=-\frac{\alpha}{2}\pm i\omega\sqrt{1-\frac{\alpha^2}{4\omega^2}}, \tag{23}$$

where $i=\sqrt{-1}$. The physical significance of this will be discussed below. It is useful here to introduce the dimensionless frictional parameter θ:

$$\theta=\frac{\alpha}{2kc_0}=\frac{\alpha}{2\omega}, \tag{24}$$

expressing the ratio between two quantities with an inverse time scale, the decay constant to the radian frequency. If we adopt the steady incompressible Poiseuille result (12) that $\beta=4$, then equation (13) gives

$$\alpha=\frac{32\nu_0}{d^2}, \tag{25}$$

and

$$\theta=\frac{16\nu_0}{d^2\omega}. \tag{26}$$

With this symbol θ equation (23) for the eigenvalues can be written in the more physical form

$$\mu_{\pm} = -\frac{\alpha}{2} \pm i k c_0 \sqrt{1-\theta^2} \,. \tag{27}$$

Small friction: For the small friction case, $\alpha < 2kc_0$, $\theta < 1$, if we write $\mu = \mu_R + i\mu_I$, then the real and imaginary parts are given by

$$\mu_R = -\frac{\alpha}{2}, \tag{28.1}$$

$$\mu_I = \pm k c_0 \sqrt{1-\theta^2} \,, \tag{28.2}$$

showing that the waves decay exponentially with a rate constant $\alpha/2$, and they propagate up and down the channel with propagation speed $C = \mu_I / k$:

$$C = c_0 \sqrt{1 - \left(\frac{\alpha}{2\omega}\right)^2} \,, \tag{29}$$

thus showing that the actual speed of propagation is not c_0 but is that quantity modified by a frictional term which also depends on the frequency of the disturbances, so that the waves exhibit the phenomenon of dispersion, where their speed depends on their length. In this case, the longer waves (ω smaller) will have a smaller speed of propagation, unlike surface gravity waves on still water. The dependence of C/c_0 on the friction parameter is shown in Figure 1.

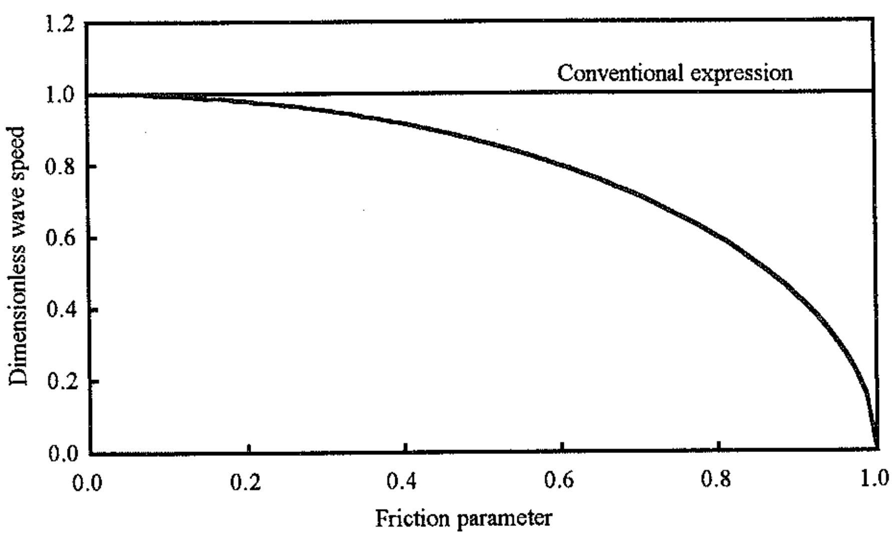

Figure 1. Dependence of wave speed C/c_0 on friction parameter α.

Large friction: For the large friction case, $\alpha > 2kc_0$, $\theta > 1$ and the real and imaginary parts are given by

$$\mu_R = -\frac{\alpha}{2} \pm \frac{\alpha}{2}\sqrt{1 - \frac{1}{\theta^2}} \,, \tag{30.1}$$

$$\mu_I = 0, \tag{30.2}$$

which is a dramatic result, that there is no imaginary part. This means that in equation (19), the expression for pressure, the coefficient of t is real so that any disturbance simply decays exponentially with rate constant given by the two values of μ_R and no waves propagate. It is easily shown that both values of μ_R are negative, and the waves are stable and do not grow with time.

Here we examine the physical conditions corresponding to the limit $\theta = 1$, that is, $\alpha/2\omega = 1$. If we adopt the case for Poiseuille flow $\beta = 4$, equation (26) gives the limiting condition that waves propagate:

$$\frac{16\nu_0}{d^2\omega} \leq 1. \tag{31}$$

If we consider the case of a water line with $\nu_0 = 0.01\ \mathrm{cm^2/s}$, $d = 1.27$ cm and a frequency of 1 Hz, $\theta = 0.016$, and the effects of friction on the propagation speed will be small. For oil of viscosity 30 times that of water and the same other parameters, we obtain $\theta = 0.5$, and the propagation speed will be substantially affected, such that $C/c_0 = 0.87$. If a complex signal being transmitted were Fourier analysed, the lowest frequency might easily be smaller than this and the propagation speed rather more profoundly affected.

5 Matrix formulation of the linear equations

Another way which can be used to examine the nature of solutions of the linear equations is to consider a matrix formulation. The equations (14.1) and (14.2) may be written in matrix form as

$$\frac{\partial}{\partial t}\begin{bmatrix} p \\ u \end{bmatrix} + \begin{bmatrix} 0 & \rho_0 c_0^2 \\ 1/\rho_0 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} p \\ u \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} p \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{32}$$

We write this as the matrix equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathrm{D}\frac{\partial \mathbf{u}}{\partial x} + \mathrm{E}\,\mathbf{u} = \mathbf{0}. \tag{33}$$

The conventional interpretation of the operator $\partial \mathbf{u}/\partial t + \mathrm{D}\partial \mathbf{u}/\partial x$ is that the eigenvalues of D give the velocity of propagation of disturbances. In fact, it is easily shown that the eigenvalues of D are $\pm c_0$, suggesting that the velocity of disturbances relative to the flow is this conventional expression. It is our assertion that the presence of the third and additional homogeneous term $\mathrm{E}\,\mathbf{u}$ in equation (33), traditionally ignored in considerations of the nature of such second-order systems, not only changes the nature of solutions into diffusive and dispersive solutions, but also changes the fundamental velocities at which disturbances propagate.

To demonstrate this we assume a solution of the form $\mathbf{u} = \mathbf{U}\exp(ikx + \mu t)$. Substituting into the matrix differential equation we obtain

$$(\mu I + ik\mathrm{D} + \mathrm{E})\,\mathbf{U}\exp(ikx + \mu t) = \mathbf{0}. \tag{34}$$

Hence,

$$(\mu I + ik\mathrm{D} + \mathrm{E})\,\mathbf{U} = \mathbf{0}, \tag{35}$$

showing that the μ are the eigenvalues of the matrix $-ik\mathrm{D}-\mathrm{E}$ with corresponding eigenvectors $\mathbf{U}$. Performing the operations we obtain for the eigenvalues:

$$\mu_{\pm} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - k^2 c_0^2}\,, \tag{36}$$

exactly the same as equation (21) obtained previously using the Telegrapher's equation, and with the same deductions as to the real and imaginary parts, the decay rate constants and the speeds of propagation. Here, however, we can extract more information which we can put to effect in deriving a computational scheme. The eigenvectors corresponding to the eigenvalues are

$$\begin{bmatrix} -i\rho_0 c_0^2 k \\ \mu_\pm \end{bmatrix}. \tag{37}$$

Hence the general solution of the differential equation is

$$\mathbf{u}(x,t) = H_1 \begin{bmatrix} -i\rho_0 c_0^2 k \\ \mu_+ \end{bmatrix} e^{ikx+\mu_+ t} + H_2 \begin{bmatrix} -i\rho_0 c_0^2 k \\ \mu_- \end{bmatrix} e^{ikx+\mu_- t}, \tag{38}$$

from which solution we extract

$$p = -i\rho_0 c_0^2 k (H_1 e^{ikx+\mu_+ t} + H_2 e^{ikx+\mu_- t}), \text{ and} \tag{38.1}$$

$$u = H_1 \mu_+ e^{ikx+\mu_+ t} + H_2 \mu_- e^{ikx+\mu_- t}. \tag{38.2}$$

It is possible that a computational scheme could be devised which could exploit this knowledge of the precise behaviour of the solution. However in the following section we exploit the very simple nature of the equations when interpreted in a characteristic sense to develop a scheme of high accuracy.

6 Numerical scheme

6.1 CHARACTERISTIC FORMULATION

Consider the nonlinear equations (15) written in the form

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho_0 c_0^2 \frac{\partial u}{\partial x} = 0, \tag{15.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} + \alpha(u)u = 0, \tag{15.2}$$

where we have written the frictional coefficient as

$$\alpha(u) = \lambda |u| / 2d, \tag{39}$$

following the notation from the linear friction case, equation (14.2), where α was a constant. Using this notation we hope to be able to consider both models. Now, multiplying equation (15.2) by a parameter Λ and adding to equation (15.1), the combined equation can be written in the form

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \frac{\Lambda}{\rho_0}\frac{\partial p}{\partial x} + \Lambda\left(\frac{\partial u}{\partial t} + \frac{\rho_0 c_0^2}{\Lambda}\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \alpha(u)u\right) = 0, \tag{40}$$

and if we write the coefficients of the x derivatives in both terms as dx/dt, such that

$$\frac{dx}{dt} = u + \frac{\Lambda}{\rho_0} = u + \frac{\rho_0 c_0^2}{\Lambda}, \tag{41}$$

then we can solve for Λ giving $\Lambda = \pm\rho_0 c_0$ and hence $dx/dt = u \pm c_0$. Equation (40) can then be written

$$\frac{\partial p}{\partial t} + \frac{dx}{dt}\frac{\partial p}{\partial x} \pm \rho_0 c_0 \left(\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} + \alpha(u)u\right) = 0,$$

such that on the characteristics, which are paths given by solutions of the differential equation

$$\frac{dx}{dt} = u \pm c_0, \tag{42.1}$$

the differential equation holds:

$$\frac{dp}{dt} \pm \rho_0 c_0 \left(\frac{du}{dt} + \alpha(u) u \right) = 0. \tag{42.2}$$

In this way we have converted the two partial differential equations into the four ordinary differential equations given by both corresponding sign alternatives in equations (42.1) and (42.2). In the case of the linear approximation, equation (14), these equations are easily shown to be

$$\frac{dx}{dt} = \pm c_0 \quad \text{and} \quad \frac{dp}{dt} \pm \rho_0 c_0 \left(\frac{du}{dt} + \alpha u \right) = 0. \tag{43}$$

As the two forms are so clearly related we will retain the nonlinear form for our discussion and whenever the linear form is required it will be a simple matter of deleting the leading u from the gradient of the characteristic and of using a constant value of α.

6.2 NUMERICAL APPROXIMATION OF CHARACTERISTIC FORMULATION

As the quantity u on the right side of equation (42.1) is a function of both x and t, we cannot obtain exact solutions in general, however as $|u| << c_0$ a convenient and good approximation is that u is constant on a characteristic, such that it is a straight line. This is exact for the linear case. We denote characteristics on which $dx/dt = u \pm c_0$ by $C_\pm$ respectively. Figure 2 shows the arrangement on the (x, t) plane. At a computational point $(x_i, t+\Delta)$ at which we require the updated values of p and u, consider the two characteristics approximated by straight lines meeting at that point. They intersect the line corresponding to time level t at $x_\pm = x_i - (u \pm c_0)\Delta$ respectively, where the value of u is that at (x_i, t).

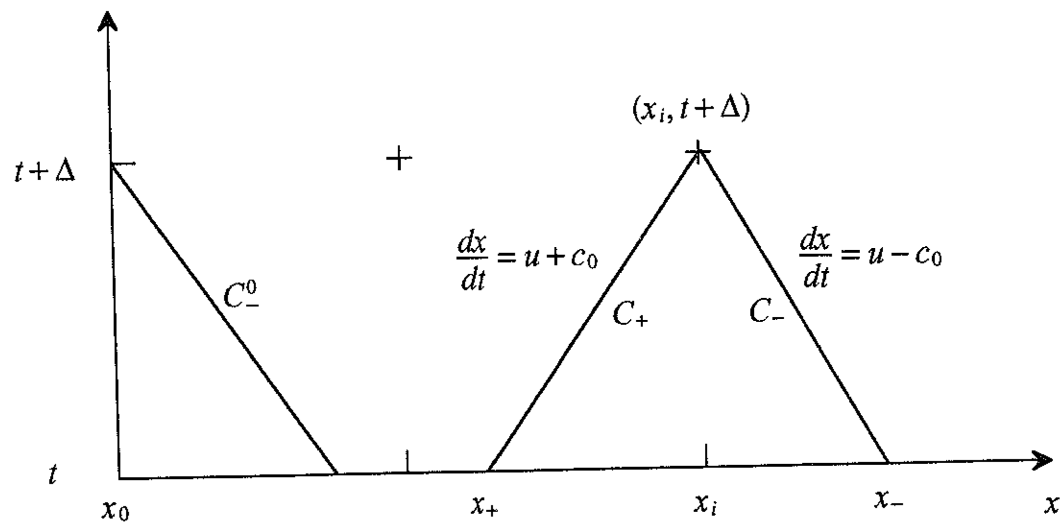

Figure 2. Computational points and typical characteristics

Now consider equation (42.2). It would be possible to approximate the derivatives by forward difference expressions, which have errors $O(\Delta)$, however this is not as accurate as the scheme which we propose here. Integrating the equation with respect to time gives

$$p \pm \rho_0 c_0 \left(u + \int_{C_\pm} \alpha(u) u \, dt \right) = \text{constant}, \tag{44}$$

and now we approximate the integral by the trapezoidal rule, which has error $O(\Delta^2)$, so that we obtain:

$$p_i(t+\Delta) \pm \rho_0 c_0\left(u_i(t+\Delta) + \frac{\Delta}{2}(\alpha(u_\pm)u_\pm + \alpha(u_i(t+\Delta))\,u_i(t+\Delta))\right) = p_\pm \pm \rho_0 c_0 u_\pm, \quad (45)$$

where we have used the notation $p_i(t+\Delta) = p(x_i, t+\Delta)$, $p_\pm = p(x_\pm, t)$ and the same notation for u. By adding the two equations (+ and - cases) we obtain an explicit expression for the updated value of p which we require:

$$p_i(t+\Delta) = \frac{(p_+ + p_-)}{2} + \frac{\rho_0 c_0(u_+ - u_-)}{2} - \rho_0 c_0 \frac{\Delta}{4}(\alpha(u_+)u_+ - \alpha(u_-)u_-)\ , \quad (46)$$

and by subtracting the two expressions we obtain an expression for the updated value of u:

$$u_i(t+\Delta) = \frac{(u_+ + u_-) + (p_+ - p_-)/\rho_0 c_0 - \Delta(\alpha(u_+)u_+ + \alpha(u_-)u_-)/2}{2 + \Delta\alpha(u_i(t+\Delta))}\ . \quad (47)$$

This equation is not as simple to implement, as in the nonlinear case the unknown $u_i(t+\Delta)$ appears in the denominator on the right side. In the linear case, α is constant, and the equation can be evaluated explicitly. In the nonlinear case, the equation could be solved as a quadratic or it could be solved iteratively using equation (47). As the term is small anyway, and the coefficient of friction is only approximately known it might be most appropriate simply to use another velocity in the denominator, for example $(u_+ + u_-)/2$.

6.3 IMPLEMENTATION USING SPLINE APPROXIMATION

To implement the scheme, all that is required is to obtain the values of u and p at points $x_\pm = x_i - (u \pm c_0)\Delta$ intermediate between the computational points, so that the problem is one of interpolation. In fact, any convenient local interpolation method could be used, and the implementation of the method would be very simple. However, given the problem which we have here, of knowing the values at a number of computational points, rather more accurate methods can be used. One option is to use spline interpolation, which has a number of advantages over conventional piecewise polynomial methods ([4] for example). The error of spline interpolation is fourth order, proportional to the fourth power of the maximum spacing between interpolation points, and is considerably less than conventional local polynomial methods.

In [5], solving equations similar to those of the present work, it was shown that a convenient way of implementing the interpolation is to use exponential splines ([6]) which have the advantages of cubic splines, but which have the added advantage that there is a free parameter which varies the degree of stiffness of the approximation and which reduces to cubic splines in one limit. For smoothly varying functions, that is the limit to use, but if, as in the present case of pressure pulses in an hydraulic line, there are some very rapid transients, then the exponential splines can describe these with very few extraneous oscillations, such as conventional methods would obtain.

6.4 BOUNDARY CONDITIONS

These can be treated in an obvious way using the characteristics formulation. For example, considering equation (45), at $x = x_0$, the left boundary on Figure 2, the characteristic C_-^0 can be used. As it is a C_- characteristic, we choose the lower of the alternatives in equation (45), with the negative signs. Then, suppose that the boundary condition at x_0 can be expressed as a function connecting p and u at any time t:

$$F_0(p_0(t), u_0(t), t) = 0, \tag{48}$$

then to obtain the updated solution we solve the pair of equations consisting of the lower alternative of equation (45) and the equation

$$F_0(p_0(t+\Delta), u_0(t+\Delta), t+\Delta) = 0. \tag{49}$$

If either $p_0(t+\Delta)$ or $u_0(t+\Delta)$ are specified explicitly, of course, the procedure is simpler. Similarly, to implement the boundary at the right end, one would solve the upper equation in (45) together with an equation similar to (49) for the pressure and velocity at the end $x=L$:

$$F_L(p_L(t+\Delta), u_L(t+\Delta), t+\Delta) = 0. \tag{50}$$

7 Results

To test the method some simple but demanding tests were performed. A single line was considered with the pressure at one end being instantaneously raised to a constant value and kept there, with a fully-reflecting boundary condition at the other end, such that the velocity is zero there and the wave should be fully reflected. The instantaneous nature of the initial transient is what makes this otherwise simple problem somewhat demanding. The computational parameters are shown in Table 1. The linear model was used. Each run on a 486-based personal computer took about 64 seconds to perform the basic calculations and about 10 seconds more to produce the hidden line plot.

	Test 1	Test 2
Length of line	12000m	
ρ_0	1000 kg/m^3	
c_0	1414 m/s	
α	0	0.2 s^{-1}
Δ	0.05 s	
Number of computational points	140	
Total number of time steps	400	
Intermit for hidden line plot	10	
Weight for exponential splines	2.00	4.00

Table 1. Computational parameters used in tests

Test 1 was without friction, to examine the ability of the computational scheme to handle an abrupt change. Results are shown in Figure 3 each line showing the pressure in the hydraulic line, successive lines at later times, where a hidden line plot has been used. It can be seen that the proposed method performs this demanding computation well, and the abrupt shock moves up and down the line with little numerical diffusion. There are wiggles at the bottom of the shock. These can be reduced by using a larger weight for the exponential splines, which, however, introduces more numerical diffusion. What is noteworthy is the ability of the scheme to propagate the shock with almost no trailing waves at all. Somehow the scheme allows the shock to pass through a computational panel and is sufficiently accurate that after it has passed, the pressure settles down to the precise value.

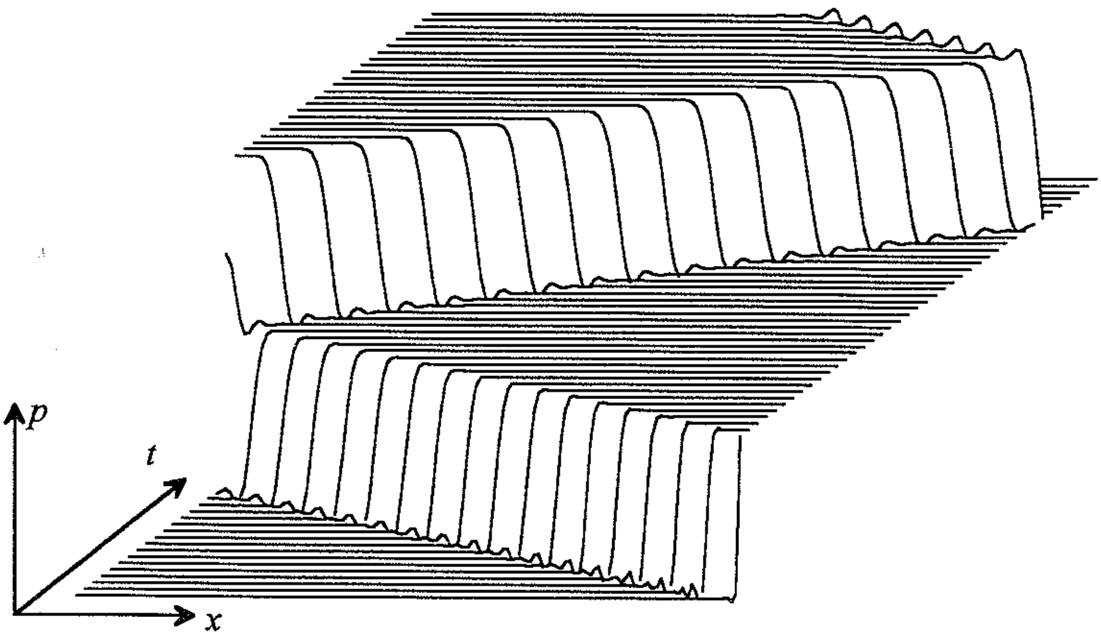

Figure 3. Instantaneous pressure surge travelling down a line with full reflection at end; no friction

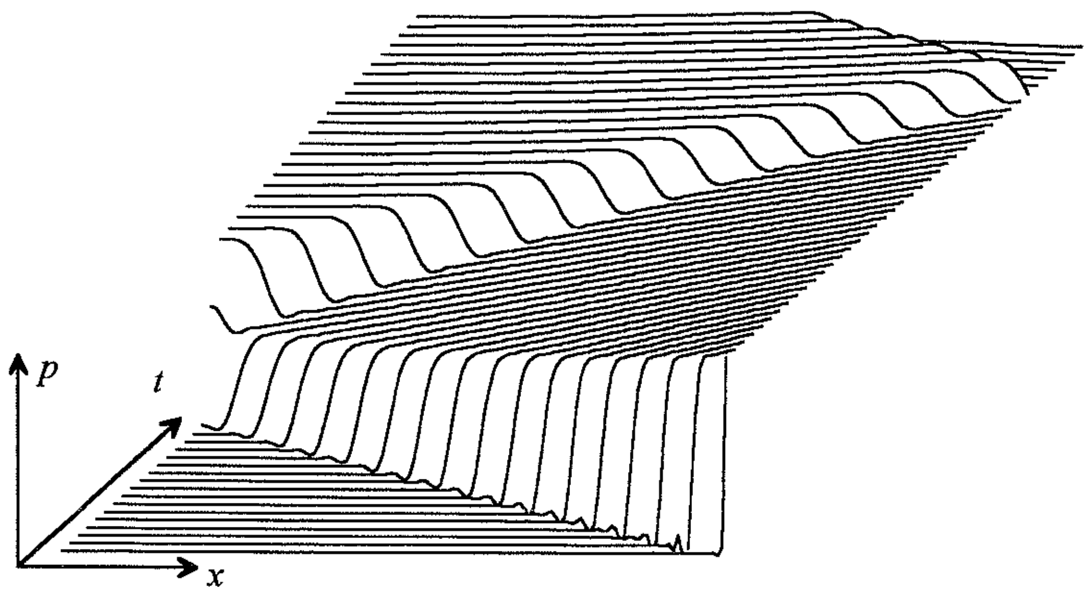

Figure 4. Instantaneous pressure surge travelling down a line with full reflection at end; finite friction $\alpha = 0.2\ s^{-1}$

The only changes in Test 2 were that a finite value of α was introduced and a different value of the exponential spline weight was used. Results are shown in Figure 4. Comparing the two figures it is clear that the effect of friction in dampening the motion is much more important than the effect which it has on the propagation speed of the front. The wavelengths associated with the sharp front are all very short (corresponding to some kind of Gibbs' phenomenon in describing a discontinuity with periodic functions), hence the frequencies associated with the front are large, and as the expression for the propagation speed of disturbances equation (29) contains the frequency in the denominator, the effect on propagation speed is relatively minor.

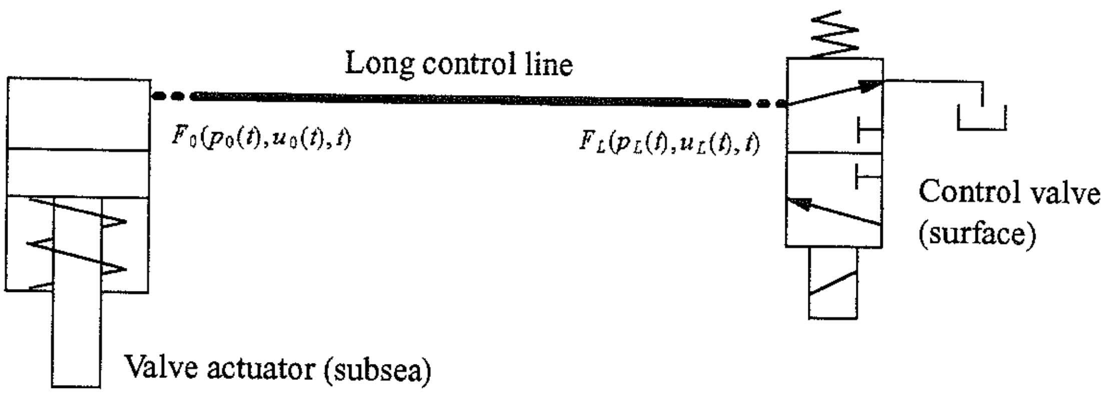

Figure 5. Diagram of control line with valves and actuators providing boundary conditions

In application to the modelling of subsea systems, the long transmission line model developed above will be combined with lumped parameter models of valves and actuators placed at the subsea on the Christmas Tree and control components located topside on the platform. The lumped parameter models will be interfaced with the transmission line by means of the end functions $F_0()$ and $F_L()$ which represent any linear or nonlinear combination of pressures and flows at the points of entry and exit, as shown in Figure 5.

8 Conclusions

In this paper we have considered the equations in physical variables including time, rather than using a Laplace transform formulation.

1. It has been shown that the linear two-dimensional equations governing the motion of a viscous incompressible fluid in an hydraulic line may be integrated to give the commonly used one-dimensional equations, with relatively few assumptions as to the velocity distribution in the line.
2. It is then shown how the linear equations may have one variable eliminated, giving a version of the Telegrapher's Equation.
3. Elementary solutions of that equation have been obtained, throwing some light on the nature of wave propagation in hydraulic lines, and showing that waves which propagate along the lines show diffusion and, possibly unexpectedly, that they also show dispersion, whereby the propagation speed depends on the wavelength of the disturbance, and longer waves propagate more slowly than shorter waves. For practical values of physical

parameters this effect will be small, but in long lines the effect may be finite, and in any case, one should be aware of the existence of the phenomenon.

4. A numerical scheme has been developed, for both linear and nonlinear formulations of the problem, which is based on a specified interval characteristic formulation. When combined with spline interpolation along the line, the resulting physical space method is fast, accurate and relatively simple, even for shock wave problems.

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WRITTEN DISCUSSION
Subsea control systems - on the nature of wave propagation in long hydraulic transmission lines
JD Fenton, JS Stecki

Question: **KA Edge**
University of Bath, UK

In section 4 of your paper you examine the large friction case and state that if the friction is sufficiently high then "no waves propagate". I have problems visualising this situation: could you explain what would happen say in the case of a piston being used to pressurise a closed-ended pipe of finite length, filled with a high viscosity fluid?

Answer:

This is an interesting question, which would be answered more completely by detailed computations. In a spirit of brevity here, we note that in the case where no waves propagate the behaviour of the solution will show decay and diffusive behaviour only. However, for most problems this will be for the longer wavelengths with low frequencies, to satisfy the inequality $\alpha > 2kc_0$. For a generalised disturbance in which there are many (Fourier) components present, the longer wavelength components will show this decay/diffusive behaviour. The shorter wavelengths will tend to violate the inequality and will show some travelling wave behaviour. The more abrupt the driving behaviour of the piston mentioned in the question, the greater the number of components necessary to describe the motion and the greater will be the travelling wave effect.

Design and Performance

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