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Nonlinear Wave Theories

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1. Introduction

The general case of water wave motion is where disturbances propagate in varying directions interacting nonlinearly, over water of possible non-uniform density which might be flowing on a shear current, over varying permeable or deformable topography. It is not possible to solve this general problem analytically. A convenient set of approximations is to assume that locally at least the bed is impermeable and flat, that the propagation of disturbances is collinear and they are of infinite length transverse to the direction of propagation such that the flow is two-dimensional, that the fluid is homogeneous and incompressible, and that the boundary layer is small such that inviscid flow theory can be used. Under these approximations it is possible to obtain analytical solutions which correspond to a single periodic wave train which propagates steadily without change of form. This is the steady wave problem, and a great deal of attention has been given to it as it has been considered to be an important and convenient model for the more general one. A recent detailed presentation of common methods has been given by Sobey *et al.* (1987).

It is possible to obtain nonlinear solutions for waves on shear flows for special cases of the vorticity distribution. For waves on a constant shear flow, Kishida and Sobey (1988), calculated a third-order solution, while Dalrymple (1974), used a numerical method. It is conceivable that in future such solutions will become more important. However, solution for more general shear flows is difficult. In many cases the details of the shear flow are not known, and the irrotational model seems to be adequate for many situations in the absence of any other information. In this article theories presented will be limited to those assuming that the flow is irrotational.

There are two main theories for steady waves which are capable of refinement. The first is Stokes theory, most suitable for waves which are not very long relative to the water depth. The second is Cnoidal theory, suitable for the other limit where the waves are long. Both theories are presented in the following sections. It should be noted that nowhere in the text of this article are the terms "Stokes wave" or "cnoidal wave" used. In each of the two theories, the waves which being described are steady waves of progression, and there is no fundamental difference between the waves themselves.

A new version of cnoidal theory is presented here, which is rather simpler to apply and which gives very much better results for fluid velocities than previous cnoidal theories. Both Stokes and cnoidal theories are presented to fifth order, which is a reasonable compromise; the accuracy is shown to be good, yet the amount of data presented in the form of numerical coefficients is not unreasonable. It is recommended that the full fifth-order theory be used in each case, as once the organisation of first-order calculations in a systematic manner has been accomplished, the extension to fifth-order involves little more effort.

The sections on the two main wave theories are followed by a description of accurate numerical methods based on Fourier approximation. These methods are capable of high accuracy over almost the whole range of wave heights and lengths. The accuracy of each of the approximate theories is then examined, in the course of which some new results are obtained. It is suggested that the two fifth-order theories jointly are capable of acceptable accuracy over most of the range of possible waves. An empirical formula for the height of the highest waves is presented and a boundary between the regions of validity of the theories is proposed. Finally a set of formulae is presented for integral properties of the wave train. These contain the results of recent work which correct previous theory.

2. Steady waves: the effects of current and the governing equations

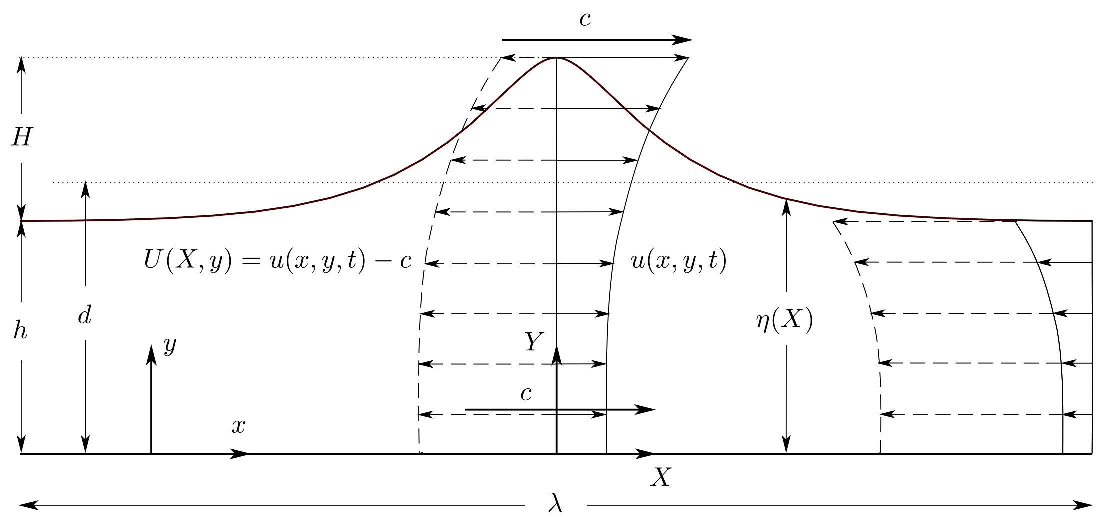

Figure 1. One wave of a steady train, showing principal dimensions and co-ordinates. Fluid velocities are shown under crest and trough, solid lines show those in the stationary frame, dashed lines show those in the frame of the wave.

Three physical dimensions uniquely define a wave train: the mean depth d, the wave crest-to-trough height H, and wavelength λ. Many presentations of theory have assumed that the wave period can replace the wave length as the third parameter identifying a wave train. However in most marine situations waves travel on a finite current which is determined by oceanographic and topographic factors. The wave speed relative to an observer depends on the current, such that waves travel faster with the current than against it. Contrary to the implicit assumptions of most presentations of steady wave theory, no theory can predict the actual wave speed. What the theories do predict, however, is the speed of the waves relative to the current.

The existence of a current has two main implications for the application of a steady wave theory. Firstly, the apparent period measured by an observer depends on the actual wave speed and hence on the current, that is, the apparent period is Doppler-shifted. This means that without explicit allowance for the current in calculations it is not possible to solve the problem uniquely, if, as in many applications, the period is known instead of the wave length. The second main effect of current is the additive effect it has on horizontal fluid velocities. To determine these velocities it is necessary to know the current. If the current is not known, then the problem is under-specified.

Consider the wave as shown in Figure 1, with a frame of reference (x, y), x in the direction of propagation of the waves and y vertically upwards with the origin on the flat bed. The waves travel in the x direction at speed c relative to this frame. It is this stationary frame which is the usual one of interest for engineering and geophysical applications. Consider also a frame of reference (X, Y) moving with the waves at velocity c, such that $x = X + ct$, where t is time, and $y = Y$. The fluid velocity in the (x, y) frame is (u, v), and that in the (X, Y) frame is (U, V). The velocities are related by $u = U + c$ and $v = V$.

In the (X, Y) frame all fluid motion is steady, and consists of a flow in the negative X direction, roughly of the magnitude of the wave speed, underneath the stationary wave profile. The mean horizontal fluid velocity in this frame, for a constant value of Y over one wavelength λ is denoted by $-\bar{U}$. It is negative because the apparent flow is in the $-X$ direction. The velocities in this frame are usually not important, they are used to obtain the solution rather more simply.

In the stationary frame of reference the time-mean horizontal fluid velocity at any point is denoted by $\bar{u}_1$, the mean current which a stationary meter would measure. (In some recent papers by the author the symbol c_E was used for this quantity. The symbol adopted here would seem to be better, as it is a mean of u velocities, and the subscript 1 is appropriate as it is related to Stokes' first approximation to wave speed.) Relating the velocities in the two co-ordinate systems gives

$$\bar{u}_1 = c - \bar{U}. \tag{1}$$

If $\bar{u}_1 = 0$ then $c = \bar{U}$, so that in this special case the wave speed is equal to $\bar{U}$. This is Stokes' first approximation to the wave speed, usually incorrectly referred to as his "first definition of wave speed", and is that relative to a frame in which the current is zero. Most wave theories have presented an expression for $\bar{U}$, obtained from its definition as a mean fluid speed. It has often been referred to, incorrectly, as "the wave speed".

A second type of mean fluid speed is the depth-integrated mean speed of the fluid under the waves in the frame in which motion is steady. If Q is the volume flow rate per unit span underneath the waves in the (X, Y) frame, the depth-averaged mean fluid velocity is $-Q/d$, where d is the mean depth. In the physical (x, y) frame, the depth-averaged mean fluid velocity, the "mass-transport velocity", is $\bar{u}_2$, given by

$$\bar{u}_2 = c - Q/d. \tag{2}$$

If there is no mass transport, $\bar{u}_2 = 0$, then Stokes' second approximation to the wave speed is obtained: $c = Q/d$. Most theoretical presentations give Q as a function of wave parameters.

In general, neither of Stokes' first or second approximations is the actual wave speed, and in fact the waves can travel at any speed. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed.

Now we consider the governing equations in the (X, Y) co-ordinate system. If the fluid is incompressible, a stream function ψ exists such that velocity components (U, V) in the (X, Y) co-ordinate system are given by $U = \partial\psi/\partial Y$ and $V = -\partial\psi/\partial X$. If in addition the fluid motion is irrotational, ψ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} = 0. \tag{3}$$

The boundary conditions to be satisfied are that

$$\psi(X, 0) = 0, \tag{4}$$

the condition that no flow pass through the bottom $y = 0$; and, that

$$\psi(X, \eta(X)) = -Q. \tag{5}$$

This equation expresses that the free surface $y = \eta(X)$ is also a streamline. Also on the free surface the pressure is constant, so that Bernoulli's equation gives

$$\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial X}\right)^2 + \left(\frac{\partial\psi}{\partial Y}\right)^2\right]_{y=\eta(X)} + g\eta(X) = R, \tag{6}$$

where g is the gravitational acceleration and R is a positive constant. It is the nonlinear Eqs. (5) and (6) which render solution of the problem difficult. Now, the two main theories will be described.

3. Stokes theory

Stokes assumed that all variation in the X direction can be represented by Fourier series and that the coefficients in these series can be written as perturbation expansions in terms of a parameter which increases with wave height. Substitution of the high order perturbation expansions into the governing Eqs. (3) - (6) and manipulation of the series yields the solution. A recent presentation of Stokes theory, retaining terms to fifth order, is that of Fenton (1985). The following description follows that paper, with the notational difference described above, that $\bar{u}_1$ replaces c_E and $\bar{u}_2$ replaces c_S.

For the wave height parameter it is most convenient to use the wave height non-dimensionalised with respect to the wave length in the form $\varepsilon = kH/2$, where k is the wavenumber $k = 2\pi/\lambda$. The theory can be presented completely in terms of this quantity and the dimensionless depth kd. Formulae for the coefficients used in the theory are presented in Table 1.

If the three dimensions, water depth d, wave height H and wave length λ are known, then all the coefficients can be calculated. However if unsteady fluid velocities are to be calculated it is still necessary to know the wave speed c or either of the mean velocities $\bar{u}_1$ or $\bar{u}_2$.

In the more usual case that the wave period τ is known instead of the wave length, then it is also necessary to know c or one of the determining values of current, $\bar{u}_1$ or $\bar{u}_2$. The first step is to determine the wave length. Stokes theory provides an equation for $\bar{U}$ as a function of H, d and λ:

$$\bar{U}(k/g)^{1/2} = C_0 + \varepsilon^2 C_2 + \varepsilon^4 C_4 + \dots\,, \tag{7}$$

where the functional dependence of the coefficients C_0, C_2 and C_4 on kd are given in Table 1. Substituting Eq. (7) and the definition of wave speed $c = \lambda/\tau$ into Eq. (1) and re-writing the equation in terms of the wavenumber throughout, we obtain

$$\left(\frac{k}{g}\right)^{1/2}\bar{u}_1 - \frac{2\pi}{\tau(gk)^{1/2}} + C_0(kd) + \left(\frac{kH}{2}\right)^2 C_2(kd) + \left(\frac{kH}{2}\right)^4 C_4(kd) + \ldots = 0, \tag{8}$$

which is a nonlinear transcendental equation for the wavenumber k, provided d, H, τ and $\bar{u}_1$ are all known.

If $\bar{u}_2$ is known instead of $\bar{u}_1$, then the equation

$$\bar{u}_2 = c - \frac{Q}{d} \tag{9}$$

can be used, together with the expression for Q from the theory:

$$Q(k^3/g)^{1/2} = C_0\, kd + \varepsilon^2(C_2\, kd + D_2) + \varepsilon^4(C_4\, kd + D_4) + \ldots, \tag{10}$$

to give a transcendental equation for k, similar to Eq. (e5):

$$\begin{aligned}\left(\frac{k}{g}\right)^{1/2}\bar{u}_2 - \frac{2\pi}{\tau(gk)^{1/2}} + C_0(kd) + \left(\frac{kH}{2}\right)^2\left(C_2(kd) + \frac{D_2(kd)}{kd}\right)& \\ + \left(\frac{kH}{2}\right)^4\left(C_4(kd) + \frac{D_4(kd)}{kd}\right) + \ldots \quad &= \quad 0.\end{aligned} \tag{11}$$

$$
\begin{aligned}
A_{11} &= 1/\sinh kd \\
A_{22} &= 3S^2/(2(1-S)^2) \\
A_{31} &= (-4-20S+10S^2-13S^3)/(8\sinh kd\,(1-S)^3) \\
A_{33} &= (-2S^2+11S^3)/(8\sinh kd\,(1-S)^3) \\
A_{42} &= (12S-14S^2-264S^3-45S^4-13S^5)/(24(1-S)^5) \\
A_{44} &= (10S^3-174S^4+291S^5+278S^6)/(48(3+2S)(1-S)^5) \\
A_{51} &= (-1184+32S+13232S^2+21712S^3+20940S^4+12554S^5-500S^6-3341S^7-670S^8) \\
&\quad /(64\sinh kd\,(3+2S)(4+S)(1-S)^6) \\
A_{53} &= (4S+105S^2+198S^3-1376S^4-1302S^5-117S^6+58S^7)/(32\sinh kd\,(3+2S)(1-S)^6) \\
A_{55} &= (-6S^3+272S^4-1552S^5+852S^6+2029S^7+430S^8)/(64\sinh kd\,(3+2S)(4+S)(1-S)^6) \\
B_{11} &= 1 \\
B_{22} &= \coth kd\,(1+2S)/(2(1-S)) \\
B_{31} &= -3(1+3S+3S^2+2S^3)/(8(1-S)^3) \\
B_{33} &= -B_{31} \\
B_{42} &= \coth kd\,(6-26S-182S^2-204S^3-25S^4+26S^5)/(6(3+2S)(1-S)^4) \\
B_{44} &= \coth kd\,(24+92S+122S^2+66S^3+67S^4+34S^5)/(24(3+2S)(1-S)^4) \\
B_{51} &= -(B_{53}+B_{55}) \\
B_{53} &= 9(132+17S-2216S^2-5897S^3-6292S^4-2687S^5+194S^6+467S^7+82S^8) \\
&\quad /(128(3+2S)(4+S)(1-S)^6) \\
B_{55} &= 5(300+1579S+3176S^2+2949S^3+1188S^4+675S^5+1326S^6+827S^7+130S^8) \\
&\quad /(384(3+2S)(4+S)(1-S)^6) \\
C_0 &= (\tanh kd)^{1/2} \\
C_2 &= (\tanh kd)^{1/2}\,(2+7S^2)/(4(1-S)^2) \\
C_4 &= (\tanh kd)^{1/2}\,(4+32S-116S^2-400S^3-71S^4+146S^5)/(32(1-S)^5) \\
D_2 &= -(\coth kd)^{1/2}/2 \\
D_4 &= (\coth kd)^{1/2}\,(2+4S+S^2+2S^3)/(8(1-S)^3) \\
E_2 &= \tanh kd\,(2+2S+5S^2)/(4(1-S)^2) \\
E_4 &= \tanh kd\,(8+12S-152S^2-308S^3-42S^4+77S^5)/(32(1-S)^5)
\end{aligned}
$$

Table 1. Coefficients used in Stokes theory in terms of hyperbolic functions of kd, including $S = \operatorname{sech} 2kd$

Eqs. (8) and (11) show that if neither $\bar{u}_1$ nor $\bar{u}_2$ is known, then properly no value of k can be found. In some cases in practice, however, it might be sufficiently accurate to assume that either $\bar{u}_1$ or $\bar{u}_2$ is zero in the absence of any other information. While this might give a value of k which is proximately correct, subsequent calculations of unsteady fluid velocities, where a knowledge of the current is necessary, will in general be in error.

The wavenumber k can be found numerically from Eq. (8) or Eq. (11) by a numerical method for solving transcendental equations, such as trial and error, the secant method or bisection. The bisection method, which is very simple to program, requires only an upper and lower bound. Other methods, however, require an initial estimate of the solution. Fenton and McKee (1989), have given a convenient approximation to the familiar linearised version of Eq. (8), in which the effects of current are ignored:

$$\frac{\sigma^2}{gk} = \tanh kd, \tag{12}$$

where the angular frequency $\sigma = 2\pi/\tau$. Their approximation is

$$k \approx \frac{\sigma^2}{g}\left(\coth\left(\sigma\sqrt{d/g}\right)^{3/2}\right)^{2/3}. \tag{13}$$

This expression is an exact solution of Eq. (12) in the limits of long and short waves, and between those limits its greatest error is 1.5%. Such accuracy is adequate if one is solving Eq. (12), which ignores effects of both currents and nonlinearities.

Having calculated the wavenumber k, and hence the dimensionless wave height $\varepsilon = kH/2$ and dimensionless depth kd, the theory may now be applied. If the wave speed c is not known it can be calculated using Eqs. (1) or (2), depending on whether $\bar{u}_1$ or $\bar{u}_2$ is known:

$$c = \bar{u}_1 + \bar{U} \qquad \text{or} \qquad c = \bar{u}_2 + Q/d, \tag{14}$$

in which $\bar{U}$ and Q are given by Eqs. (7) and (10) respectively.

Fluid velocities in the (x, y) frame are given by $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$, where the velocity potential ϕ is given by

$$\phi(x, y, t) = (c - \bar{U})x + C_0(g/k^3)^{1/2} \sum_{i=1}^{5} \varepsilon^i \sum_{j=1}^{i} A_{ij} \cosh jky \sin jk(x - ct) + \ldots\,, \tag{15}$$

in which the coefficients A_{ij} are given in Table 1.

The surface elevation $\eta(x, t)$ is given by

$$k\eta(x, t) = kd + \sum_{i=1}^{5} \varepsilon^i \sum_{j=1}^{i} B_{ij} \cos jk(x - ct) + \ldots\,. \tag{16}$$

By applying Bernoulli's theorem, here done most conveniently in the frame in which motion is steady, an expression for the pressure can be found:

$$\frac{p(x, y, t)}{\rho} = R - gy - \frac{1}{2}\left[(u - c)^2 + v^2\right]\,, \tag{17}$$

in which ρ is the fluid density and the Bernoulli constant R is given by

$$Rk/g = \frac{1}{2}C_0^2 + kd + \varepsilon^2 E_2 + \varepsilon^4 E_4 + \ldots\,. \tag{18}$$

A detailed examination of the coefficients in Table 1 for the shallow water limit $kd \to 0$ shows that the coefficients tend to behave like increasingly negative powers of kd as higher order terms are considered, in this limit as $kd \to 0$. This is an unfortunate result for the application of the theory in shallow water, for it means that the contributions of the higher order terms will tend to dominate, and results obtained will not be accurate. In fact it can be shown (Fenton, 1985) that for each higher order term in ε the coefficients behave like an extra power of $(kd)^{-3}$, thus the *effective* expansion parameter is $\varepsilon/(kd)^3$ in the shallow water (long wave) limit. This is proportional to the Ursell number $H\lambda^2/d^3$, and shows that the Stokes theory should only be applied magnitudes of both ε and $\varepsilon/(kd)^3$ are small. The magnitude of the latter should be monitored if the Stokes theory is to be used in shallow water. A detailed examination of the limits of accuracy of the theory is given in Section 5 below.

4. Cnoidal theory

The cnoidal theory for the steady water wave problem follows from a shallow water approximation, in which it is assumed that the waves are much longer than the water is deep. In its derivation, cnoidal theory makes no approximation based on wave height, however most presentations of the theory recast the series expansions obtained in terms of the dimensionless wave height H/h where h is the water depth under the trough. A first order solution shows that the surface elevation is proportional to $\mathrm{cn}^2(z|m)$, where $\mathrm{cn}(z|m)$ is a Jacobian elliptic function of argument z and modulus m and which gives its name to the theory. This solution shows the long flat troughs and narrow crests characteristic of waves in shallow water. In the limit as $m \to 1$ the solution corresponds to the infinitely-long solitary wave.

Various versions of cnoidal theory have been presented. Fenton (1979) gave a fifth-order theory, and showed that in previous theories the effective expansion parameter was not H/h, but H/mh, where m is the parameter defined above. It can be shown that in the limit as short waves are approached, or as infinitesimal waves are considered, this expansion parameter varies like $(d/\lambda)^2$. For this quantity to be small and for the series results to be valid, the short wave limit is excluded. In this way the cnoidal theory breaks down in deep water (short waves) in a manner complementary to that in which Stokes theory breaks down in shallow water (long waves).

Cnoidal theory has not been applied as widely as it might have been. One possible reason is that the theory makes extensive use of Jacobian elliptic functions and integrals which have been perceived to be difficult to calculate. This difficulty has been caused by the fact that conventional formulae are very poorly convergent, if convergent at all, in the limit $m \to 1$, precisely the limit in which cnoidal theory is most appropriate. However, alternative formulae can be obtained which are most accurate and very quickly convergent in the limit of $m \to 1$. This has been done by Fenton and Gardiner-Garden (1982). The formulae are dramatically convergent, even for values of m not in that limit. Convenient approximations to these formulae can be obtained and are given here in Table 2. It is remarkable that provided $m \geq 1/2$, the simple approximations given in the table are accurate to five significant figures. For the case $m < 1/2$, when cnoidal theory becomes less valid, conventional approximations could be used, for which reference can be made to Fenton and Gardiner-Garden or to standard references for the usual formulae. However, cnoidal theory should probably be avoided in this case.

Table 2. Approximations for elliptic functions and integrals in the case most appropriate for cnoidal theory, $m \geq 1/2$.

Elliptic integrals

Complete elliptic integral of the first kind $K(m)$

$$K(m) \approx \frac{2}{\left(1+m^{1/4}\right)^2} \log \frac{2\left(1+m^{1/4}\right)}{1-m^{1/4}}$$

Complementary elliptic integral of the first kind $K'(m)$

$$K'(m) \approx \frac{2\pi}{\left(1+m^{1/4}\right)^2}$$

Complete elliptic integral of the second kind $E(m)$

$$E(m) = K(m)\, e(m), \text{ where}$$

$$e(m) \approx \frac{2-m}{3} + \frac{\pi}{2KK'} + 2\left(\frac{\pi}{K'}\right)^2 \left(-\frac{1}{24} + \frac{q_1^2}{\left(1-q_1^2\right)^2}\right),$$

where $q_1(m)$ is the complementary nome $q_1 = e^{-\pi K/K'}$.

Jacobian elliptic functions

$$\operatorname{sn}(z|m) \approx m^{-1/4} \frac{\sinh w - q_1^2 \sinh 3w}{\cosh w + q_1^2 \cosh 3w},$$

$$\operatorname{cn}(z|m) \approx \frac{1}{2}\left(\frac{m_1}{m\, q_1}\right)^{1/4} \frac{1 - 2q_1 \cosh 2w}{\cosh w + q_1^2 \cosh 3w},$$

$$\operatorname{dn}(z|m) \approx \frac{1}{2}\left(\frac{m_1}{q_1}\right)^{1/4} \frac{1 + 2q_1 \cosh 2w}{\cosh w + q_1^2 \cosh 3w},$$

in which $w = \pi z / 2K'$.

Another disincentive to the use of cnoidal theory was provided by some of the results given in Fenton (1979) (which was for the special case of zero current, $\bar{u}_1 = 0$). In that paper results were presented to fifth order, which required the presentation of many coefficients, rendering application rather daunting. More of a disincentive, however, was provided by the comparisons made with experimental measurements of fluid velocities under the crests of waves. For smaller waves, of height $H/d \approx 0.4$, the theory gave quite good agreement with experiment, but for higher waves, $H/d \approx 0.5$, the velocity profiles showed rather severe oscillations, and agreement with experiment was poor. It was found that at higher order the results were even worse.

While preparing this article, unsatisfied with the complexity of the theory and the surprisingly-poor results, the author discovered that two major modifications can be made to the cnoidal theory, the first which makes it somewhat easier to apply, and the second gives much better results for fluid velocities underneath waves.

The first simplification which can be made is suggested by the fact that for waves which are not low and/or short, the values of the parameter m used in practice are very close to unity indeed. This suggests the simplification that, in all the formulae, wherever m appears explicitly, it be replaced by $m = 1$, which results in much shorter formulae. In the presentation below, this has been done, although explicit appearances of the elliptic functions $\mathrm{cn}()$, $\mathrm{sn}()$ and $\mathrm{dn}()$, and the elliptic integrals $K(m)$, $E(m)$ and their ratio $e(m)$ have been retained, as these quantities show strong, even singular, variation in the $m \to 1$ limit.

The use of the $m = 1$ approximation for typical values of m is rather more accurate than the conventional approximations on which the theory is based, namely the neglect of higher powers of the wave height or the shallowness. For example, $m = 0.9997$ for a wave of height 40% of the depth and a length 15 times the depth; in this case the error introduced by neglecting the difference between m^6 and 1^6 (0.002) in first-order terms is less than the neglect of sixth-order terms not included in the theory ($0.4^6 = 0.004$). In the various formulae which follow, the orders of the neglected terms are not shown. Mostly they are of order $(H/d)^6$ and $1 - m^6$.

If the wave height and length and the water depth are known, then the parameter m can be found by solving the transcendental equation:

$$\frac{\lambda}{d} - \frac{4K(m)}{(3H/d)^{1/2}} \left[1 + \frac{H}{d}\left(\frac{5}{8} - \frac{3}{2}e\right) + \left(\frac{H}{d}\right)^2 \left(-\frac{21}{128} + \frac{1}{16}e + \frac{3}{8}e^2\right) + \left(\frac{H}{d}\right)^3 \left(\frac{20127}{179200} - \frac{409}{6400}e + \frac{7}{64}e^2 + \frac{1}{16}e^3\right) + \left(\frac{H}{d}\right)^4 \left(-\frac{1575087}{28672000} + \frac{1086367}{1792000}e - \frac{2679}{25600}e^2 + \frac{13}{128}e^3 + \frac{3}{128}e^4\right)\right] = 0, \tag{19}$$

in which $e = e(m) = E(m)/K(m)$, for which a convenient expression is given in Table 2. The variation of the left side with m is very rapid in the limit as $m \to 1$, and gradient methods such as the secant method for this might break down. The author prefers to use the bisection method, using as the initial range $m = 10^{-12}$ to $m = 1 - 10^{-12}$.

If instead of the wavelength, the wave period and the current are known, then formulae based on Eqs. (1) or (2) can be used. In this case it is simpler to present separate expansions for the quantities which appear in the equations. Several expressions must be evaluated.

Eq. (1) can be shown to give

$$\frac{\bar{u}_1}{\sqrt{gd}} + \frac{\bar{U}}{\sqrt{gh}}\left(\frac{h}{d}\right)^{1/2} - \frac{1}{\tau\sqrt{g/d}}\frac{h}{d}\frac{2K(m)}{\alpha(H/h, m)} = 0, \tag{20}$$

in which h/d is the ratio of the depth underneath the trough h to the mean depth as given by Fenton (1979) (from his Equation 4.8 and Table A.3):

$$\frac{h}{d} = 1 + \frac{H}{d}(-e) + \left(\frac{H}{d}\right)^2 \frac{e}{4} + \left(\frac{H}{d}\right)^3 \left(-\frac{e}{25} + \frac{e^2}{4}\right) + \left(\frac{H}{d}\right)^4 \left(\frac{573}{2000}e - \frac{57}{400}e^2 + \frac{e^3}{4}\right) + \left(\frac{H}{d}\right)^5 \left(\frac{-302159}{1470000}e + \frac{1779}{2000}e^2 - \frac{123}{400}e^3 + \frac{e^4}{4}\right). \tag{21}$$

Having calculated the trough depth, the expansion quantity H/h used in conventional presentations of cnoidal theory can be calculated:

$$\frac{H}{h} = \frac{H/d}{h/d}. \tag{22}$$

The quantity α is given by

$$\alpha = \left(\frac{3}{4}\frac{H}{h}\right)^{1/2}\left(1-\frac{5}{8}\frac{H}{h}+\frac{71}{128}\left(\frac{H}{h}\right)^2-\frac{100627}{179200}\left(\frac{H}{h}\right)^3+\frac{16259737}{28672000}\left(\frac{H}{h}\right)^4\right). \tag{23}$$

Finally in equation (20) the quantity $\bar{U}/\sqrt{gh}$ is given by

$$\begin{aligned}\frac{\bar{U}}{\sqrt{gh}} &= 1+\frac{H}{h}\left(\frac{1}{2}-e\right)+\left(\frac{H}{h}\right)^2\left(-\frac{3}{20}+\frac{5}{12}e\right)+\left(\frac{H}{h}\right)^3\left(\frac{3}{56}-\frac{19}{600}e\right)\\ &+\left(\frac{H}{h}\right)^4\left(-\frac{309}{5600}+\frac{3719}{21000}e\right)+\left(\frac{H}{h}\right)^5\left(\frac{12237}{616000}-\frac{997699}{8820000}e\right).\end{aligned} \tag{24}$$

The parameter m is deeply embedded in this set of equations, however each can be evaluated directly, and methods such as bisection can be applied to obtain a solution.

In the other case, where the depth-integrated mean current $\bar{u}_2$ is known, the equation to solve is

$$\frac{\bar{u}_2}{\sqrt{gd}}+\frac{Q}{\sqrt{gh^3}}\left(\frac{h}{d}\right)^{3/2}-\frac{1}{\tau\sqrt{g/d}}\frac{h}{d}\frac{2K(m)}{\alpha(H/h,m)}=0, \tag{25}$$

using some of the quantities introduced above and the expression for the dimensionless discharge underneath the wave:

$$\frac{Q}{\sqrt{gh^3}}=1+\frac{1}{2}\frac{H}{h}-\frac{3}{20}\left(\frac{H}{h}\right)^2+\frac{3}{56}\left(\frac{H}{h}\right)^3-\frac{309}{5600}\left(\frac{H}{h}\right)^4+\frac{12237}{616000}\left(\frac{H}{h}\right)^5. \tag{26}$$

Having solved for m iteratively, the cnoidal theory can now be applied. The free surface elevation η is given by

$$\begin{aligned}\frac{\eta}{h} &= 1+\frac{H}{h}\,\mathrm{cn}^2(\alpha(x-ct)/h|m)+\left(\frac{H}{h}\right)^2\left(-\frac{3}{4}\,\mathrm{cn}^2+\frac{3}{4}\,\mathrm{cn}^4\right)\\ &+\left(\frac{H}{h}\right)^3\left(\frac{5}{8}\,\mathrm{cn}^2-\frac{151}{80}\,\mathrm{cn}^4+\frac{101}{80}\,\mathrm{cn}^6\right)\\ &+\left(\frac{H}{h}\right)^4\left(-\frac{8209}{6000}\,\mathrm{cn}^2+\frac{11641}{3000}\,\mathrm{cn}^4-\frac{112393}{24000}\,\mathrm{cn}^6+\frac{17367}{8000}\,\mathrm{cn}^8\right)\\ &+\left(\frac{H}{h}\right)^5\left(\frac{364671}{196000}\,\mathrm{cn}^2-\frac{2920931}{392000}\,\mathrm{cn}^4+\frac{2001361}{156800}\,\mathrm{cn}^6-\frac{17906339}{1568000}\,\mathrm{cn}^8+\frac{1331817}{313600}\,\mathrm{cn}^{10}\right),\end{aligned} \tag{27}$$

where in all cases the argument of the cnoidal functions is $\alpha(x-ct)/h$ and the modulus is m.

Perhaps the most useful result of any wave theory is that which predicts the fluid velocities underneath the waves. Results for these velocities presented by Fenton (1979), to fifth order were not good for waves higher than 50% of the depth. Below, new formulae for fluid velocities will be presented. These constitute the second modification to cnoidal theory mentioned above.

The shallow water theory used to develop cnoidal theory does not make use of the wave height as expansion parameter. Rather it is expressed in terms of α, the stretching or shallowness parameter given in Eq. (23). Cnoidal theory traditionally recasts the series obtained to give series in terms of the wave height. While preparing this article the author considered the expressions for fluid velocity in the original α (actually α^2) formulation and compared them with experiment. It was found that the results were very much better than those presented previously, even with the additional $m = 1$ approximation. Here, the formula for velocity components are presented; the results of such comparisons will be presented in Section 5. It is recommended that these expressions be used in practice.

i	j	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$
1	0	$-1/2$	1				
2	0	$-19/40$	$3/2$	-1			
	1		$-3/2$	$9/4$			
3	0	$-55/112$	$71/40$	$-27/10$	$6/5$		
	1		$-9/4$	$75/8$	$-15/2$		
	2		$3/8$	$-45/16$	$45/16$		
4	0	$-11813/22400$	$53327/42000$	$-13109/3000$	$1763/375$	$-197/125$	
	1		$-213/80$	$3231/160$	$-729/20$	$189/10$	
	2		$9/16$	$-327/32$	$915/32$	$-315/16$	
	3		$-3/80$	$189/160$	$-63/16$	$189/64$	
5	0	$-57159/98560$	$-144821/156800$	$-1131733/294000$	$757991/73500$	$-298481/36750$	$13438/6125$
	1		$-53327/28000$	$1628189/56000$	$-192481/2000$	$11187/100$	$-5319/125$
	2		$213/320$	$-13563/640$	$68643/640$	$-5481/32$	$1701/20$
	3		$-9/160$	$267/64$	$-987/32$	$7875/128$	$-567/16$
	4		$9/4480$	$-459/1792$	$567/256$	$-1215/256$	$729/256$

Table 3. Coefficients Φ_{ijl} used in expressions for velocity components from cnoidal theory

It is more convenient to present the results in terms of δ, rather than in terms of α, where

$$\delta = \frac{4}{3}\alpha^2, \tag{28}$$

and where α is given by Eqn. (h3). Fluid velocities in the (x, y) frame are given by

$$\frac{u(x,y,t)}{\sqrt{gh}} = \frac{c}{\sqrt{gh}} - 1 + \sum_{i=1}^{5} \delta^i \sum_{j=0}^{i-1} \left(\frac{y}{h}\right)^{2j} \sum_{l=0}^{i} \mathrm{cn}^{2l}(\alpha(x-ct)/h|m)\Phi_{ijl}, \tag{29}$$

and

$$\begin{aligned}\frac{v(x,y,t)}{\sqrt{gh}} &= 2\alpha\ \mathrm{cn}(.|.)\,\mathrm{sn}(.|.)\,\mathrm{dn}(.|.) \times \\ &\sum_{i=1}^{5} \delta^i \sum_{j=0}^{i-1} \left(\frac{y}{h}\right)^{2j+1} \sum_{l=1}^{i} \mathrm{cn}^{2(l-1)}(\alpha(x-ct)/h|m)\frac{l}{2j+1}\Phi_{ijl},\end{aligned} \tag{30}$$

in which the coefficients Φ_{ijl} in the expansions for fluid velocities are given in Table 3. If the wave speed c is not known it can be calculated using Eqs. (1) or (2), depending on whether $\bar{u}_1$ or $\bar{u}_2$ is known:

$$c = \bar{u}_1 + \bar{U} \qquad \text{or} \qquad c = \bar{u}_2 + Q/d,$$

in which $\bar{U}$ and Q are given by Eqs. (24) and (26) respectively.

It will be seen in Section 5 that this theory predicts velocities under wave crests accurately over a wide range of wave conditions.

By applying Bernoulli′s theorem in the frame in which motion is steady, Eqn.(g6) can be used to give an expression for the fluid pressure at a point. In this case, the Bernoulli constant R is given by cnoidal theory with the $m = 1$ approximation:

$$\frac{R}{gh} = \frac{3}{2} + \frac{1}{2}\frac{H}{h} - \frac{1}{40}\left(\frac{H}{h}\right)^2 - \frac{3}{140}\left(\frac{H}{h}\right)^3 - \frac{3}{175}\left(\frac{H}{h}\right)^4 - \frac{2427}{154000}\left(\frac{H}{h}\right)^5. \tag{31}$$

5. Fourier approximation methods

A limitation to the use of both Stokes and cnoidal theories has been that they have been widely believed, correctly in view of the evidence to date, to be not accurate for all waves. In Section 6 below an examination of the accuracy of these theories is presented, and it is suggested that, contrary to previous belief, fifth-order theory in the versions as presented above are of acceptable engineering accuracy almost everywhere within the range of validity of each. To obtain highly-accurate results from either theory, however, it would be necessary to obtain very high-order expansions for velocity as a function of position, which has not yet been done. In cases where it might be necessary to obtain results of high accuracy, where a structure of major importance is to be designed and where design data are accurately known, or where it is necessary to use a method which is valid in both deep and shallow water, then numerical solution of the full nonlinear equations is a better option.

The usual method, suggested by the basic form of the Stokes solution, is to use a Fourier series which is capable of accurately approximating any periodic quantity, provided the coefficients in that series can be found. A reasonable procedure, then, instead of assuming perturbation expansions for the coefficients in the series as is done in Stokes theory, is to calculate the coefficients numerically by solving the full nonlinear equations. This Fourier approach would be expected to break down in the limit of very long waves, when the spectrum of coefficients becomes broad-banded and many terms have to be taken. Also, if the highest waves are approached, the crest becomes more and more sharp, and the spectrum also becomes broad. Despite these limitations this approach would be expected to be more accurate than either of the perturbation expansion approaches described above, because its only approximations would be numerical ones, and not the essential analytical ones of the perturbation methods. Also, increasing the order of approximation would be a relatively trivial matter.

It is this approach which lies behind the methods of Chappelear (1961), Dean (1965), and Rienecker and Fenton (1981). Each method assumes a Fourier series in which each term identically satisfies the field equation throughout the fluid and the boundary condition on the bottom. The values of the Fourier coefficients and other variables for a particular wave are then found by numerical solution of the nonlinear equations obtained by substituting the Fourier series into the nonlinear boundary conditions. Chappelear used the velocity potential for the field variable and also introduced a Fourier series for the surface elevation. By using instead the stream function for the field variable and point values of the surface elevations Dean obtained a rather simpler set of equations (and called the method "stream function theory"). In both these approaches the solution of the equations proceeded by a method of successive corrections to an initial estimate such that the least-squares errors in the surface boundary conditions were minimised. Rienecker and Fenton presented a method which gave somewhat simpler equations, where the nonlinear equations were solved by Newton's method, such that the equations were satisfied identically at a number of points on the surface rather than minimising errors there.

A simpler method and computer program have been given by Fenton (1988). The major simplification is that all the partial derivatives necessary are obtained numerically. In application of the method to waves which are high, in common with other versions of the Fourier approximation method, it was found that it is sometimes necessary to solve a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. For very long waves these methods can occasionally converge to the wrong solution, that of a wave 1/3 of the length, which is obvious from the Fourier coefficients which result, as only every third is non-zero. This problem can be avoided by using the sequence of height steps.

Results from these numerical methods show that accurate solutions can be obtained with Fourier series of 10-20 terms, even for waves close to the highest, and they seem to be the best way of solving any steady water wave problem where accuracy is important. Sobey *et al.* (1987), made a comparison between contemporary versions of the numerical methods. They came to the conclusion that there was little to choose between them.

6. Results and comparisons

The range over which periodic solutions for waves can occur is given in Figure 2, which shows limits to the existence of waves determined by computational studies.

The upper limit of height/depth, H_m/d, is shown by the filled squares, which are the results of Williams (1981), taken from his Table 7. These show the highest waves for which solutions could be obtained

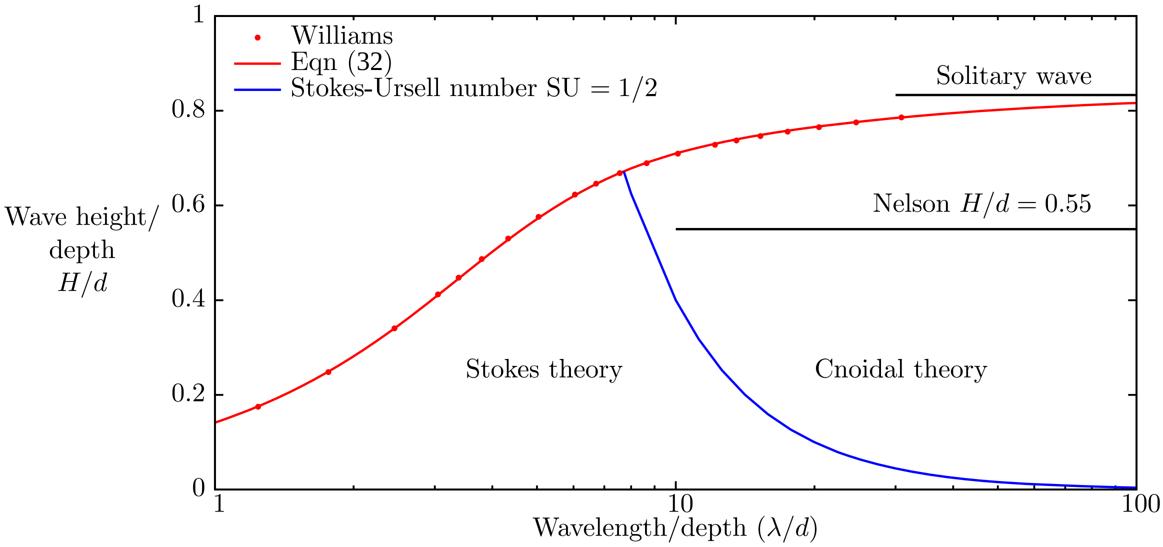

Figure 2. The region in which solutions for steady waves can be obtained, showing Williams' experimental points, the empirical curve for the highest waves, Eqn. (32), contours of errors of Stokes and cnoidal theories for the volume flux under the crest, and Hedges' proposed demarcation line between Stokes and cnoidal theories, Eqn. (33b).

using very high-order Stokes expansions and incorporating the crest singularity analytically. The results are believed to be highly accurate. Williams' result, that in the short wave limit (deep water), the ratio of H_m/λ is 0.141063 is shown (a curve on this semi-log plot). The opposite limit, in which the highest solitary wave has a height $H_m/d = 0.83322$ is also shown (Hunter and Vanden-Broeck, 1983).

For engineering purposes it would be convenient to have an expression for the greatest wave height as a function of wavelength and depth. To do this, a rational approximation can be fitted to Williams' points. If this is done, using the four points at values of λ/d of 30.89, 7.56, 2.45 and 0.624, the equation obtained is:

$$\frac{H_m}{d} = \frac{0.141063\,\frac{\lambda}{d} + 0.0095721\left(\frac{\lambda}{d}\right)^2 + 0.0077829\left(\frac{\lambda}{d}\right)^3}{1 + 0.0788340\,\frac{\lambda}{d} + 0.0317567\left(\frac{\lambda}{d}\right)^2 + 0.0093407\left(\frac{\lambda}{d}\right)^3}. \tag{32}$$

The leading coefficient in the numerator is such that the approximation gives the correct behaviour in the limit of short waves, $H_m/\lambda \rightarrow 0.141063$. Also, the ratio of the coefficients of the cubic terms is 0.83322, such that in the limit $\lambda/d \rightarrow \infty$, the correct result is obtained. The approximation is shown on Figure 2, and deviates by no more than 0.4% from any of the other points given by Williams. Equation (32) may be useful in practice, although of course the six-figure accuracy of the coefficients is hardly necessary. Having established this boundary, it is possible to examine the accuracy of the various theories over the range of possible solutions.

Figure 3 shows results for the horizontal fluid velocity under the crest, $u(0, y)$, often one of the more important quantities used for design purposes. The experimental results are those of Le Méhauté *et al.* (1968), obtained by measurements of the motion of particles over a finite period of time in a closed flume such that $\bar{u}_2 = 0$. The experimental results have been included, although the real comparison here is between the approximate Stokes and cnoidal theories and the accurate results of the Fourier approximation method. Because of the Lagrangian mode of measurement, the experimental velocities in the vicinity of the crest tend to be averaged, and if anything the experiments underestimate the actual instantaneous maximum fluid velocity, which seems to be the case in the figure. The three curves are for the highest set of waves, referred to as "just below breaking" and "limit waves" by Le Méhauté *et al.*. Results from the Fourier approximation method and the experimental results seem to agree consistently. In Figure 3(a), for an intermediate wave of dimensionless period $\tau\sqrt{g/d} = 8.59$, ($\lambda/d \approx 8.2$ from the accurate Fourier = solution), both the Stokes theory of Section 2 and the Cnoidal theory of Section 3

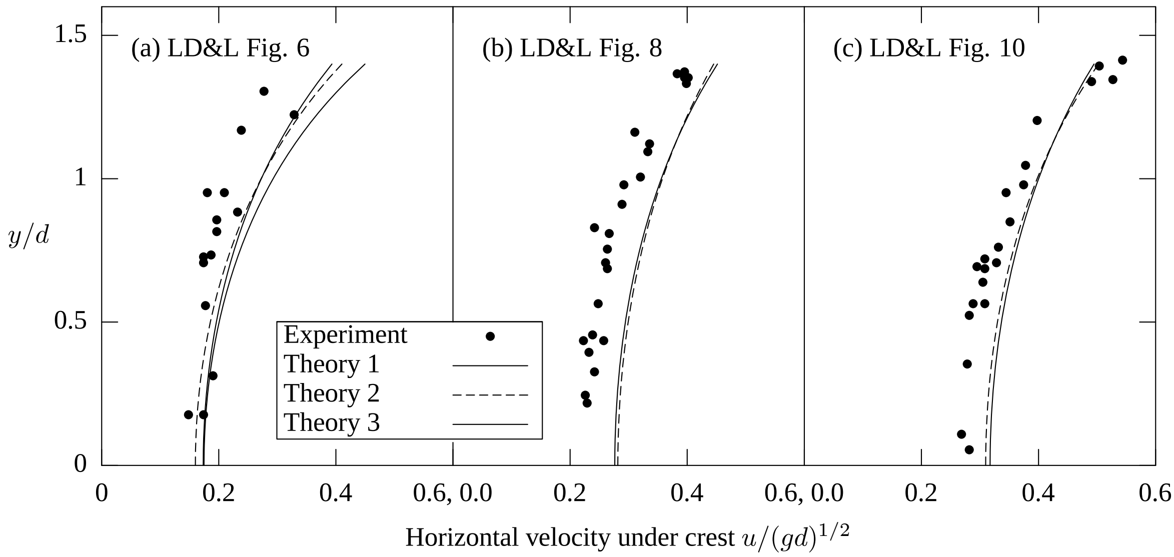

Figure 3. Fluid velocity under crest. Comparison between theories and experimental results of Le Méhauté *et al.* (a) $H/d = 0.499$, $\tau\sqrt{g/d} = 8.59$, (b) $H/d = 0.522$, $\tau\sqrt{g/d} = 15.9$, (c) $H/d = 0.548$, $\tau\sqrt{g/d} = 27.3$

slightly underestimate the fluid velocities. This wave is almost exactly in the middle of the narrow region on Figure 2 where neither theory is particularly accurate. In Figures 3(b) and (c), where the waves are longer ($\lambda/d \approx 17$ and 31 respectively), no solution could be found using the Stokes theory. It will be shown below that these waves are too long for Stokes theory to be applied.

The new cnoidal theory presented in Section 3, however, gives results which agree very closely with the accurate Fourier results for the longer waves of Figures 3(b) and (c). This is something of a surprise. The previous high-order cnoidal theory as presented by Fenton (1979), gave rather poor results for these higher waves. It would seem that the new cnoidal theory of Section 3 should replace the previous version. These results suggest that for waves of height up to at least half the water depth, the two fifth-order methods, each in its respective range of validity, are capable of sufficient accuracy for most engineering purposes.

Although these waves are somewhat lower than the theoretical highest, there is evidence that they may, in fact, be close to the highest realisable in practice. Nelson (1987), has shown from a number of experiments on mild slopes, that in the limit as the slope goes to zero the maximum wave height observed was only $H/d = 0.55$. Further evidence for this conclusion is provided by the results of Le Méhauté *et al.*, whose maximum wave height tested was $H/d = 0.548$, described as "just below breaking". It seems that there may be enough instabilities at work that real waves propagating over a flat bed cannot approach the theoretical limit given by Eq. (32). This is very fundamental for the application of the present theories. If indeed the highest waves do have a height to depth ratio of only 0.55, it seems that both fifth-order theories in Sections 2 and 3 are capable of giving accurate results over all possible waves.

Here the accuracy of the theories for waves over a wider range will be examined. To do this, a number of wave solutions were obtained, over most of the theoretically possible domain shown in Figure 2. It was found that, solving for 40 increasing wave heights at each wavelength, the Fourier method usually failed to converge on the last, so that Fourier solutions were obtained up to within 2.5% of the highest theoretical waves defined by Eq. (32). The criterion for accuracy in testing the theories was chosen to be the integral from the bed to the crest of the horizontal fluid velocity under the crest (the instantaneous discharge under the crest). The Fourier method was chosen to be the standard for comparison. Results are shown on Figure 2, in the form of contours of the error in the crest discharge, and are rather encouraging. It can be seen that over most of the diagram, the appropriate theory, Stokes for short waves, cnoidal for long, gives results to within 1% , with a narrow band where the error is more than 5% . This is almost certainly good enough for practical application. It is noteworthy, given the different natures of the two theories, that each loses accuracy in the same region and in a similar manner, just where the other starts to gain accuracy, and the error contours of each are roughly parallel to each other. Thus the two theories seem to be rather more fortuitously complementary than has been realised. There is a region in the middle of Figure 2, however, where the best accuracy attainable is between 5 and 10% only. The experimental conditions of Figure 3(a) are precisely in the middle of this region, and so it provides the worst case for waves of this height.

It is noteworthy that over most of the domain the error contours are not almost horizontal, as might have been expected from theories in which the fundamental expansion parameter is the wave height. It is well-known that Stokes and cnoidal theories break down in the inappropriate wavelength limits, and the diagram shows precisely where this occurs. What is not widely appreciated, although pointed out by Fenton (1979), is that in the limit of small amplitude waves, cnoidal theory ceases to converge. The diagram shows, for example, that to solve a low and long wave with $H/d = 0.1$ with a length as great as $\lambda/d = 15$, it is better to use Stokes theory. This violates the naive limitation of $\lambda/d = 10$ as suggested by Fenton (1979 & 1985) to demarcate the regions of validity of the two theories.

In the original version of this review, the author wrote:

> *A rather better line of demarcation is the solid line shown on Figure 2, halfway between the two 5% contours, which has the equation*
>
> $$\frac{\lambda}{d} = 21.5e^{-1.87H/d}. \tag{33}$$
>
> *For waves longer than this, cnoidal theory should be used, and for shorter waves, Stokes theory.*

Hedges (1995) showed that a better boundary between Stokes and cnoidal theories' areas of application is

$$\mathbf{U} = \frac{H\lambda^2}{d^3} = 40, \tag{33b}$$

where **U** is the *Ursell number*

$$\mathbf{U} = \frac{H/d}{(d/\lambda)^2} = \frac{\text{"Nonlinearity" (measure of height)}}{\text{"Shallowness" (measure of depth/length)}},$$

which can be used to characterise waves. Those with a large Ursell number are generally long high waves, and cnoidal theory is best, whereas for small Ursell number (deeper water), Stokes theory is most applicable. This is shown on Figure 2. The Fourier approximation method works well for all waves up to within about 1% of the highest.

To examine the performance of the theories for waves close to the theoretical highest in the vicinity of this demarcation line, crest velocity profiles for three waves were examined, each with a height equal to 97.5% of the maximum as given by Eq. (32). They straddle the point of intersection between Eqs. (32) and (33b). Results are shown in Figure 4.

Part (a) of the figure shows the profile for $\lambda/d = 6$. Even the new cnoidal theory is almost acceptable, surprisingly for a wave as short as this and which is almost breaking. The Stokes theory is very accurate, except in the vicinity of the crest. For shorter waves other profiles were computed using the Stokes theory, and the results were everywhere excellent. All, except near the case of Figure 4(a), near the demarcation line, were visually obscured everywhere by the accurate Fourier results. It seems that throughout the **ft ac12** of the diagram, for waves shorter than that given by Eq. (33b), the Stokes theory can be used with great accuracy. However, it quickly loses accuracy for longer waves: Figure 4(b) for $\lambda/d = 8$ shows that cnoidal theory is more accurate, while in Figure 4(c) for $\lambda/d = 10$, Stokes theory fails miserably.

Considering the results from cnoidal theory, it can be seen that the excellent results of Figure 3 for waves of H/d about 0.55 are no longer obtained, and that for the waves in Figure 4 of height about 0.68, finite

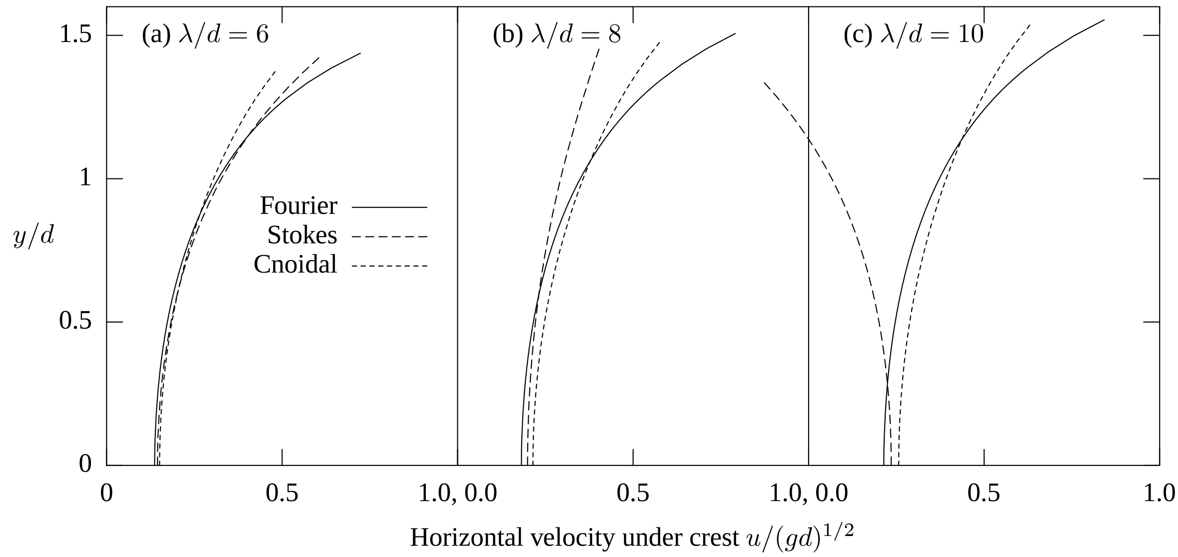

Figure 4. Fluid velocity under crest for waves in the intermediate length region,with height 97.5% of the highest possible for that length

deviations from the accurate Fourier results are noticeable. However, the cnoidal results are not grossly in error, as can be seen in Figures 4(b) and (c), and the integral of u with y obtained from the cnoidal theory for these longer waves was always accurate to within 3%. When solutions for longer waves were obtained, the results for the highest waves all looked like Figure f4(c), and as expected, in this $m \to 1$ limit, the theory did not lose any more accuracy for longer waves. For $H/d < 0.55$, as shown in Figure 3, the results were always excellent, however it seems that for solutions of high accuracy for waves closest to the highest theoretically possible, it is necessary to use the Fourier approximation method.

7. Integral properties of waves

There are several other important properties of a steady wave train which are characteristic of the wave train as a whole, in addition to the discharge Q, the Bernoulli constant R, and $\bar{U}$ the mean fluid velocity at a constant level, all used above. The additional properties include the mean wave momentum, kinetic and potential energy, radiation stress, energy flux and mean square of the velocity on the bed and are generally given as the mean values per unit area.

Klopman (1990), has presented general formulae for these integral quantities in the frame through which waves pass at velocity c. His expressions correct some previously-presented expressions. These extra integral quantities can be calculated in terms of quantities which have been defined above and for which expressions from Stokes and cnoidal theories have been given above, with the exception of the potential energy V:

$$V = \overline{\int_d^\eta \rho g y \, dy} = \frac{1}{2}\rho g \left(\overline{\eta^2} - d^2\right) . \tag{34}$$

Here, expressions are given for V from both Stokes and cnoidal theories. Substituting Eq. (16) into Eq. (34) and performing the manipulations and integration gives

Fifth-order Stokes theory approximation for V:

$$V = \frac{1}{16}\rho g H^2 (1 + \varepsilon^2 (2B_{31} + B_{22}^2) + O(\varepsilon^4)). \tag{35}$$

Fenton (1979) presented a formula for V using cnoidal theory. It is in keeping with the $m = 1$ approximation of Section 3 to provide that approximation here, giving

Fifth-order cnoidal theory approximation for V:

$$\begin{aligned}\frac{V}{\rho g H^2} &= \frac{e}{3} - \frac{e^2}{2} + \left(\frac{H}{h}\right)\left(-\frac{e}{10} + \frac{e^2}{4}\right) + \left(\frac{H}{h}\right)^2\left(\frac{9}{2800}e - \frac{57}{800}e^2\right) \\ &\quad + \left(\frac{H}{h}\right)^3\left(-\frac{4369}{42000}e + \frac{593}{2000}e^2\right) + O((H/h)^4). \end{aligned} \tag{36}$$

It should be noted that most other integral quantities for which Fenton gave formulae were for the special case $\bar{u}_1 = 0$. The following supersedes the expressions given.

Now, all the quantities defined above, may be used to obtain values for the quantities below. Here, each quantity is defined and a formula presented for it. In all cases the overbar denotes averaging over one wavelength, and the results are for a unit width normal to the plane of the flow, such that the integral quantities are per unit plan area:

There are a couple of corrections here, made after the original article was published

Mean wave momentum

$$I = \overline{\int_0^\eta \rho u \, dy} = \rho\left(cd - Q\right).$$

Mean kinetic energy

$$T = \overline{\int_0^\eta \frac{1}{2}\rho(u^2 + v^2)\, dy} = \frac{1}{2}\left(cI - \bar{u}_1 Q\right).$$

Mean square of bed velocity

$$\overline{u_b^2} = \overline{u(x,0,t)^2} = 2(R - gd) - c(c - 2\bar{u}_1).$$

Mean radiation stress

$$S_{xx} = \overline{\int_0^\eta (p + \rho u^2)\, dy} - \frac{1}{2}\rho g d^2 = 4T - 3V + \rho \overline{u_b^2} d - 2\bar{u}_1 I.$$

Mean energy flux

$$F = \overline{\int_0^\eta \left(p + \frac{1}{2}\rho(u^2 + v^2) + \rho g(z - d)\right) u\, dy} = c(3T - 2V) + \frac{1}{2}\overline{u_b^2}(I + \rho c d) - 2c\bar{u}_1 I.$$

Momentum flux

$$S = \overline{\int_0^\eta \left(p + \rho(u - c)^2\right)\, dy} = S_{xx} - 2cI + \rho d\left(c^2 + \frac{1}{2}gd\right).$$

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List of symbols

Symbol	Definition
A_{ij}	Velocity potential coefficients, Stokes theory.
B_{ij}	Surface elevation coefficients, Stokes theory
c	Wave speed.
$\mathrm{cn}()$	Elliptic function.
C_i	Coefficients for mean fluid velocity, Stokes theory.
d	Mean water depth.
$\mathrm{dn}()$	Elliptic function.
D_i	Coefficients for volume flux, Stokes theory.
$e(m)$	$= E(m)/K(m)$, Ratio of elliptic integrals
$E(m)$	Elliptic integral of the second kind.
F	Mean energy flux (wave power) per square metre.
g	Gravitational acceleration.
H	Wave height (crest to trough)
h	Water depth under wave trough.
H_m	Maximum wave height.
i	Integer variable used in sums *etc.*
I	Mean wave momentum per square metre.
j	Integer variable used in sums *etc.*.
k	$= 2\pi/\lambda$, Wavenumber.
$K(m)$	Elliptic integral of the first kind
$K'(m)$	$= K(1-m)$, Complementary elliptic integral.
l	Integer variable used in sums *etc.*.
m	Parameter of elliptic functions and integrals.
m_1	$= 1-m$, Complementary modulus
p	Pressure.
q_1	$= \exp(-K/K')$, Complementary nome of elliptic functions.
Q	Volume flux per unit span perpendicular to flow.
R	Bernoulli constant (energy per unit mass).
$\mathrm{sn}()$	Elliptic function.
S	Momentum flux per unit span perpendicular to flow.
S_{xx}	Mean radiation stress.
t	Time.
T	Mean kinetic energy per square metre.
u	Velocity component in x co-ordinate of frame fixed to bed.
$\bar{u}_1$	Mean value of u, averaged over time at a fixed point.
$\bar{u}_2$	Mean value of u over depth, averaged over time.
u_b	Value of u on the bottom.
U	Velocity component in X co-ordinate.
$\bar{U}$	Mean value of U over a line of constant elevation.
v	Velocity component in y co-ordinate of frame fixed to bed.
V	Velocity component in Y co-ordinate **or**, Mean potential energy per square metre.
w	$= z/2K$, Dummy variable.
x	Horizontal co-ordinate in frame fixed to bed.
X	Horizontal co-ordinate in frame moving with wave crest.
y	$= Y$, Vertical co-ordinate in frame fixed to bed.
Y	Vertical co-ordinate in frame moving with wave crest.
z	Dummy argument used in elliptic function formulae.
α	Shallowness parameter used in cnoidal theory, also appears as a coefficient of the x co-ordinate.
δ	$= 4\alpha^2/3$, equal to H/h at first order.
ε	$= kH/2$, Dimensionless wave height.
η	Water depth.
λ	Wavelength.

ρ	Fluid density.
σ	$= 2\pi/\tau$, Angular frequency of waves.
τ	Wave period (in time).
ϕ	Velocity potential.
Φ_{ijl}	Velocity coefficients in cnoidal theory.
ψ	Stream function.
$O()$	Landau order symbol: ”at least of the order of”.