

CHAPTER 15

Polynomial approximation and water waves

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A different approach to the solution of water wave problems is considered. Instead of using an approximate wave theory combined with highly accurate global spatial approximation methods, as for example in many applications of linear wave theory, a method is developed which uses local polynomial approximation combined with the full nonlinear equations. The method is applied to the problem of inferring wave properties from the record of a pressure transducer, and is found to be capable of high accuracy for waves which are not too short, even for large amplitude waves. The general approach of polynomial approximation is well suited to problems of a rather more general nature, especially where the geometry is at all complicated. It may prove useful in other areas, such as the nonlinear interaction of long waves, shoaling of waves, and in three dimensional problems, such as nonlinear wave refraction and diffraction.

1. Introduction

Conventional linear wave theory, for example, uses a physical approximation which is of a low order of accuracy, neglecting terms which have a magnitude of the order of the square of the wave steepness. In most applications of this theory, however, a global spatial approximation method is used. Often the wave field is approximated by a full spectral representation (Fourier series), so that the spatial approximation is of very high accuracy, and is valid throughout the region of interest. This not only leads to an apparent large discrepancy between the accuracy by which the physical system is modelled and the numerical accuracy used in that model, but also severe limitations as to the boundary geometries which can be considered. In fact, anything other than a flat bed is usually very difficult to treat.

Long wave theory uses approximations based on the limitation that the motion have a large horizontal length scale relative to the water depth, so that the variation of velocities etc. with elevation in the fluid is relatively small. In this case there is a greater degree of harmony between the level of physical and numerical approximation. However, to higher orders of approximation, shallow water (long wave) theory quickly becomes unmanageable as the equations become complicated, and almost invariably higher derivatives occur at each level of approximation. For example, the Boussinesq equations and their relations involve third derivatives at first order, and so on. This

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compares with the the full nonlinear equations, where there are no derivatives higher than the first in the boundary conditions. Another feature of long wave theory, as in methods which use wave steepness expansions (Stokes' methods), is that the presence of anything other than a flat bed quickly complicates the method and equations.

This paper is an initial attempt to turn in the direction of conventional numerical methods for the simulation of nonlinear waves, so that the level of physical and numerical approximation is consistent, and that as far as possible routine computational techniques can be used. In particular, local polynomial approximation methods are to be used, which are of a finite order of accuracy, but which are much simpler to use, especially in their incorporation into the full nonlinear boundary conditions. They have a further major advantage, in that they can be used to satisfy other boundary conditions locally, so that irregular boundaries can be treated, including problems of wave interactions with solid boundaries of a possibly abrupt nature, or of a gentle nature such as an irregularly varying bottom.

In this paper, it is attempted to explore where methods of local polynomial approximation might be used, by considering one problem only, that of analyzing the data obtained by a pressure transducer, and inferring other wave properties from that data, for example the surface elevation, the fluid velocity components at other points in the fluid, and possibly spectra of these and other gross quantities associated with the waves.

The information obtained by a pressure transducer is $p(t_n)$, the pressure at a finite number of instants t_n , $n = 1, 2, \dots, N$. From these pressure readings it is simple to calculate the mean and to infer the mean depth. This can then be used as a length scale for non-dimensionalisation. Throughout the rest of this paper, all quantities are to be taken as having been non-dimensionalised with respect to the mean depth d , gravitational acceleration g , and/or fluid density ρ . (For example, pressure p is " $p/\rho g d$ " in dimensional terms, horizontal velocity u is actually " u/\sqrt{gd} ", and so on.) The conventional approach based on linear wave theory would be to take the signal $p(t_n)$, obtain its discrete Fourier transform P_j , for $j = 0, \pm 1, \dots, \pm N/2$, use linear wave theory to find the corresponding harmonic components, for example U_j and V_j , of the fluid velocity, and then to obtain the actual velocities $u(x,y,t)$ and $v(x,y,t)$, in the horizontal and vertical coordinate directions x and y respectively. A limiting feature of linear theory is that all components of the waves are travelling at the speed corresponding to that phase as given by linear theory, and that there are no nonlinear interactions at all. Particularly in near-shore regions, with the observed tendency of long waves to travel as (non-linear) waves of translation, where the individual components are bound to the main wave and travel with its speed, this is an unnecessarily limiting assumption.

This is not as critical a problem as the fundamental ill-conditioning of the problem as posed, of inferring fluid motions governed by an elliptic equation (Laplace) from boundary data specified on one level only, that of the pressure transducer. This occurs if the probe is located far below the point at which the velocities are

required or it is desired to find the surface elevations $\eta(t_n)$, from the inferred spectrum Y_j of the free surface elevation. The transfer function connecting Y_j and P_j is proportional to $\cosh k(j)/\cosh k(j)y_p$, where k is the wavenumber given by the linear dispersion relation for the j th harmonic of the signal. For higher frequency components the transfer function varies like $\exp(k(j)(y-y_p))$, and using the short wave approximation for the linear dispersion relation, this varies as $\exp(j\omega^2(y-y_p))$, where ω is 2π divided by the total time of the record. It is clear that the transfer function grows remarkably quickly with j , corresponding to higher frequency components. Even for smooth records with spectra P_j which decay quite quickly in j , this exponential growth of the transfer function with j completely destroys any accuracy for harmonic components shorter than the water depth. The method is really only suited to long waves in shallow water. Unfortunately it is for these conditions that linear wave theory is not particularly appropriate, as the waves are likely to be nonlinear and to be long, giving rise to the presence of higher harmonics with their attendant ability to destroy the meaningful part of the signal.

Proceeding in the other direction does not present the same set of problems, as noted above. For example, if the surface elevation is recorded, to give the Y_j , and it is desired to calculate velocities deeper in the fluid, then the transfer function works so as to dramatically reduce the U_j relative to the Y_j for higher frequency components, and the process is well-conditioned. For example, Vis (1980) and Daemrich, Eggert & Kohlhasse (1980) used this approach, and found that for waves which were not high the linear theory gave good results for the fluid velocities.

Use of the spectral method outlined above does have some further practical problems. To implement the method it is usually necessary to resort to a number of techniques of data analysis, which degrade the information provided by the original signal. These techniques include trend removal, multiplying by a "window" to remove spurious components due to end discontinuities, filtering and so on.

The method used in this paper, based on local low-degree polynomial approximation does not overcome the fundamental ill-conditioning of the problem, but in water of finite depth the approach is much less susceptible to the problems described above. The application of local approximation for this problem was originated by Nielsen (1986), who used an approach based on local interpolation by trigonometric functions combined with linear theory.

2. Theory

Throughout this work it is assumed that the waves are travelling over an impermeable bed which is locally flat with coordinate origin on that bed, that all motion is two-dimensional, and that the fluid is incompressible and the fluid motion irrotational such that a complex velocity potential w exists, $w = \phi + i\psi$, where ϕ is velocity potential and ψ is stream function, which is an analytic function of the complex coordinate $z = x + iy$. The coordinate origin is taken to be on the bed, beneath the pressure probe. As the entire discussion is based on

local approximation we can introduce a local time t , which is zero at t_n , the instant at which the pressure reading is taken. The velocity components (u, v) are given by $u - i v = dw/dz$. The approximation is made here that the motion locally is propagating without change in the x direction with a speed c , which is as yet unknown. Hence, variation with x and t can be combined in the form $x - ct$. Locally, this is a reasonable assumption, as the time scale of distortion of the wave as dispersion and nonlinearity take effect is considerably larger than the local time over which the theory is required to be valid.

A principle of local polynomial approximation is adopted, such that in the vicinity of the pressure probe, throughout the depth of fluid, the complex velocity potential $w(x, y, t)$ and the free surface $\eta(x, t)$ are given by polynomials of degree M :

$$w(x, y, t) = \phi(x-ct, y) + i \psi(x-ct, y) = \sum_{j=0}^M \frac{a_j}{j+1} (z-ct)^{j+1}, \quad (2.1)$$

where $z = x + iy$, and

$$\eta(x, t) = \sum_{j=0}^M b_j (x-ct)^j. \quad (2.2)$$

Expansion (2.1) satisfies the governing equation for irrotational flow of an incompressible fluid (Laplace's equation) identically throughout the flow. The bottom boundary condition $v(x, 0, t) = 0$ is satisfied if the coefficients a_j are real only, as the b_j are. It remains to satisfy the boundary conditions on the free surface, that the pressure is constant and that particles remain on the surface. Here we use the approximation again that motion is steady in a coordinate system $(x-ct, y)$.

The steady kinematic equation that the value of ψ is constant on the surface $y = \eta$ is

$$\psi(x-ct, \eta(x-ct)) = -Q, \quad (2.3)$$

where Q is a constant. The steady Bernoulli equation is

$$R = \frac{1}{2} \left| \frac{dw}{d(z-ct)} \right|_s^2 + \eta, \quad (2.4)$$

where R is a constant, and the subscript s denotes the surface $y = \eta$.

Also, in the frame in which motion is steady Bernoulli's equation can be written for the point $(0, y_p)$, where the pressure probe is located. The pressure around that point can be expressed as a Taylor series in $x-ct$:

$$p(x, y_p, t) = R - \frac{1}{2} \left| \frac{dw}{d(z-ct)} \right|_{y_p}^2 - y_p \quad (2.5.1)$$

$$= \sum_{j=0}^M p_j (x-ct)^j. \quad (2.5.2)$$

The coefficients p_j can be found from the pressure readings $p(t_n)$. Details of this will be presented further below.

Substitution of the series (2.1) and (2.2) into equations (2.3), (2.4) and (2.5) gives polynomials in $x-ct$. These polynomials must be valid locally for all values of $x-ct$, hence the coefficients of each power of $x-ct$ must agree across the equation. This gives a system of nonlinear equations in the unknown coefficients a_0, a_1, a_2, \dots , and b_0, b_1, b_2, \dots . The equations are in terms of the coefficients p_0, p_1, p_2, \dots , which are assumed to have been calculated from the pressure readings.

It is feasible to produce the equations by hand calculation for $M = 2$. However for larger values of M the amount of calculation becomes prohibitive. The symbolic algebra manipulation package MACSYMA was used on the computer in the School of Mathematics at the University of New South Wales to produce solutions for $M = 2$ and $M = 4$. MACSYMA has the facility of producing executable code, and the output could be used to evaluate the full set of equations.

3. Equations

The equations can be written in the form

$$K_1 = K_2 = K_3 = K_4 = \dots = 0, \quad D_1 = D_2 = D_3 = D_4 = \dots = 0,$$

and

$$P_0 = P_1 = P_2 = P_3 = P_4 = \dots = 0, \quad (3.1)$$

where the quantities K , D and P come from the kinematic equation (2.3), the dynamic equation (2.4) and the pressure equation (2.5) respectively. The subscripts 0, 1, 2 etc. refer to the power of $x-ct$ of which this term is coefficient.

For a quadratic level of approximation, $M = 2$, the quantities are defined by:

$$K_1 = -b_0^2 b_1 a_2 + a_0 b_1 + b_0 a_1,$$

$$K_2 = -b_0^2 a_2 b_2 + a_0 b_2 - b_0 b_1^2 a_2 + b_0 a_2 + a_1 b_1,$$

$$D_1 = 2b_0^3 b_1 a_2^2 - 2a_0 b_0 b_1 a_2 + b_0^2 a_1 a_2 + b_0 a_1^2 b_1 + b_1 + a_0 a_1,$$

$$D_2 = 2b_0^3 a_2^2 b_2 - 2a_0 b_0 a_2 b_2 + b_0 a_1^2 b_2 + b_2 + 3b_0^2 b_1^2 a_2^2 + b_0^2 a_2^2 - a_0 b_1^2 a_2 \\ + 2b_0 a_1 b_1 a_2 + a_0 a_2 + a_1^2 b_1^2 / 2 + a_1^2 / 2,$$

$$P_0 = a_2^2 y_p^4 / 2 - a_0 a_2 y_p^2 + a_1^2 y_p^2 / 2 - b_0^4 a_2^2 / 2 + a_0 b_0^2 a_2 - b_0^2 a_1^2 / 2 \\ + p_0 - b_0 + y_p,$$

$$p_1 = a_1 a_2 y_p^2 + p_1 + a_0 a_1,$$

$$p_2 = a_2^2 y_p^2 + p_2 + a_0 a_2 + a_1^2/2. \quad (3.2)$$

This system of equations is overdetermined, in that it contains seven nontrivial equations in the six unknowns a_0 , b_0 , a_1 , b_1 , a_2 and b_2 .

For the case $M = 4$, where quartic variation is allowed, a system of 13 equations in 10 unknowns results. The system is similarly overdetermined. In this case, the equations are rather longer and space does not permit their inclusion here.

4. Solution of equations

The problem is, for each of the N values of the $p(t_n)$, take several adjacent values, find the local approximating polynomial by finding the p_j for $j = 0, \dots, M$, and solve the equations to give the a_j, b_j for $j = 0, \dots, M$. This enables a complete local solution for the velocity field to be obtained, from the a_j , and the free surface variation from the b_j .

The solution procedure is as follows.

Step 1: Initially, assume that the disturbances everywhere travel at the speed of linear long waves $c = 1$ ($c = \sqrt{g \times \text{mean depth}}$ in physical terms).

Step 2: For successive iterations until the process converges, repeat Steps 3 to 6. (In the calculations described in Section 5, 3 iterations were sufficient for four-figure accuracy).

Step 3: For each of the times t_n , n from 1 to N , perform Steps 4 and 5.

Step 4 Calculation of p_j for $j = 0, \dots, M$: Consider equation (2.5.2) written for the point at which the pressure probe is located, $x = 0$, $y = y_p$:

$$p(0, y_p, t) = \sum_{j=0}^M p_j (-ct)^j. \quad (4.1)$$

To find the coefficients in this series it would be possible to use simple finite difference methods based on the $M+1$ pressure readings $p(0, y_p, t_m)$ in the vicinity of t_n , for $m = -N/2, \dots, 0, \dots, +N/2$. (Here an ambiguous but convenient notation t_m has been used for discrete values of the local time such that $t_0 = 0$, whereas t_n will continue to be used to refer to instants at which pressure readings have been taken. This use of local time is important in the simplifications which lead to equations (4.2.1) and (4.2.2).) As it is intended that the method be used in practice, where there are likely to be irregularities in the input data, such interpolation of the pressure record is a rather dangerous procedure, as interpolating polynomials fitted to experimental points exhibit wild fluctuations. A better procedure would be to use a least-squares procedure using K points, where K is rather

greater than $M+1$. It can be shown (see standard numerical analysis textbooks) that the least squares procedure leads to the matrix equation

$$\begin{bmatrix} K & \Sigma t_m & \dots \\ \Sigma t_m & \Sigma t_m^2 & \dots \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} p_0 \\ -p_1 c \\ \dots \end{bmatrix} = \begin{bmatrix} \Sigma p(t_m) \\ \Sigma t_m p(t_m) \\ \dots \end{bmatrix}.$$

In this expression all the summations are over K values of m . If approximation by a quartic is used, this matrix equation involves five equations in five unknowns, and the solution is complicated. A useful simplification can be made, however. Here and throughout the rest of this work it will be assumed that the points are obtained at equal intervals of time Δ , K is chosen to be an odd number, and points are distributed symmetrically about the point of calculation, such that m varies from $m = -(K-1)/2$ to $m = +(K-1)/2$. Then all the sums over odd powers of t_m become zero, and the system can be written as two separate systems, one of third-order and the other second-order. With a little re-arrangement they can be written:

$$\begin{bmatrix} K & \Sigma m^2 & \Sigma m^4 \\ \Sigma m^2 & \Sigma m^4 & \Sigma m^6 \\ \Sigma m^4 & \Sigma m^6 & \Sigma m^8 \end{bmatrix} \begin{bmatrix} p_0 \\ p_2 c^2 \Delta^2 \\ p_4 c^4 \Delta^4 \end{bmatrix} = \begin{bmatrix} \Sigma p(t_m) \\ \Sigma m^2 p(t_m) \\ \Sigma m^4 p(t_m) \end{bmatrix}, \tag{4.2.1}$$

and

$$\begin{bmatrix} \Sigma m^2 & \Sigma m^4 \\ \Sigma m^4 & \Sigma m^6 \end{bmatrix} \begin{bmatrix} -p_1 c \Delta \\ -p_3 c^3 \Delta^3 \end{bmatrix} = \begin{bmatrix} \Sigma m p(t_m) \\ \Sigma m^3 p(t_m) \end{bmatrix}. \tag{4.2.2}$$

If quadratic approximation $M = 2$ is used, then the last row and column of the matrices are simply deleted.

The coefficients in the square matrices are simply sums over the powers of the integers, from $m = -(K-1)/2$ to $m = +(K-1)/2$, and analytical expressions can be found for them. However the numerical values are easier to obtain by simple summation than by these complicated formulae, particularly as once the level of approximation M and the number of points K in the least squares fit are decided, the coefficients are simple integer constants for all points of all pressure records. In any case, it is easy to write down the inverses of the matrix equations presented here. That is a standard mathematical task, and will not be done here. Of course, they too are constant, once M and K are adopted.

Although the use of interpolation rather than least squares approximation is not recommended, for that case, where the number of

points is such that $K = M + 1$, all the matrix formulae presented above are still valid, and one is free to use either interpolation or a least-squares fit simply by choosing the value of K .

Having found the coefficients p_j , the equations can now be solved. However, the above formulae are in terms of the speed c with which the local disturbance is travelling. It is not known a priori, and has to be found iteratively, as described in Step 6.

Step 5 Solve for the $a_j, b_j, j = 0, \dots, M$, using Method 5a or 5b below: This is the most complicated part of the solution process. To obtain the "best" solution of an overdetermined system of equations (which can never be satisfied identically because there are insufficient free variables), one procedure is to find the solution which minimises the sum of the squares of the errors. This "least-squares" procedure for the system of nonlinear equations is mentioned in Step 5b below. For the calculations described in this paper, it was found that a very simple method gave excellent results. This is described below.

5a. Direct iteration solution of the equations

In this case, sufficient of the equations were discarded, one in the case of $M = 2$, and three for $M = 4$, to give the same number of unknowns as equations, N say. Newton's method could be used for this system of nonlinear equations, and is known to converge quickly. However, it involves the calculation of a matrix, whose elements are the derivative of every equation with respect to every variable, plus the numerical solution of that matrix equation at each iteration. Fortunately, however, a much simpler method was found to work for the nonlinear system in this work, which gave results which were within 1% of those using the full overdetermined set of equations.

The solution procedure is the nonlinear equivalent of Gauss-Seidel iteration, whereby each equation is written for one unknown, whose influence dominates that equation, and the set of equations evaluated repeatedly until a solution is obtained. As the next value of each variable is calculated, its new value is substituted in subsequent equations. In the calculations it was found that the procedure worked well, both for $M = 2$ and $M = 4$.

Examining the "P" equations in (3.2) it can be seen that, for example, $P_1 = a_1 a_2 y^2 + p_1 + a_0 a_1 = 0$ can be re-arranged to give the equation $a_1^1 = a_1^1 - P_1/a_0^1$. With initial estimates of the coefficients, the right side can be evaluated, and substituted for a_1 . Other equations can be similarly rewritten. The scheme which was used in the present work was to recalculate each of the a_j from the pressure equations and each of the b_j from the dynamic equations, for j from 1 to M . In practice it was found that the order in which this was done was not important, but it is recommended that the procedure work from M to 1, so that the coefficients which are smaller in magnitude are iterated first. The P_0 equation was rewritten in terms of b_0 .

If the kinematic equations are examined, it is not so obvious that any one term dominates, and it was felt that if any equations were to be deleted it should be these. The procedure adopted was, that if

locally the linear contributions were greatest, then $K_1 = 0$ should be used. At such a point, if it were a point of inflection in the pressure record, p_2 would be zero, as would all the a_2 and b_2 so that $K_2 = 0$ would be satisfied identically. Similarly near a crest or trough, the quantities with subscript 1 would be close to zero, and it would be better to use $K_2 = 0$. The criterion adopted was simply whether $|p_1|$ or $|p_2|$ was greater.

Rewriting the kinematic equations as above would give the scheme for K_1 for example, $a_0 = a_0 - K_1/b_1$. In practice, this was found to give an iteration scheme which was occasionally unstable. Luckily, a simple remedy was found to give very reliable scheme. This was to multiply the previous expression by a_0 and to take the square root to give

$$a_0 = -\sqrt{a_0(a_0 - K_1/b_1)}$$
 or $a_0 = -\sqrt{a_0(a_0 - K_2/b_2)}$,
 whether $|p_1|$ or $|p_2|$ respectively was greater. The value of a_0 is negative, and of a similar magnitude to c .

To commence iteration it is desirable to have some estimate of the solution. The following were found to work well in practice:

$$b_0 = p_0 - p_2 p_0^2, \quad a_0 = -\sqrt{b_0}, \quad \text{and} \quad a_j = b_j = p_j, \quad \text{for } j = 1, \dots, M.$$

These are exact for no wave motion. In subsequent passes of the iteration terminating in Step 6, the results from the previous pass were used.

The iteration scheme can be summarised:

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For successive iterations until solution converges
{
  For j=M to 1 in steps of -1
  {
    
$$a_j = a_j - \frac{p_j}{a_0}$$

    
$$b_j = b_j - D_j$$

  }
  If ( $|p_1| > |p_2|$ )  $a_0 = -\sqrt{a_0(a_0 - K_1/b_1)}$ 
  Else  $a_0 = -\sqrt{a_0(a_0 - K_2/b_2)}$ 
  
$$b_0 = b_0 + p_0$$

}
    
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In practice it was found that convergence of this scheme was sure and rapid. Four iterations were enough to have the solution converge to four significant figures.

5b. Newtonian iteration for overdetermined systems

A more complicated but theoretically-sounder method than that described in Step 5a is to obtain that solution which minimises the errors of the complete set of equations, rather than identically satisfying some of those equations. The method is much more complicated and

takes more computational time. It was implemented by the author, but was found to give results which were little different from those obtained from the method of Step 5a. Comparisons are not presented here. It is recommended that the simpler method be used in practice.

Step 6 Recalculate the speed c: Having processed all the points $n = 1, \dots, N$, excluding those at the beginning and end of the pressure record which cannot be treated by scheme (4.2), it is now possible to estimate the speed of propagation of the disturbances. Locally, there is no information provided by the act of measuring the pressures, and it is necessary to make rather ad hoc assumptions.

From equation (2.1), it can be easily shown that the horizontal fluid velocity in the frame of the moving disturbance, at $x = 0$ and $t = 0$ is $a_0 - a_2 y^2 + a_4 y^4$, hence the fluid velocity in the frame of reference of a stationary observer is

$$u(0, y, t_n) = c + a_0 - a_2 y^2 + a_4 y^4. \quad (4.3)$$

This provides us with a means of calculating the speed of propagation c , for if the mean current (fluid velocity) at a point of elevation y is known, denoted by \bar{u} , this is the mean of u over all the t_n , $n = 1, \dots, N$. (In the absence of any knowledge of \bar{u} , a value of zero would be a reasonable approximation.) Substituting into equation (4.3) we have

$$c = \bar{u} - \bar{a}_0 + \bar{a}_2 y^2 - \bar{a}_4 y^4, \quad (4.4)$$

where the \bar{a}_j denotes the mean of the a_j averaged over all the n . Clearly the result shows that the value of c obtained depends upon the value of y at which the expression is evaluated. However, as the a_2 and a_4 express fluctuations of velocity about a mean, in general their means will be small and one would expect the deduced value of c to show little variation with y .

In most cases it might be rather more realistic to use the depth-integrated value of the current, denoted by \bar{U} . By integrating (4.3) from 0 to b_0 , the surface elevation at time t_n , and taking the ensemble mean over all the n , then

$$c = \bar{U} - \bar{a}_0 + \frac{\bar{a}_2}{3} \bar{b}_0^2 - \frac{\bar{a}_4}{5} \bar{b}_0^4 \quad (4.5)$$

With the new value of c as obtained in this step, return to Step 2, unless the process has converged. If it has converged, then

Step 7 Solution: The most useful results are the surface elevations, represented by the b_0 , and the fluid velocity, given by

$$u = c + a_0 + a_1(x-ct) + a_2((x-ct)^2 - y^2) + a_3((x-ct)^3 - 3(x-ct)y^2) + a_4((x-ct)^4 - 6(x-ct)^2 y^2 + y^4),$$

and

$$v = -a_1 y - 2a_2(x-ct)y - a_3(3(x-ct)^2 y - y^3) - a_4(4(x-ct)^3 y - 4(x-ct)y^3).$$

If it were desired, now a Fourier transform of the surface profile could be taken, to obtain the spectral data.

5. Results

Comparisons were made with accurate numerical solutions of steadily-progressing waves in water of constant depth. Solutions were obtained using a Fourier approximation method (Rienecker & Fenton, 1981; Fenton, 1987), giving values of pressure on the sea bed at equally-spaced intervals and the corresponding surface elevations. Using the pressure values the method of Section 4 above was applied. For both quadratic and quartic approximations 7 adjacent points were used to fit the polynomials at each point, that is, $K = 7$.

The family of steady waves is a two-parameter one. To obtain an idea of the accuracy of the local polynomial approximation method, a traverse through this two-dimensional space was made, considering four waves. The height/depth increased from 0.25, 0.333, 0.5, to 0.667 while the wavelength/depth simultaneously was 3, 5, 10 to 15. It would be expected that the polynomial method would work best for long waves, while for shorter waves the variation with y tends to exponential and it would not be appropriate.

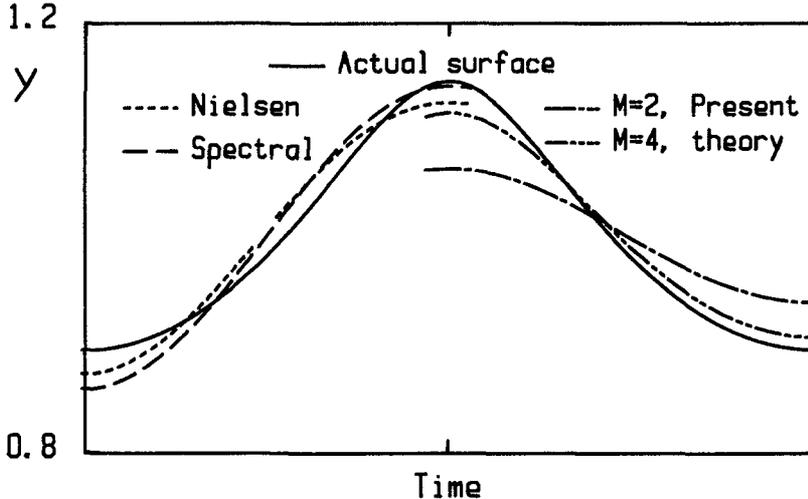


Figure 1. Actual and calculated free surface of a steady wave of height 1/4 of the depth and a length 3 times the depth.

Results are shown in Figures 1 to 4. On each figure the free surface obtained from the numerical solutions is shown as a solid line. There

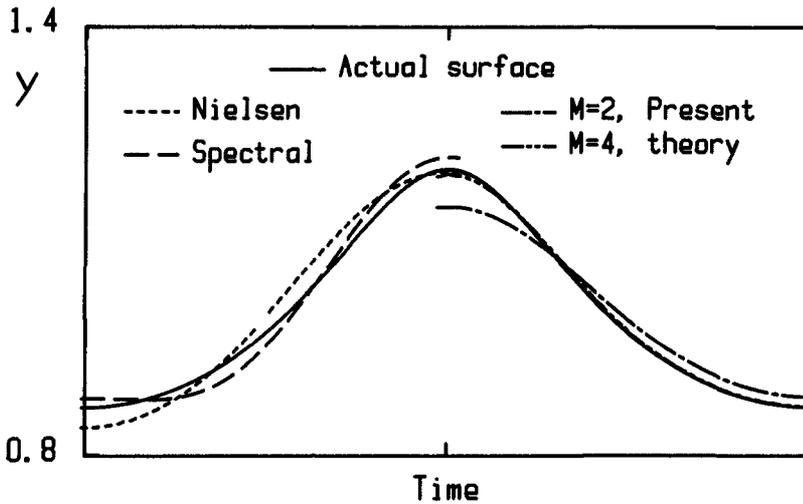


Figure 2. Actual and calculated free surface of a steady wave of height $1/3$ of the depth and length 5 times the depth.

are two curves plotted on each half of the wave, plotted according to four methods of inferring the free surface from the pressure results. The actual pressure record is not shown. On the left half of the wave is shown two sets of results from linear theory. The long dashed line shows the results of applying linear wave theory using the full spectral representation. The resulting Fourier series was truncated at the last term where the successive contributions were still decreasing. After that point the results would be meaningless. The short dashed line shows results obtained from the simple empirical approximation of Nielsen (1986), able to be presented in a single formula. On the right half of the wave are shown results from the present theory, for the quadratic ($M = 2$) and quartic ($M = 4$) approximations.

It can be seen that the use of linear wave theory and a full spectral representation gives poor results for the longer and higher waves. Nielsen's local approximation method, based on fitting of a trigonometric function to part of the pressure signal is capable of good accuracy near the crest and trough of the waves. In fact, the agreement at the crests is remarkably good, considering the approximations implicit in the method. At the wave crests, the method is usually more accurate than the polynomial approximation method. Over part of the wave, however, near the inflection points, the fitting of a sinusoid breaks down, and the method gives wildly divergent results or none at all.

The results for the polynomial approximation method are shown on the right sides of the figures. For a relatively short wave (3 times the depth) the polynomial approximation is poor (Figure 1), as expected. It would be necessary to include higher order approximation

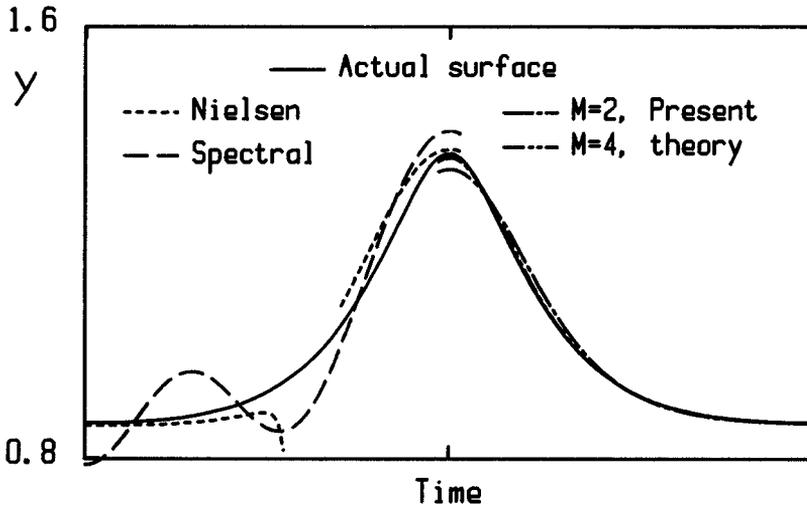


Figure 3. Actual and calculated free surface of a steady wave of height $1/2$ of the depth and length 10 times the depth.

for waves this short. The situation changes quickly as longer waves are considered. For a wave which is only 5 times the water depth, Figure 2 shows that the present method is capable of high accuracy, but that it is necessary to use quartic approximation. As longer waves are considered, the quadratic approximation becomes more acceptable. Figure 3 shows a wave 10 times as long as the depth with a height of 0.5 of the depth. It is clear that the quartic approximation is excellent, usually being obscured by the surface on the plot of this scale. Finally, Figure 4 shows a very demanding case, of a wave which is 15 times the water depth in length, and a height of $2/3$ of the depth. This is probably close to breaking. Except at the very crest, the $M = 4$ results are excellent, with the $M = 2$ results slightly less so. Altogether it seems that the method proposed here, with quartic approximation, is capable of high accuracy for waves up to close to the breaking point, provided that they are no shorter than about 5 times the water depth. This range of accuracy is very much greater than that of conventional wave theories and suggests that the local polynomial method, based on no approximations other than the truncation at finite degree, is capable of accurate description of steady water waves.

Peter Nielsen has kindly made available a set of his experimental results so that an indication of the robustness of the method in practice might be had. Waves were generated in a laboratory flume, and at a station two wave gauges were used to measure the free surface elevations and a pressure probe was located on the bottom. A comparison of the different methods and experimental results is given in Figure 5. The wave is high, about half the mean depth, and has the sharp crest and flat trough associated with long waves. It can be seen that Nielsen's local trigonometric approximation method works rather better

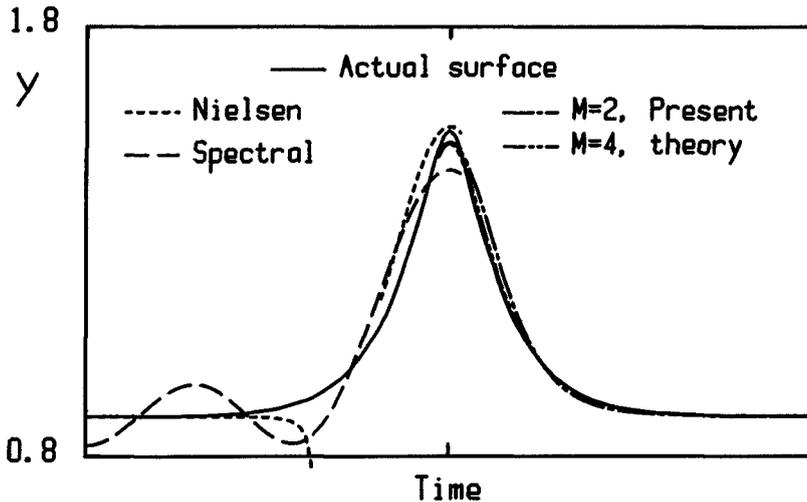


Figure 4. Actual and calculated free surface of a steady wave of height $2/3$ of the depth and length 15 times the depth.

than the $M = 2$ approximation of the polynomial theory near the crest of the wave. However, the $M = 4$ level of the theory seems to be very accurate indeed. Almost everywhere the double chain-dashed line is obscured by the two lines corresponding to the wave gauge results, even at the crest. In this case, however, as the experimental results showed the expected small irregularities, it was found necessary to use the least-squares method with 13 points. For any fewer points it was found that the irregular data was such that the method of solving the equations did not converge. With 13 or more points, however, it was robust and accurate.

6. Conclusions

It has been shown that relatively low degrees of polynomial approximation can describe waves accurately in water of finite depth. This has the potential of allowing solution of wave problems by polynomial means, which allows the use of the full nonlinear boundary conditions and which may allow the incorporation of other more-general geometries.

For the problem considered here, the method has provided a technique for obtaining free surface data from pressure data, which obviates the need for Fourier transform programs and has important computational advantages over traditional methods based on Fourier analysis.

The examples presented here have not demonstrated the accuracy and robustness of the proposed method over a wide range of waves and under field conditions, however, they do suggest that the method could be used in a variety of applications.

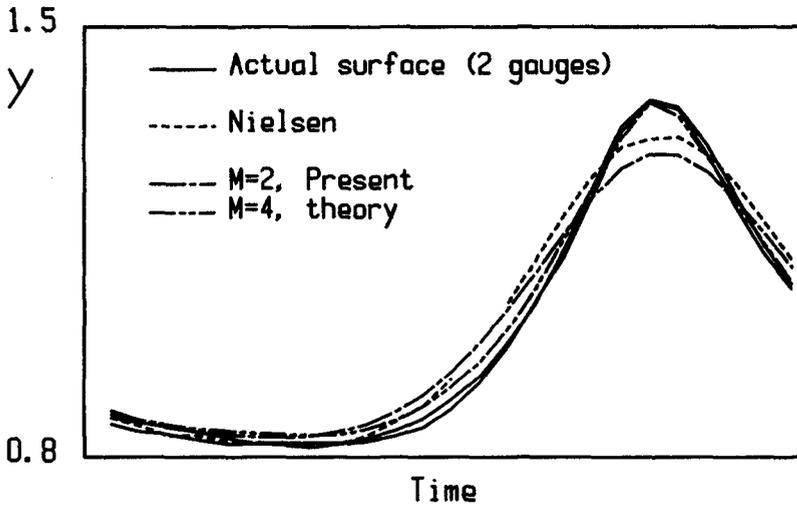


Figure 5. Experimental and computational results for laboratory wave

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