WAVE FORCES ON VERTICAL WALLS

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ABSTRACT

Formulae are given for the force exerted on vertical walls by the reflection of water waves with an arbitrary angle of incidence. The variation of the loads with all design variables show a number of unusual features, including the fact that the maximum force per unit length can be caused by obliquely-incident waves rather than standing waves. It is important for design that the whole range of possible wave conditions be considered. A method is developed for the numerical solution of the problem, which unlike the theory on which the above-mentioned formulae are based, solves the stated problem exactly. Results from the approximate formulae are compared with those from the numerical method, and are found to be surprisingly accurate over a wide range of wave conditions.

INTRODUCTION

Despite the plethora of solutions for the problem of standing waves, or the reflection of waves by walls, relatively little attention has been given to the analytical determination of the loads exerted by the waves on the walls, a marine problem of some importance. Battjes (1982) used linear theory to obtain expressions for the loading on long vertical-walled structures produced by short-crested waves with an assumed distribution of incidence angle. Kachoyan & McKee (1986) have developed a general method, also based on linear theory, for calculating the forces exerted by normally-incident waves on walls of steep but otherwise arbitrary profile.

It would seem obvious that the largest forces on walls would be caused by waves which approach the wall normally, with crests parallel to the wall, setting up a standing wave system. However Kuznetzov, in an unpublished work outlined by Silvester (1974), reported the surprising phenomenon that forces due to obliquely-incident waves can exceed considerably those due to normally-incident waves. Higher-order solutions of the problem of the reflection of periodic obliquely-incident waves by a vertical wall have been obtained in recent years. Hsu, Tsuchiya & Silvester (1979 - subsequently referred to here as HT&S) obtained a solution to third order, via a Stokes-type of theory, in which spatial variation in the two horizontal directions was represented by a double Fourier series, the coefficients of the series being given as expansions in a quantity related to the ratio of wave height to length. HT&S presented third-order expansions for the velocity

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potential, the free surface elevation, and pressures within the fluid. This procedure was extended by Roberts (1983), who used computer manipulation of the series to obtain solutions to 27th order. He presented some results for gross properties of the wave system, such as frequency and energies, as well as for the free surface. It was found that a phenomenon of resonant interactions between the fundamental components and their harmonics led to complications in the use of the perturbation expansions.

A different approach was adopted by Roberts & Schwartz (1983). Instead of obtaining the coefficients in the Fourier series by computer manipulation of perturbation expansions, they obtained the coefficients numerically by solving the system of nonlinear algebraic equations which resulted by substituting the Fourier series into the nonlinear boundary conditions at a finite number of points.

Fenton (1985) examined HT&S's third order solution and applied a numerical test to show that the solution was correct to third order, with the exception of some higher order terms for the fluid pressure. The solution was then recast in terms of the wave height/length ratio itself. The solution for the pressure was obtained, and shown numerically to be correct to third order. Then, the pressure was integrated to give third-order formulae for the force and moment on a vertical wall, which could be used for design purposes. The expressions so obtained showed several unusual features: the maximum force per unit length on the wall was found to be caused by obliquely-incident waves, rather than standing waves, as found by Kuznetzov; the second-order contribution to the load may be larger than that at first order without the solution being invalidated; the greatest net force is that directed offshore under the wave troughs; and that the greatest onshore force does not necessarily occur under wave crests. It was shown that, despite the apparent convenience of explicit formulae for wave loads, that the problem of determining the maximum load for design purposes is one of finding that maximum in a space of variables which includes the wave height, wave length or period, angle of incidence, and the wall length relative to the wave length.

It is the main aim of this paper to test the expressions obtained by Fenton, by using a numerical solution of the problem, similar to that of Roberts & Schwartz, which makes no essential analytical approximations other than truncation of the Fourier series. A couple of innovations make the method more accurate and less demanding of computer time than the previous one. From these numerical solutions, expressions for the wave loads are obtained, and these numerical results compared with the theoretical expressions and with experimental results. The theoretical expressions are shown to be accurate over a wide range of wave conditions.

FORMULATION OF THE PROBLEM

Consider a layer of fluid bounded below by an impermeable bed and on one side by a vertical wall. The coordinate origin is at the wall, such that it is at the mean level of the free surface; the x coordinate is along the wall, the y coordinate is normal to the

wall into the fluid, and the z coordinate is vertically upwards. The equation of the sea floor is $z = -d$, where d is the mean depth, and the equation of the wall is $y = 0$. It is assumed that the fluid flow is incompressible and irrotational, such that a velocity potential ϕ exists, where the velocity u is given by $u = \nabla\phi$ and that there is no flow through the floor or the wall:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \text{ throughout the fluid,} \quad (1)$$

$$\phi_z = 0 \text{ on the bed, } z = -d, \text{ and} \quad (2)$$

$$\phi_y = 0 \text{ on the wall, } y = 0, \quad (3)$$

where the subscripts denote partial differentiation. For the case of the total reflection of a periodic wave train obliquely-incident on the wall, such that all motion is periodic in time and in both x and y directions, a solution for ϕ can be assumed

$$\frac{\phi(x,y,z,t)}{(g/k^3)^{1/2}} = (gk)^{1/2}\,\phi_{00}\,t + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\phi_{ij}\sin i(mkx-\omega t)\cos jnky\,\frac{\cosh\alpha_{ij}k(z+d)}{\cosh\alpha_{ij}kd}, \quad (4)$$

in which the quantities used to non-dimensionalise where necessary are the gravitational acceleration g, and the wave number of the incident waves $k = 2\pi/L$, where L is the wave length. The ϕ_{ij} are dimensionless coefficients of the double Fourier series shown, $m = \sin\theta$ and $n = \cos\theta$ where θ is the angle of incidence ($\theta = 0$ for standing waves and $\theta = 90^\circ$ for waves progressing steadily parallel to the wall), ω is the angular frequency of the motion, $\omega = 2\pi/T$ where T is the period, and the α_{ij} are dimensionless coefficients,

$$\alpha_{ij}^2 = i^2m^2 + j^2n^2.$$

It can easily be verified that (4) satisfies equations (1), (2) and (3).

The remaining equations to be satisfied are those on the free surface: the dynamic and kinematic conditions such that the pressure on the surface is zero and that fluid particles on the surface $z = \eta(x,y,t)$ remain on the surface. Respectively, the surface equations are

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \quad \text{on } z = \eta, \quad (5)$$

and,

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \quad \text{on } z = \eta, \quad (6)$$

where the subscripts denote partial differentiation. An expansion for the surface elevation, similar to (4), is assumed:

$$k\eta(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\eta_{ij}\cos i(mkx-\omega t)\cos jnky \quad (7)$$

in which the η_{ij} are dimensionless coefficients, and where $\eta_{00} = 0$ because the origin is located at the mean level. In the Stokes type of solution as used by HT&S, the coefficients ϕ_{ij} and η_{ij} are found as asymptotic expansions in a quantity related to the wave steepness $\delta = kH_s/2 = \pi H_s/L$, the coefficients in the expansions being given functions of the dimensionless mean depth d/L and the angle of incidence θ. HT&S obtained the coefficients to third order in the wave height. From these solutions, they obtained expressions for the pressure p(x,y,z,t) at any point in the fluid.

FORMULAE FOR WAVE FORCES

The expressions obtained by HT&S were converted by Fenton (1985) into expansions in terms of the dimensionless wave height δ itself, and the expressions for pressure at y=0 were integrated with respect to z from -d to η to give formulae for the force and moment about the base of the wall per unit length along the wall. The expressions obtained for P_*, the net force per unit length due to the waves (total force/unit length = $P_* + \rho g d^2/2$, ρ the fluid density) are:

$$\frac{P_*(x,t)}{\rho g d^2} = \delta F_{11}\cos(mkx-\omega t) + \delta^2(F_{20} + F_{22}\cos 2(mkx-\omega t)) + \frac{1}{2}\delta^3(F_{31}\cos(mkx-\omega t) + F_{33}\cos 3(mkx-\omega t)), \tag{8}$$

where the fourth and higher order terms have been neglected, and where the coefficients are given by:

$$(kd)^2F_{11} = \omega_0^2 = \tanh kd,$$

$$(kd)^2F_{20} = \frac{1}{4} + \omega_0\beta_1(-kd(1+n^2) + \frac{1}{2}(1+m^2)\sinh 2kd),$$

$$(kd)^2F_{22} = \frac{1}{4} + \omega_0\beta_1(kd(1+m^2) - \frac{1}{2}n^2\sinh 2kd) + \omega_0\beta_2\sinh 2kd + \frac{\omega_0\beta_3}{m}\sinh 2mkd,$$

$$\frac{(kd)^2}{\omega_0}F_{31} = \frac{3}{4}\omega_0 + \omega_2 - \omega_0 b_4 + \beta_1(-3n^2 + (1+3m^2)\cosh 2kd) + \beta_2(2\cosh 2kd + n^2 - \frac{1}{3}(1+m^2)\frac{\sinh 3kd}{\sinh kd}) + \beta_3[2\cosh 2mkd + \frac{2m}{4m^2-1}(\frac{(1-2m^2)\sinh 2mkd}{\omega_0^2} - m\cosh 2mkd)] + \beta_{13}\frac{\sinh\gamma_1 kd}{\gamma_1},$$

$$\frac{(kd)^2}{\omega_0}F_{33} = \frac{1}{4}\omega_0 + \beta_1(1+m^2-n^2\cosh 2kd) + \beta_2(2\cosh 2kd - 1 - m^2 + \frac{1}{3}n^2\frac{\sinh 3kd}{\sinh kd}) + \beta_3[2\cosh 2mkd + \frac{2m}{4m^2-1}(3m\cosh 2mkd-(1+2m^2)\frac{\sinh 2mkd}{\omega_0^2})] + 3\beta_{31}\frac{\sinh\gamma_3 kd}{\gamma_3} + \beta_{33}\sinh 3kd. \tag{9}$$

In these expressions, formulae for the coefficients ω_0, β_i and γ_i for i=1,2,3 are given by HT&S. In addition to (8) and (9), Fenton gave similar expressions for the net moment per unit length of wall due to the waves. Here, attention will be limited to the force only.

These formulae for the loads exhibit several phenomena, some of which are quite unexpected, which are all due to one particular term in the second order contribution to the pressure, where for small angles of incidence, the pressure at second order decays very slowly with depth in the fluid, unlike the first order terms. This can mean that, integrated over the depth, the second order contributions to the force can become as large or greater than those at first order. The fact that the troublesome term is of negative sign has some interesting consequences: the greatest onshore forces due to the waves are not due to normally-incident waves, but obliquely-incident waves, that the net forces which are greatest in magnitude are those under the wave troughs, directed offshore. Another consequence is that the force as a function of time has such a large secondary component that it may not have a local maximum at the crest, which is instead a local minimum, but at points on either side of the crest. The existence of all these phenomena is demonstrated in Figure 1, showing for given water depth and wave height and length, how the forces at the wave trough, crest and intermediate maximum if any, vary with the angle of incidence and how they compare with each other in magnitude.

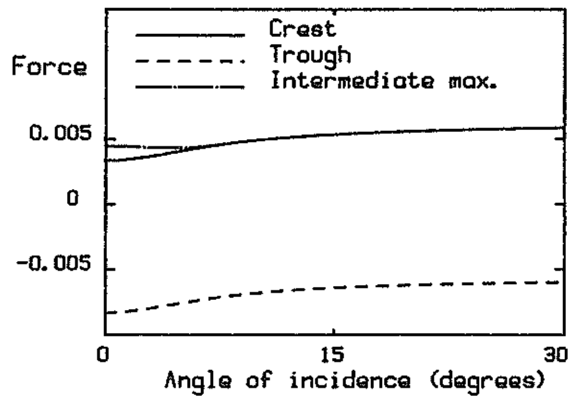

Figure 1. Variation of the force per unit length with angle of incidence for each of three parts of the wave, (i) for the crest $\chi = mkx-\omega t = 0$, (ii) for the trough $\chi = \pi$, and (iii) for the intermediate maximum. Values of the force plotted are $P_*(\chi)/\rho g d^2$. The whole diagram is for the case $H_s/L = 0.05$, $L/d = 1$.

In the original paper by Fenton (1985) it was shown that expressions (8) and (9) gave quite reasonable agreement with experimental results over a wide range of wave conditions. However, it was emphasized that the expressions might be inaccurate for waves which are high or long. That high waves should not be well described by the theory is to be expected, as it is a theory in terms of an asymptotic expansion in wave height, which would not be expected to be valid for high waves. The breakdown in the other limit, that of long waves, is not immediately obvious, but was justified mathematically by the author. In the limit of long waves, the effective expansion parameter becomes, not the nominal H_s/L but $H_s L^2/d^3$, and for long waves as L/d becomes large, this cannot remain small. In view of these possible limitations to the theory, it was considered important to test the formulae given, by solving the wave problem using a method which required no essential analytical approximation, and to compare the approximate theory with the results so obtained.

NUMERICAL SOLUTION BY FINITE FOURIER SERIES

The series (4) and (7) can be substituted into the nonlinear boundary conditions (5) and (6), and for a particular wave system, solved numerically to give values of the Fourier coefficients. To give a finite system, it is necessary to truncate the series in (4) and (7) at some finite integer N, and then solve enough equations so that the coefficients as far as ϕ_{NN} and η_{NN} can be found.

Roberts & Schwartz (1983) used this procedure, and showed that a number of symmetries exist, such that half the coefficients are identically zero: $\phi_{ij} = \eta_{ij} = 0$ when i+j is odd. They then satisfied (5) and (6) at a number of points on a square grid, within one of the basic symmetry triangles, and used a "pseudo-Newtonian" iteration method to solve for the Fourier coefficients.

In the present work, the procedure of Roberts & Schwartz was used initially, but because the equations so obtained were often ill-conditioned, an alternative procedure was devised which was more robust numerically. Instead of obtaining equations by satisfying (5) and (6) at a finite number of points, thus using a collocation method, the discrete sine and/or cosine transformations of those equations over a rectangular grid were obtained and required to be zero, which gave a system of equations which were much more robust. This idea is due to Bryant, who adopted the approach in a sequence of papers (see, for example, Bryant (1982)).

Further details of the method adopted are rather lengthy, and will not be presented here. The results of the numerical solution are, for a particular wave, numerical values of the coefficients ϕ_{ij} and η_{ij} and ω.

CALCULATION OF LOADS ON WALL

Consider the pressure equation for the pressure at any point within the fluid:

$$\frac{p}{\rho}(x,y,z,t) = -gz - \frac{\partial\phi}{\partial t} - \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right),$$

where ρ is the fluid density. The dynamic surface boundary condition was obtained from this condition, where p = 0 on $z = \eta$. If the Fourier coefficients have been found, as described above, then the series (4) and (7) can be substituted, and the pressure found at any point. On the wall, where y = 0, the integral of the pressure from z = -d to $z = \eta$ gives the force per unit length. Substituting and performing the integration gives:

$$\frac{P_*(x,t)}{\rho g d^2} = -\frac{1}{2}(k\eta)^2 - \phi_{00}(k\eta + kd)$$

$$+ \omega(gk)^{1/2} \sum_{i=1}^{N} \sum_{j=0}^{N} \frac{i\,\phi_{ij}}{\cosh \alpha_{ij} d} \cos i\chi \, Q(\alpha_{ij})$$

$$- \frac{1}{4} \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{I=0}^{N} \sum_{J=0}^{N} \frac{\phi_{ij}\,\phi_{IJ}}{\cosh \alpha_{ij} d \cosh \alpha_{IJ} d} \times$$

$$[m^2 iI \cos i\chi \cos I\chi \,(Q(\alpha_{ij}+\alpha_{IJ}) + Q(\alpha_{ij}-\alpha_{IJ})) +$$

$$\alpha_{ij}\alpha_{IJ} \sin i\chi \sin I\chi \,(Q(\alpha_{ij}+\alpha_{IJ}) - Q(\alpha_{ij}-\alpha_{IJ}))],$$

in which $Q(\alpha) = \sinh \alpha(d+\eta) / \alpha$. In the case $\alpha = 0$, $Q(\alpha)$ takes the limiting value $d+\eta$.

The evaluation of this expression is a time-consuming process. To enable the analytical integration to be performed in its derivation, it was necessary to convert the product of two series, each of order N^2 terms, into a quadruple series of order N^4 terms, giving a corresponding increase in computational cost. The process of finding the maximum force on the wall exerted by a particular wave is a one-dimensional optimisation problem, for the variation of P_* as a function of x and t is a function of them combined only in the form $\chi = mkx - \omega t$. In the present work, a procedure was adopted where an estimate of the value of χ_m, the phase where the force takes on a maximum, was obtained from a formula given by Fenton (1985), and a simple search procedure followed to find the maximum force according to the numerical solution. Often $\chi_m = 0$, such that the maximum coincides with the wave crest.

COMPARISON WITH EXPERIMENT

Nagai (1969) reported the results of many experiments on standing waves. They were performed over a wide range of wave lengths and wave heights. The waves varied in length from 1.15 times the depth (effectively deep water) to 15 times the depth (effectively shallow water). The range of wave heights was from $H_s/L = 0.022$ to 0.182, some even breaking. Forces on the wall were found by measuring pressure at some 10 points, integrating numerically, and subtracting the hydrostatic component. The simulation of the standing wave problem by Nagai is open to some doubt, as the waves used were set up by generating a periodic wave train, which was reflected by the wall, the pressures being obtained from three waves between the first and the tenth. There seems to have been no attempt to set up a periodic standing wave.

Figure 2 contains a comparison between the maximum force predicted by the numerical method described above, the value given by equations (8) and (9) from the third-order theory, and the experimental values measured by Nagai. If there is any standard for comparison, it is the results from the numerical method, which has the capability of solving the idealized irrotational incompressible periodic standing wave problem almost exactly. The first figure compares the experimental values with those of the numerical method. It can be seen that agreement generally is good, considering the wide range of conditions over which the experiments were performed, the difficulty of performing such experiments, and the fact that the experiments were not on a perfectly periodic standing wave system. It can be seen that there is a consistent trend in the results for the experimental values to deviate from those obtained numerically for higher waves. This almost certainly has a simple explanation, that in the experiments, only the height H of the incident wave train was measured, and it was asserted that the standing wave heights "were approximately 2H". As no other values for H_s were provided, the value of 2H was adopted as the basis for the numerical solutions in this work. This is probably incorrect at third order. The actual H_s used in the experiments would be given by an expression like $H_s = 2H(1 + a_1H^2 + \ldots)$, where a_1 is independent of H, so that the approximation $H_s = 2H$ would be accurate for small waves, and increasingly inaccurate for larger waves, as suggested by the trend in the top figure. Over all the points, however, agreement between the "exact" numerical solution and experiment is good.

The next figure, whose production was the main aim of this work, compares the numerical results with the theoretical third-order expressions (8) and (9), using Nagai's data set. It can be seen that the 122 points, most of which cannot be distinguished from each other, fall very close to the line of agreement, over the wide range of wave lengths and heights used in Nagai's experiments. The close agreement, considering that the approximate theory is only of third order is quite remarkable. This is particularly so, given that the breakdown of Stokes-type theories in shallow water is well known, and would have been expected here as well. Indeed, almost all the points which deviate noticeably from the line of agreement are for long waves. Even though some of the waves were of a length some 15 times the water depth, the theory is still accurate to within some 10%.

CONCLUSIONS

A numerical method was developed for the accurate solution of the problem of the reflection of periodic waves by a vertical wall, and was used to calculate the maximum force on the wall. The results were compared with an explicit expression for the force based on a third-order theory. That expression was found to be highly accurate over a wide range of wave conditions and to agree well with experimental results. It could be used routinely for practical purposes with some degree of confidence.

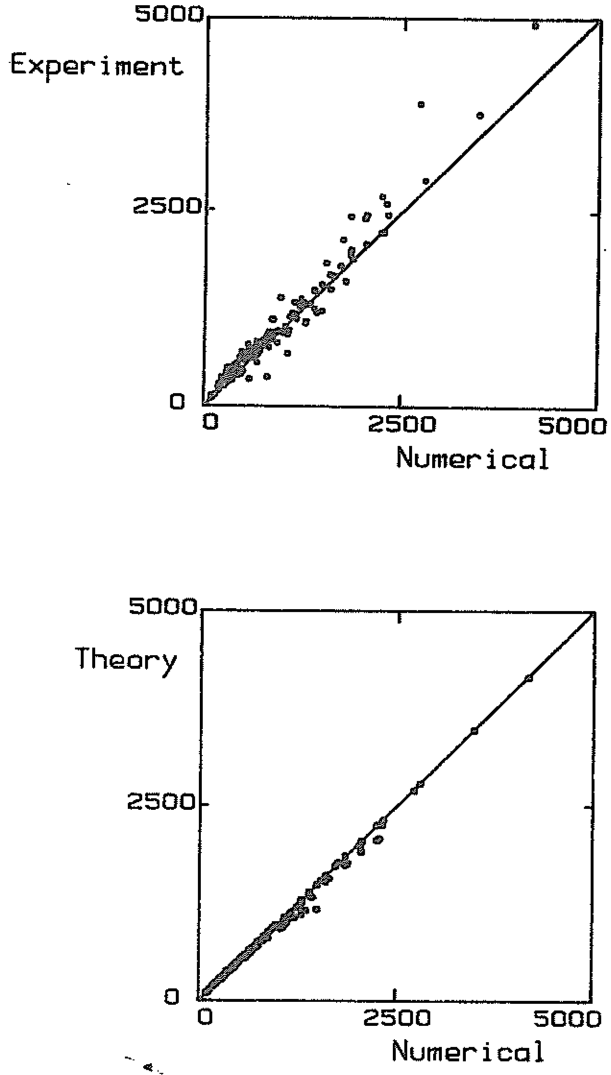

Figure 2. Maximum force on a wall due to standing waves, in gm.wt./cm.; the Experimental values are taken from Table of Nagai (1969), the "Numerical" values for those cases from the numerical method described above, and the "Theoretical" value from equations (8) & (9).

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