Wave Forces on Vertical Walls

By John D. Fenton[1]

Abstract: Formulas are presented to third order in wave height for the force and moment exerted on a vertical wall by the complete reflection of waves with an arbitrary angle of incidence. These expressions show a number of unusual features, some of which have been found previously for the special case of standing waves. They include the following: the maximum force per unit length is caused by obliquely-incident waves rather than standing waves; the second-order contribution to the load may be larger than that at first order without invalidating the solution; the greatest net force is that directed offshore under the wave troughs; and the greatest onshore force sometimes does not occur under wave crests. The formulas presented make the problem of determining the maximum load for design purposes one of finding the maximum of a given function in a space which includes as its dimensions the wave height, wave length or period, angle of incidence, and the wall length relative to the wavelength.

Introduction

An important marine problem is the determination of the forces exerted on a vertical wall when waves are reflected by the wall. There seem to be, however, no explicit higher-order formulas for the forces and moments on the wall. Most work on this problem has been for the case where the waves approach the wall with their crests parallel to the wall, so that on reflection, a standing wave system is set up. While it might be thought that this would give the maximum load on the wall, there is some evidence that the most extreme pressures are caused by waves which are obliquely-incident on, and reflected by the wall. The main aim of this paper is to produce formulas for the force and moment on the wall, and to make recommendations about the determination of wave loads for practical problems.

The standing wave problem, where the angle of incidence of the waves is zero, has been solved to third order by Tadjbakhsh and Keller (15) and to fourth order by Goda (6), through Stokes-type solutions in which the coefficients of a double Fourier series in space and time are found by expressing them as a perturbation expansion in a parameter related to wave steepness, and then solving by use of Taylor expansions of the boundary conditions about the undisturbed free surface. Goda examined in some detail the results for the pressure in the fluid, and compared them with experiment. As with Miche [see, e.g., (8)] and Rundgren (13) it was found for sufficiently deep water and high waves, that the second-order second-harmonic terms can play an important role in the solution, and that maximum pressures do not always occur under the wave crests, but sometimes at points adjacent to them. Goda found that the

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fourth-order theory agreed well with experiment over a wide range of conditions.

Tsuchiya and Yamaguchi (16) made a comparison of the various orders of solution with experiment, and found that first and second-order theories had quite limited regions of validity, but that third and fourth-order theories were accurate over a considerably larger region. Nagai (10) made comparisons between experiment and irrational theories, and found that the theories were invalid. For severe conditions, such as for high waves, and, surprisingly, for deep water, it was observed that the rational Stokes-type theories also gave poor agreement with experiment.

For the case where waves approach the wall obliquely, when the wave crest strikes the wall, it is reflected obliquely, and the resulting wave system is that of short-crested waves, the intersecting wave crests forming a doubly-periodic diamond-shaped pattern when viewed from above. In one limit, of normal approach to the wall, a pure standing wave is set up, and in the other limit—of travel parallel to the wall—the waves are steady waves of translation.

The conventional theoretical approach to the solution of the short-crested wave problem has been via an essentially Stokes-type theory, most suited to deeper water [although Bryant (2) has obtained solutions for the shallow water limit correct to first order in wave height/water depth]. Chappelear (3) and Hsu, Tsuchiya and Silvester (7) have obtained third-order solutions. Roberts (11) and Roberts and Schwartz (12) have studied the problem in considerable detail. They used a Fourier approximation method to solve numerically, then computer manipulation of the series to obtain high-order solutions. It was found that a phenomenon of sub-harmonic resonance occurs for an infinity of values of the angle of incidence, with the result that the series are not convergent. This is unimportant in practice, however, and does not present a problem for the third-order solution.

The preceding papers did not concern themselves with the loads exerted on a vertical wall by a short-crested wave system; perhaps, the quantity associated with this problem is of greatest interest to engineers. Battjes (1), however, has produced expressions based on linear theory for the forces produced by short-crested waves with an assumed angular distribution incident on long structures. Of some ominous import is the unpublished work of Kuznetzov, outlined by Silvester (14), who reported the surprising phenomenon that forces due to obliquely-incident waves can exceed, considerably, those for normal approach.

In this paper, a procedure is given for the determination of the loads (force and moment) exerted on a wall by a short-crested wave system, including standing waves, based on third-order theory. An examination of Hsu, Tsuchiya and Silvester's solution is made, and it is found to be correct to third order, except for expressions for the pressure in the fluid. Their solution is recast to give a third-order solution in terms of the wave height, which makes the first step in the application of the theory simpler if the wave period, rather than the wavelength, is specified. Expressions are given for the fluid pressure, and the whole solution is tested numerically and shown to be correct to third order. The expressions for the pressure are then integrated to give formulas for the force and moment per unit length of wall, and these are, in turn, integrated to give expressions for the loads on a finite length of wall. It is shown that at second order, these expressions are able to describe several phenomena, including the fact that obliquely-incident waves can exert larger forces than standing waves, first-order theory is sometimes grossly in error, maximum onshore forces do not always occur under the wave crest, offshore forces under the trough are usually greater anyway, and the highest wave is not always that which exerts the greatest force. The determination of the maximum force on the wall for design purposes is shown to require the optimization of a function in four-dimensional space. The formulas presented are compared with experiment, and generally good agreement is found, except for very long waves. A critique of the theory and its application to design is given, with a review of when it is not reasonable to use the theory at the level presented.

Solution of Hsu, Tsuchiya and Silvester

The notation used throughout this paper will be that of Hsu, Tsuchiya and Silvester (7), subsequently referred to as HT&S. This work relies extensively on that paper, and reference can be made to it for a diagram of the physical situation, and for many of the formulas used. They are long and complicated, and need not be repeated here.

Consider a layer of fluid bounded below by an impermeable bed, and on one side by a wall. The coordinate origin is at the wall, such that it is at the mean level of the free surface; the x coordinate is along the wall; the y coordinate is normal to the wall into the fluid; and the z coordinate is vertically upwards. The equation of the sea floor is $z = -d$, in which d = mean depth; and the equation of the wall is $y = 0$. It is assumed that the fluid flow is incompressible and irrotational, and that there is no flow through the floor or the wall. The boundary conditions on the free surface are the dynamic and kinematic conditions such that the pressure on the surface is zero and the fluid particles on the surface remain on the surface.

HT&S obtained a solution to this problem for the case of a short-crested wave system, in which two trains of periodic waves can be identified, each of the same height and length, one propagating towards the wall, and the other away. The incident and reflected waves each have a wavenumber, $k = 2\pi/L$, in which L = wavelength. The solution to the problem comprises the following equations:

$$\phi\left(\frac{k^3}{g}\right)^{1/2} = \epsilon \sin(mkx - \sigma t)\left[\omega_0 \frac{\cosh k(z + d)}{\sinh kd} \cos nky + \frac{1}{2}\epsilon^2\beta_{13} \cosh \gamma_1 k(z + d) \cos 3nky\right] + \epsilon^2 \sin 2(mkx - \sigma t)[\beta_2 \cosh 2k(z + d) \cos 2nky + \beta_3 \cosh 2mk(z + d)] + \frac{1}{2}\epsilon_3 \sin 3(mkx - \sigma t)[\beta_{31} \cosh \gamma_3 k(z + d) \cos nky + \beta_{33} \cosh 3k(z + d)] + \epsilon^2\beta_1\sigma t] + 0(\epsilon^4) \quad (1)$$

in which ϕ = velocity potential, such that fluid velocity $\underline{u} = \nabla\phi$, and

$$k\eta = \cos(mkx - \sigma t)\left[\left(\epsilon + \frac{1}{2}\epsilon^3 b_{11}\right)\cos nky + \frac{1}{2}\epsilon^3 b_{13}\cos 3nky\right]$$

$$+ \epsilon^2 \cos 2(mkx - \sigma t)(b_1 \cos 2nky + b_2) + \epsilon^2 b_3 \cos 2nky$$

$$+ \frac{1}{2}\epsilon^3 \cos 3(mkx - \sigma t)(b_{31}\cos nky + b_{33}\cos 3nky) + 0(\epsilon^4) \qquad (2)$$

in which η = elevation of free surface above mean level; and

$$\sigma = (gk)^{1/2}\left(\omega_0 + \frac{1}{2}\epsilon^2\omega_2\right) + 0(\epsilon^4) \qquad (3)$$

In these equations, ϵ = a dimensionless quantity which is the first coefficient in the double Fourier series for η, Eq. 2. To first order, it is equal to the dimensionless amplitude. The Landau symbol, 0( ), is used to show the order of the neglected terms. Other quantities introduced in these equations are $m = \sin\theta$, $n = \cos\theta$, in which θ = angle of incidence and of reflection, measured from the normal to the wall; and σ = angular frequency of wave motion, $\sigma = 2\pi/T$, in which T = wave period. The coefficients, ω_i, β_{ij}, γ_i and b_{ij} = functions only of kd (=2 × π × depth/wavelength) and θ. Formulas for each of these dimensionless coefficients are given by HT&S.

The solution, a double Fourier series in the horizontal variables, and a perturbation expansion in terms of a parameter related to wave steepness, is essentially a Stokes type of solution, which is not suitable for long waves in shallow water. For a review of this limitation for steady waves, see Ref. 4. By comparing the formulas given by HT&S for terms of different orders, e.g., β_2 and β_{33}, it can be shown that the effective expansion parameter is actually $\epsilon/\sinh^3 kd$, so that if kd is small, as for long waves, this is effectively $\epsilon/(kd)^3$, the Ursell parameter, which clearly becomes large in this limit of small kd. If a shallow water situation is encountered, e.g., $L/d > 10$, then the use of higher-order terms may make the solution less, rather than more accurate. The magnitude of $\epsilon/(kd)^3$ should be carefully monitored.

It is simply verified that ϕ as given by Eq. 1 satisfies Laplace's equation throughout the fluid, and the boundary conditions on the seabed and the wall. The nonlinear boundary conditions on the free surface are

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \quad \text{on} \quad z = \eta \qquad (4)$$

$$\text{and} \quad \eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \quad \text{on} \quad z = \eta \qquad (5)$$

in which the subscripts denote partial differentiation.

A simple numerical check can be made that the solution satisfies Eqs. 4 and 5 to third order, following the technique used by Fenton in Ref. 4, which is a variant of Richardson extrapolation to the limit. For given arbitrary values of kd and θ, values of all the numerical coefficients in the expansions can be calculated. Then for given arbitrary values of x, y and t, and for a given small numerical value of ϵ, denoted here by ϵ_1, the numerical value of all quantities in Eqs. 4 and 5 can be calculated from Eqs. 1–3. If it is assumed that the error terms in the third-order expansions for ϕ, η and σ are proportional to ϵ_1^{μ}, in which μ is not presupposed to be 4, then evaluating the left sides of Eqs. 4 and 5 does not give zero, but a number, e.g., e_1, which is proportional to ϵ_1^{μ}. Similarly a value of ϵ_2 gives an error of e_2. It is simply shown that

$$\frac{e_2}{e_1} = \left(\frac{\epsilon_2}{\epsilon_1}\right)^{\mu} + 0(\epsilon_1, \epsilon_2) \quad (6)$$

giving

$$\mu = \frac{\log\left(\frac{e_2}{e_1}\right)}{\log\left(\frac{\epsilon_2}{\epsilon_1}\right)} + 0(\epsilon_1, \epsilon_2) \quad (7)$$

which yields the order of the error terms.

This was done for a particular case, in which $\epsilon_1 = 0.01$; $\epsilon_2 = 0.02$; and the value of μ calculated for each of the boundary conditions Eqs. 4 and 5 over 8 equispaced values of $mkx - \sigma t$ for each of 8 equispaced values of nky, each over one complete period. In all cases, the value of μ obtained was between 3.89 and 4.02, suggesting strongly that the errors in HT&S's solution are, indeed, of order ϵ^4, and that their results are correct to third order.

At the end of their paper, HT&S also presented formulas for the pressure, p, within the fluid, obtained from the substitution of Eqs. 1–3 into the pressure equation

$$\frac{p}{\rho} = -gz - \phi_t - \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \quad (8)$$

in which ρ = fluid density. The resulting expression for the pressure at any point in the fluid is

$$\frac{kp}{\rho g} = -kz + \epsilon\left(p_0 + \epsilon p_1 + \frac{1}{2}\epsilon^2 p_2\right) + 0(\epsilon^4) \quad (9)$$

in which HT&S give expressions for the dimensionless terms, p_0, p_1 and p_2, as functions of kd, θ, x, y, z, and t. These can be checked numerically by substituting $z = \eta$ into Eq. 9, with η given by Eq. 2, to calculate the pressure on the free surface, which should be zero. This was tested in the same manner as described previously, and the surprising result was obtained that $\mu = 2.0$, showing that the expressions given by HT&S for pressure (their Eqs. 89 and 90) are wrong at second order.

The writer repeated their calculations and found where the expressions are incorrect. Their Eq. 89 for p_1 contains a trivial typographical error (an m^3 should be m^2). However, Eq. 90 for p_2 was found to be considerably in error. One of the aims of the present work is to produce an alternative solution which is easier to apply, so that instead of presenting a corrected version of p_2 here, that will be done as part of a third-order expansion in wave height in the next section.

It should also be pointed out here that the present theory ceases to be valid before the $m \rightarrow 1$ limit is reached. This is because for waves which are glancingly oblique, regular reflection of the waves seems not to be possible. For the case of shallow waves, it may be replaced by Mach reflection, whereby the crests of the incident and reflected waves intersect away from the wall—this junction being joined to the wall by a single crest. This phenomenon has been studied for the case of the solitary wave; for a recent work see Melville (9). For the case of waves in deeper water, the problem has been studied by Roberts (11) and by Yue and Mei(17). For waves near the wall, the structure may be quite different from that of the short-crested wave system. The present theory is most applicable to waves which do not approach the wall glancingly.

Third-Order Solution in Terms of Wave Height

It was shown in Ref. 4 that for practical application of Stokes' theory for steady water waves, it is easier to use an expansion in terms of the dimensionless wave height itself, rather than as are most published solutions, in terms of one of the first Fourier coefficients. Also, in Ref. 4, it was shown that to apply the theories, it is essential to know the actual wave speed, or the current, or the mass flux under the waves, if the wave period is specified as a design parameter. Similar comments relate to the theory of short-crested waves as presented by HT&S, as follows.

Consider a physical problem in which the quantities known are the period, T, of the motion, the water depth, d, and the crest-to-trough wave height, H_s, of the short-crested or standing wave. (In this paper, the symbol, H_s, is used instead of "H_{sc}" as used by HT&S, as it can apply to long-crested standing waves as well as short-crested waves.) Eqs. 72 and 76 of HT&S are, in fact, a pair of simultaneous nonlinear equations which when given this information, make it possible to solve for the wave number, k, and the expansion parameter, ϵ, after which all expressions, such as Eqs. 1–3, can be evaluated. This solution of the simultaneous equations is often difficult. Here an analytical solution is obtained which obviates most of the difficulties by recasting all the series in terms of an expansion parameter—the dimensionless wave amplitude—which has somewhat more direct physical significance than ϵ. Eq. 72 of HT&S is

$$\frac{1}{2}kH_s = \epsilon\left[1 + \frac{1}{2}\epsilon^2(b_{11} + b_{13} + b_{31} + b_{33})\right] + 0(\epsilon^5) \quad \text{(10)}$$

In this paper, the dimensionless wave amplitude, $kH_s/2$, will be denoted by δ, and the sum of the dimensionless coefficients, $b_{11} + b_{13} + b_{31} + b_{33}$, represented by b_4, a function only of kd and θ. Eq. 10 is easily reverted to give

$$\epsilon = \delta - \frac{1}{2}\delta^3 b_4 + 0(\delta^5) \quad \text{(11)}$$

and this can be substituted into all the expressions such as Eqs. 1–3 to give a third-order solution in terms of δ. Before doing this, however, a procedure is developed for the initial step in the application of the theory, if the period is known instead of the wavelength. If the wavelength is known, of course, this step can be avoided and Eq. 13 simply used to calculate ω as follows.

Consider Eq. 76 of HT&S

$$L = \frac{2\pi}{k} = \frac{gT^2}{2\pi}\,\omega_0(\omega_0 + \epsilon^2\omega_2) + 0(\epsilon^3) \ldots \qquad (12)$$

This equation contains the implicit assumption that the waves are not traveling on a current. It is, of course, quite possible that the whole wave system is being carried by a current moving parallel to the wall, such that the observed wave period at a fixed point is the Doppler-shifted period. The value of σ as given by HT&S [Eq. 3 here and which appears in $(mkx - \sigma t)$ in Eqs. 1 and 2], is the apparent radian frequency as would be measured by an observer moving such that as the waves propagate past, the mean fluid speed is zero, and the whole theory is for that particular case.

Here the theory is generalized to allow for the effect of a current. If there is a current, U, relative to the frame of interest, parallel to the wall and positive in the $+x$ direction, then the actual observed frequency in that frame, denoted here by ω, is $\omega = \sigma + Umk$, in which σ is given by Eq. 3. Using Eq. 3 and substituting Eq. 11 gives

$$\omega = \frac{2\pi}{T} = Umk + (gk)^{1/2}\left(\omega_0 + \frac{1}{2}\,\delta^2\omega_2\right) + 0(\delta^3) \ldots \qquad (13)$$

From this point, throughout this work, the quantity, ω, is that defined in this equation—the actual Doppler-shifted frequency in the frame of interest. If U = zero, then $\omega = \sigma$ and Eq. 12 can be recovered. Eq. 13 is now rewritten in dimensionless form, ignoring the omitted terms, as

$$Um\left(\frac{k}{g}\right)^{1/2} + \omega_0(kd,\theta) + \frac{1}{2}\left(\frac{kH_s}{2}\right)^2\omega_2(kd,\theta) - \frac{2\pi}{T(gk)^{1/2}} = 0 \ldots \qquad (14)$$

in which the functional dependence of ω_0 and ω_2 on kd and θ has been indicated, and the actual details of that dependence are given by HT&S. Eq. 14 shows that it is necessary to know the current, U, if the period of the motion is specified. Provided U, T, d and H_s are known, Eq. 14 makes the solution of a practical problem simpler, as it is a single equation in one unknown, k, which can be solved by any of the standard methods for solving nonlinear equations, such as the secant or bisection methods, or simply by trial and error. The use of Newton's method is probably unreasonable here, as the derivative of the left side of Eq. 14, with respect to the unknown k, is very complicated. To start any of the iteration methods, it is necessary to know an initial approximation for k. This can be found by taking the linearized, deep water, zero current approximation to Eq. 14, which yields $k \approx 4\pi^2/gT^2$.

For standing waves, $U = 0$, but in the more general case of oblique incidence, a value for U must be known to solve Eq. 14. If it is not, then a reasonable approximation is $U = 0$, although the value of k so obtained is in error (possibly small) at *first* order, and there is certainly no justification in using third-order contributions in the subsequent application of the theory, although, as will be described, it is sometimes important to include the second-order terms.

Once k is known, then all physical quantities can be calculated using the third-order solution as set out in the following. The frequency, ω, is given by Eq. 13. The expression for the velocity potential, ϕ, such that the fluid velocity $\underset{\sim}{u} = \nabla\phi$ is

$$\phi(x,y,z,t) = Ux + \left(\frac{g}{k^3}\right)^{1/2}\left[\delta^2\beta_1\omega t + \sum_{i=1}^{3}\frac{\delta^i}{(i-1)!}\sum_{j=1}^{i}\sin j(mkx - \omega t)\sum_{1=0}^{i}A_{ijl}\cos lnky\cosh\alpha_{jl}k(z+d)\right] + 0(\delta^4) \qquad (15)$$

in which the dimensionless coefficients, A_{ijl}, are known in terms of the coefficients, β_{ij}, given by HT&S as follows: $A_{111} = \omega_0/\sinh kd$; $A_{220} = \beta_3$; $A_{222} = \beta_2$; $A_{311} = -b_4A_{111}$; $A_{313} = \beta_{13}$; $A_{331} = \beta_{31}$; and $A_{333} = \beta_{33}$. All other such coefficients are zero. The coefficients, α_{jl}, are given by $\alpha_{jl}^2 = j^2m^2 + l^2n^2$. The expression for the free surface elevation is

$$k\eta(x,y,t) = \sum_{i=1}^{3}\frac{\delta^i}{(i-1)!}\sum_{j=0}^{i}\cos j(mkx - \omega t)\sum_{l=0}^{i}B_{ijl}\cos lnky + 0(\delta^4) \qquad (16)$$

in which the nonzero coefficients are $B_{111} = 1$; $B_{202} = b_3$; $B_{220} = b_2$; $B_{222} = b_1$; $B_{311} = -(b_{13} + b_{31} + b_{33})$; $B_{313} = b_{13}$; $B_{331} = b_{31}$; $B_{333} = b_{33}$; and the b_{ij} are as given by HT&S. The expression for the pressure is

$$\frac{kp(x,y,z,t)}{\rho g} = -kz + \sum_{i=1}^{3}\frac{\delta^i}{(i-1)!}\sum_{j=0}^{i}\cos j(mkx - \omega t)\sum_{l=0}^{i}C_{ijl}(z)\cos lnky + 0(\delta^4) \qquad (17)$$

in which the nonzero $C_{ijl}(z)$ are

$$C_{111}(z) = \frac{\cosh k(z+d)}{\cosh kd};\quad C_{200}(z) = \omega_0\beta_1[\cosh 2k(z+d) - 1];\quad C_{202}(z) = \omega_0\beta_1[m^2\cosh 2k(z+d) - n^2];\quad C_{220}(z) = \omega_0\beta_1[m^2 - n^2\cosh 2k(z+d)] + 2\omega_0\beta_3\cosh 2mk(z+d);\quad C_{222}(z) = \omega_0\beta_1 + 2\omega_0\beta_2\cosh 2k(z+d);$$

$$\frac{\sinh kd}{\omega_0}C_{311}(z) = (\omega_2 - \omega_0 b_4)\cosh k(z+d) - \beta_2\cosh 3k(z+d) + m\beta_3[(1-m)\cosh(2m-1)k(z+d) - (1+m)\cosh(2m+1)k(z+d)];$$

$$\frac{\sinh kd}{\omega_0}C_{313}(z) = \beta_{13}\sinh kd\cosh\gamma_1k(z+d) + \beta_2[n^2\cosh k(z+d) - m^2\cosh 3k(z+d)];\quad \frac{\sinh kd}{\omega_0}C_{331}(z) = 3\beta_{31}\sinh kd\cosh\gamma_3k(z+d) + \beta_2[n^2\cosh 3k(z+d) - m^2\cosh k(z+d)] + m\beta_3[(1-m)\cosh(2m+1)k(z+d) - (1+m)\cosh(2m-1)k(z+d)]$$

and

$$\frac{\sinh kd}{\omega_0}C_{333}(z) = 3\beta_{33}\sinh kd\cosh 3k(z+d) - \beta_2\cosh k(z+d) \qquad (18)$$

For the case of standing waves, $\theta = 0$, Miche obtained the unusual result that as the limit of an infinitely-deep ocean is considered, the second-order contribution to the pressure does not decay exponentially as $z \to -\infty$, but remains finite to the floor of the ocean [see Longuet-Higgins (8)]. Here we examine this limit for the more general case of obliquely-incident waves, using the preceding solution. In the limit of standing waves, some of the formulas given by HT&S become indeterminate (numerator and denominator of a quotient both go to zero). It is recommended that in all applications in this limit, a small finite value such as $\theta = 0.01$ be used.

Detailed analysis of the formulas given by HT&S gives the result that in the deep water limit, $kd \to \infty$, for all angles of incidence, $\omega_0 \to 1$, $\beta_1 \to 0$, $\beta_2 \to 0$. It can then be shown that all the $C_{ijl}(z) \to 0$ exponentially as $kz \to -\infty$, with the notable exception of $C_{220}(z)$, the coefficient of a second-order term which fluctuates in time at a frequency double that of the fundamental. In this limit, it can be shown that

$$C_{220}(z) \sim \frac{-1 + m^2}{2 - m} e^{2mkz} \qquad (19)$$

This shows what happens in the limit as the angle of incidence becomes very small, so that $m \to 0$; the coefficient of z in the exponent in Eq. 19 becomes small, and thus the pressure decays slowly with depth. In the standing wave limit $\theta = 0$; $m = 0$; $C_{220}(z) = -1/2$; and the result found by Miche is obtained—that as $z \to -\infty$, the fluctuating pressure does not decay with depth

$$\frac{kp}{\rho g} \to -kz - \frac{1}{2}\delta^2 \cos 2\omega t + 0(\delta^4) \qquad (20)$$

This is important for force calculations for almost-normally-incident waves in deeper water, for it means that the second-order component of pressure, although smaller than the first-order contribution near the surface, when integrated over the whole depth may give a load of greater magnitude than that at first order. It will be seen that this gives rise to a number of interesting phenomena, which make the determination of maximum loads on the wall a rather challenging problem. (As an almost facetious example, it should perhaps be pointed out to designers of concrete dams with almost-vertical walls, that wind waves reflected by the walls are responsible for an additional pulsating pressure over the whole dam face and submerged ground surface.)

The fact that the pressure fluctuations are finite on the bottom for very deep water only for the case $\theta = 0$, is simply explained by the physical interpretation of the result for standing waves. For that case, the center of gravity of the water is actually moving up and down as the waves do. To cause the motion, there has to be a pulsating pressure on the bottom, however deep, with a frequency twice that of the fundamental. When the case of nonzero θ is considered, as in the present work, it is always possible to reduce the whole motion to a steady flow by considering a frame moving along the wall, in which the new coordinate, $\hat{x}$, along the wall is such that $mk\hat{x} = mkx - \omega t$. As the motion is steady in this frame, the center of gravity does not move vertically, and there is no finite component of fluctuating pressure on the floor of the deep ocean.

Numerical Test of Accuracy of the Solution

All coefficients were calculated for the particular case of $kd = 2.0$ and $\sin\theta = 0.6$. The numerical values of the coefficients are presented here as a means of checking any computer programs which may be written from this work. The numerical values, rounded to six decimal places, of the coefficients for which formulas are given by HT&S are in Table 1, and in Table 2 are those which are defined in this paper.

To test the accuracy of the solution, the Richardson extrapolation method described in the previous section was used, with errors being shown to be of order, μ, with μ found to lie between 3.89 and 4.04. It was felt that more convincing evidence was needed that the errors were indeed of fourth order, and a more refined version of the Richardson extrapolation method was devised, as follows.

If it is assumed that the error, e, in any boundary condition is given by

$$e = a_1\delta^u + a_2\delta^{\mu+1} + 0(\delta^{\mu+2}) \quad \ldots \quad (21)$$

in which the unknown constants, a_1 and a_2, do not depend on δ, then for two successive δ_1 and δ_2

$$\frac{e_1}{e_2} = \left(\frac{\delta_1}{\delta_2}\right)^{\mu}\left[1 + \frac{a_2}{a_1}(\delta_1 - \delta_2) + 0(\delta_1^2, \delta_2^2, \delta_1\delta_2)\right] \quad \ldots \quad (22)$$

By solving for a_2/a_1, and considering another δ_3, giving e_3, such that $\delta_1 - \delta_2 = \delta_2 - \delta_3$, it is simply shown that

TABLE 1.—Values of Dimensionless Coefficients from Formulas Given by HT&S for the Special Case, $kd = 2.0$, $m = 0.6$

kd	2.000000	m	0.600000	n	0.800000
ω_0	0.981849			ω_2	0.180261
γ_1	2.473863			γ_3	1.969772
β_1	−0.009330	β_2	0.001064	β_3	−0.040209
β_{13}	−0.000530	β_{31}	−0.002673	β_{33}	−0.000041
b_1	0.288901	b_2	−0.040956	b_3	0.156809
b_{11}	0.040836	b_{13}	0.298081	b_{31}	−0.114346
b_{33}	0.233606				

TABLE 2.—Values of Coefficients for which Formulas Are Given in this Paper, for Same Special Case as Table 1

b_4	0.458177	A_{111}	0.270716	A_{220}	−0.040209
A_{222}	0.001064	A_{311}	−0.124036	A_{313}	−0.000530
A_{331}	−0.002673	A_{333}	−0.000041	B_{111}	1.000000
B_{202}	0.156809	B_{220}	−0.040956	B_{222}	0.288901
B_{311}	−0.417341	B_{313}	0.298081	B_{331}	−0.114346
B_{333}	0.233606	$C_{111}(0)$	1.000000	$C_{200}(0)$	−0.241007
$C_{202}(0)$	−0.084197	$C_{220}(0)$	−0.281963	$C_{222}(0)$	0.047894
$C_{311}(0)$	0.090135	$C_{313}(0)$	−0.056858	$C_{331}(0)$	−0.260703
$C_{333}(0)$	−0.025165				

Note: C_{ijl} were evaluated at $z = 0$.

$$\left(\frac{e_1}{e_2}\right)\left(\frac{\delta_2}{\delta_1}\right)^{\mu} = \left(\frac{e_2}{e_3}\right)\left(\frac{\delta_3}{\delta_2}\right)^{\mu} + 0(\delta_1^2, \delta_1\delta_2, \ldots) \quad (23)$$

and subsequently, that

$$\mu = \frac{\log\left(\dfrac{e_2^2}{e_1 e_3}\right)}{\log\left(\dfrac{\delta_2^2}{\delta_1\delta_3}\right)} + 0(\delta_1^2, \delta_1\delta_2, \ldots) \quad (24)$$

For $\delta_1 = 0.008$, $\delta_2 = 0.010$ and $\delta_3 = 0.012$, the solution given in Eqs. 13, 15 and 16 (for $U = 0$) was tested by calculating the values of the left sides in Eqs. 4 and 5, at 16 points of a rectangular grid over one whole period of the wave motion, with arbitrary origin. The values of the independent variables used were $mkx - \omega t = 0.11 + i\pi/2$; $i = 0, 1, 2, 3$; and $nky = 0.1311 + j\pi/2$; $j = 0, 1, 2, 3$. The results obtained were that $\mu = 4.000$ for each point calculated, providing strong evidence that the theory as given in Eqs. 13, 15 and 16 is correct to third order.

Eqs. 17 and 18 for the pressure were tested by evaluating the pressure at the free surface from Eqs. 16–18, and the small values of the pressures so obtained used as the errors e_i in Eq. 24. The values of μ obtained were the same as previously mentioned; $\mu = 4.000$ for all points, strongly suggesting that Eq. 17 and the $C_{ijl}(z)$ as presented in Eq. 18 are correct.

This was repeated for several different values of the incident wave angle, from $m = 0$ (or in fact, from $m = 0.01$), corresponding to standing waves, to $m = 0.99$, for glancingly oblique waves. In all cases, values of $\mu = 4.000$ were obtained. For the extreme limit of $m = 0.9999$, when some of the coefficients, such as β_{13}, are poorly conditioned computationally, values of $\mu = 3.998$ were obtained, showing the formal accuracy of the solution, even though not strictly applicable in this limit of steadily propagating waves, as described previously. Eq. 17 for $p(x,y,z,t)$ will now be used to obtain expressions for forces and moments on the wall.

Forces and Moments on the Vertical Wall

The force exerted by the waves on the wall, and the moment about the base of the wall, both per unit of distance in the x direction, are given by

$$\text{Force} = \int_{-d}^{\eta} p(x,0,z,t)\, dz, \quad \text{and}$$

$$\text{Moment} = \int_{-d}^{\eta} (d + z)\, p(x,0,z,t)\, dz \quad (25)$$

Eq. 17 gives the functional dependence of p on the independent variables. The substitution of $y = 0$, and the subsequent integration with respect to z are simply performed. This gives series which are little more complicated than Eq. 17 itself. For example, the first terms of the expansion for the force are

$$\frac{k^2}{\rho g} \times \text{Force} = \frac{1}{2}(k^2d^2 - k^2\eta^2) + \delta \cos(mkx - \omega t)\frac{\sinh k(\eta + d)}{\cosh kd} + \ldots \qquad (26)$$

In this equation, the elevation, η, appears in a highly nonlinear manner. While it would be possible to use such an expression, more information can be had by substituting Eq. 16 for η and performing all the series manipulations to yield explicit power series for the force and moment on the wall, as functions of x and t. Here, the force, p_*, and moment, M_*, per unit length of the wall *due to the waves* are introduced. To obtain the total force and moment on the wall per unit length, the hydrostatic contribution of the undisturbed water, $\rho g d^2/2$ and $\rho g d^3/6$, respectively, must be added to each. In situations where the wall is backed by undisturbed water, P_* and M_* = the net force and moment on the wall. All subsequent discussions of force are for these net forces and moments.

The equations for the loads are

$$\frac{P_*}{\rho g d^2} = \delta F_{11}\cos(mkx - \omega t) + \delta^2[F_{20} + F_{22}\cos 2(mkx - \omega t)]$$

$$+ \frac{1}{2}\delta^3[F_{31}\cos(mkx - \omega t) + F_{33}\cos 3(mkx - \omega t)] + 0(\delta^4) \ldots\ldots\ldots\ldots\ (27)$$

and $$\frac{M_*}{\rho g d^3} = \delta G_{11}\cos(mkx - \omega t) + \delta^2[G_{20} + G_{22}\cos 2(mkx - \omega t)]$$

$$+ \frac{1}{2}\delta^3[G_{31}\cos(mkx - \omega t) + G_{33}\cos 3(mkx - \omega t)] + 0(\delta^4) \ldots\ldots\ldots\ldots\ (28)$$

in which the coefficients are given by

$$(kd)^2 F_{11} = \omega_0^2 = \tanh kd;$$

$$(kd)^2 F_{20} = \frac{1}{4} + \omega_0\beta_1\left[-kd(1 + n^2) + \frac{1}{2}(1 + m^2)\sinh 2kd\right];$$

$$(kd)^2 F_{22} = \frac{1}{4} + \omega_0\beta_1\left[kd(1 + m^2) - \frac{1}{2}n^2\sinh 2kd\right]$$

$$+ \omega_0\beta_2\sinh 2kd + \frac{\omega_0\beta_3}{m}\sinh 2mkd;$$

$$\frac{(kd)^2}{\omega_0}F_{31} = \frac{3}{4}\omega_0 + \omega_2 - \omega_0 b_4 + \beta_1[-3n^2 + (1 + 3m^2)\cosh 2kd]$$

$$+ \beta_2\left[2\cosh 2kd + n^2 - \frac{1}{3}(1 + m^2)\frac{\sinh 3kd}{\sinh kd}\right]$$

$$+ \beta_3\left\{2\cosh 2mkd + \frac{2m}{4m^2 - 1}\left[\frac{(1 - 2m^2)\sinh 2mkd}{\omega_0^2} - m\cosh 2mkd\right]\right\}$$

$$+ \beta_{13}\frac{\sinh \gamma_1 kd}{\gamma_1}; \quad \frac{(kd)^2}{\omega_0}F_{33} = \frac{1}{4}\omega_0 + \beta_1(1 + m^2 - n^2\cosh 2kd)$$

$$+\beta_2\left(2\cosh 2kd - 1 - m^2 + \frac{1}{3}n^2\frac{\sinh 3kd}{\sinh kd}\right)$$

$$+\beta_3\left\{2\cosh 2mkd + \frac{2m}{4m^2-1}\left[3m\cosh 2mkd - (1+2m^2)\frac{\sinh 2mkd}{\omega_0^2}\right]\right\}$$

$$+3\beta_{31}\frac{\sinh\gamma_3 kd}{\gamma_3} + \beta_{33}\sinh 3kd \quad \ldots\ldots \quad (29)$$

The coefficients for the moment are not much more complicated, being able to be written in terms of

$$I_\nu = \int_{-d}^{0}(d+z)\cosh\nu k(z+d)\,dz = \frac{kd\sinh\nu kd}{\nu} + \frac{1-\cosh\nu kd}{\nu^2} \quad \ldots\ldots \quad (30)$$

These coefficients are

$$(kd)^3 G_{11} = \omega_0^2 kd + \operatorname{sech} kd - 1;$$

$$(kd)^3 G_{20} = \frac{kd}{4} + \omega_0\beta_1\left[(1+m^2)I_2 - \frac{1}{2}(1+n^2)k^2d^2\right];$$

$$(kd)^3 G_{22} = \frac{kd}{4} + \omega_0\beta_1\left[\frac{1}{2}(1+m^2)k^2d^2 - n^2 I_2\right] + 2\omega_0\beta_2 I_2 + 2\omega_0\beta_3 I_{2m};$$

$$(kd)^3 G_{31} = \frac{1}{4} + \frac{3}{4}\omega_0^2 kd + \omega_0\beta_1 kd[-3n^2 + (1+3m^2)\cosh 2kd]$$

$$+ 2\omega_0 kd(\beta_2\cosh 2kd + \beta_3\cosh 2mkd) + \frac{\omega_0}{\sinh kd}[(\omega_2 - \omega_0 b_4 + n^2\beta_2)I_1$$

$$- \beta_2(1+m^2)I_3 - m\beta_3(1+m)I_{2m+1} + m\beta_3(1-m)I_{2m-1} + \beta_{13}\sinh kd\, I_{\gamma_1}];$$

$$(kd)^3 G_{33} = \frac{1}{12} + \frac{1}{4}\omega_0^2 kd + \omega_0 kd[\beta_1(1+m^2-n^2\cosh 2kd) + 2\beta_2\cosh 2kd$$

$$+ 2\beta_3\cosh 2mkd] + \frac{\omega_0}{\sinh kd}[-\beta_2(1+m^2)I_1 + (n^2\beta_2 + 3\beta_{33}\sinh kd)I_3$$

$$+ m\beta_3(1-m)I_{2m+1} - m\beta_3(1+m)I_{2m-1} + 3\beta_{31}\sinh kd\, I_{\gamma_3}] \quad \ldots\ldots \quad (31)$$

Table 3 is presented here for the same case as for Tables 1 and 2, giving values of the F_{ij} and G_{ij} which might prove helpful as a check for others. Vertical walls of marine structures are usually part of a large

TABLE 3.—Values of Dimensionless Coefficients in Expressions for Force and Moment for Case kd = 2.0, m = 0.6

F_{11}	0.241007	G_{11}	0.298464 ÷2
F_{20}	0.027512	G_{20}	0.067998 ÷2
F_{22}	−0.006521	G_{22}	0.042579 ÷2
F_{31}	−0.069648	G_{31}	−0.042986 ÷2
F_{33}	−0.029919	G_{33}	−0.026496 ÷2

structure which acts as a whole, so that instead of the force per unit length, it is the total force on the wall or part of the wall which is important. The total force, P, on a finite length of wall, W, can be found by integrating Eq. 27 with respect to a dummy variable, equivalent to x, between $x - W/2$ and $x + W/2$, to give

$$\frac{P}{\rho g d^2 W} = \delta F_{11} \operatorname{sinc}\left(\frac{mW}{L}\right) \cos \chi + \delta^2 \left[F_{20} + F_{22} \operatorname{sinc}\left(\frac{2mW}{L}\right) \cos 2\chi\right]$$

$$+ \frac{1}{2}\delta^3 \left[F_{31} \operatorname{sinc}\left(\frac{mW}{L}\right) \cos \chi + F_{33} \operatorname{sinc}\left(\frac{3mW}{L}\right) \cos 3\chi\right] + 0(\delta^4) \quad \ldots\ldots \quad (32)$$

in which χ = the phase $(mkx - \omega t)$; the origin for x is at the center of the wall; and the sinc function, described in introductory texts for the theory of Fourier transforms, is defined here by

$$\operatorname{sinc}\left(\frac{jmW}{L}\right) = \frac{\sin\left(\dfrac{\pi jmW}{L}\right)}{\dfrac{\pi jmW}{L}} \quad \ldots\ldots \quad (33)$$

in which j = 1, 2 or 3 in Eq. 32. The only difference between Eq. 27 for the local force per unit length, P_*, and Eq. 32 for the mean force per unit length, P/W, is the presence of the sinc functions. In the limit, as $W/L \rightarrow 0$ (when the length of wall considered goes to zero, to give the local force per unit length), or $m \rightarrow 0$ (for standing waves, when there is no variation along the wall at any instant), the values of the sinc functions go to unity, and Eq. 27 is recovered. An equation for M, the total moment on the finite length of wall, can be found similarly, giving an equation similar to Eq. 29, with M_* being replaced by M/W and each coefficient, G_{ij}, being multiplied by sinc (jmW/L).

Determination of Maximum Loads

Eq. 32 gives the force on the wall as a function of five independent variables: (1) Wave height/length, via $\delta = kH_s/2 = \pi H_s/L$; (2) water depth/wavelength, via $kd = 2\pi d/L$; (3) angle of incidence of the waves θ; (4) wall length/wave length W/L; and (5) the phase of the wave motion $\chi = (mkx - \omega t)$. The problem of obtaining the maximum possible force and moment on a wall becomes an optimization problem, of determining the maximum value of the function in this five-dimensional space.

It is possible to find the values of χ for which P takes on an extreme value by differentiating Eq. 32 with respect to χ and equating the result to zero. Using elementary trigonometry, it can be shown that the extreme values of χ satisfy the transcendental equation

$$\sin \chi \left[s_1 F_{11} + \frac{1}{2}\delta^2 (s_1 F_{31} - 3 s_3 F_{33}) + 4\delta s_2 F_{22} \cos \chi + 6\delta^2 s_3 F_{33} \cos^2 \chi\right] = 0 \quad (34)$$

in which s_j is used to represent sinc (jmW/L). This equation has three solutions:

1. $\chi = 0$, corresponding to the force on a wall when a wave crest is at the middle of the wall. This value of the force is denoted by $P(0)$, and is given by substituting $\chi = 0$ into Eq. 32 to yield

$$\frac{P(0)}{\rho g d^2 W} = \delta s_1 F_{11} + \delta^2(F_{20} + s_2 F_{22}) + \frac{1}{2}\delta^3 (s_1 F_{31} + s_3 F_{33}) + 0(\delta^4) \quad (35)$$

The value of $P_*(0)$, the force per unit length under the wave crest, is obtained by setting all the s_j to 1 in this equation.

2. $\chi = \pi$, corresponding to the force, denoted here by $P(\pi)$, when a wave trough is at the middle of the wall. Substitution back gives

$$\frac{P(\pi)}{\rho g d^2 W} = -\delta s_1 F_{11} + \delta^2(F_{20} + s_2 F_{22}) - \frac{1}{2}\delta^3(s_1 F_{31} + s_3 F_{33}) + 0(\delta^4) \quad (36)$$

The value of $P_*(\pi)$, the force per unit length under the wave trough, is obtained by setting all the s_j to 1 in this equation.

3. Another solution is obtained when the term in square brackets in Eq. 34 is zero, giving a quadratic equation in $\cos \chi$, the solution of which is denoted here by $\cos \chi_m$, in which m = "maximum." If $q_1 = s_1 F_{11} + 1/2\ \delta^2(s_1 F_{31} - 3 s_3 F_{33})$; $q_2 = 2\delta s_2 F_{22}$; and $q_3 = 6\delta^2 s_3 F_{33}$, it can be shown that the single admissible solution of the quadratic is

$$\cos \chi_m = \frac{q_2}{q_3}\left[\left(1 - \frac{q_1 q_3}{q_2^2}\right)^{1/2} - 1\right] \quad (37)$$

This solution has real solutions for the phase, χ_m, only if $q_1 q_3 \leq q_2^2$, and if the absolute value of $\cos \chi_m$ is less than unity, giving the intermediate extreme value of the force

$$\frac{P(\chi_m)}{\rho g d^2 W} = \delta s_1 F_{11} \cos \chi_m + \delta^2[F_{20} + s_2 F_{22}(2 \cos^2 \chi_m - 1)]$$

$$+ \frac{1}{2}\delta^3[s_1 F_{31} \cos \chi_m + s_3 F_{33}(4 \cos^3 \chi_m - 3 \cos \chi_m)] + 0(\delta^4) \quad (38)$$

The value of $P_*(\chi_m)$, the force per unit length under this extremum, is obtained by setting all the s_j to 1 in this equation. The significance of this intermediate maximum is made clearer if third-order terms are ignored in the square brackets of Eq. 34, which gives a linear equation with solution

$$\cos \chi_m = \frac{-s_1 F_{11}}{4\delta s_2 F_{22}} \quad (39)$$

If the second-order second-harmonic contribution to the force, measured by $\delta^2 s_2 F_{22}$, is sufficiently large compared with the first-order first-harmonic contribution, $\delta s_1 F_{11}$, then Eq. 39 has real solutions and the graph of force as a function of phase develops secondary humps—the extreme value of which are given by Eq. 38. It will be seen in the following, that for conditions under which the present theory is valid, that if F_{22} is large enough, it is negative, so that the value of $\cos \chi_m$ is positive, and the intermediate extremum is a maximum which occurs in the vicinity of the crest. For this situation, when the phase is zero, the force, $P(0)$, corresponding to the crest is, in fact, a local minimum. For standing waves, this has been measured and considered by Rundgren (13), Nagai (10), and Goda (6).

The optimization problem has been reduced from a continuous problem in five dimensions to three separate problems, each in four dimensions. Whereas it was relatively simple to find extrema in phase, the determination of extrema in the other variables is somewhat more complicated, because they possess some internal local extrema and some extrema on the boundaries of the space of all possible wave conditions. The complexity of each is exemplified by Eq. 35 for $P(0)$. The leading order term, $\delta s_1 F_{11}$, is a function which increases monotonically as kd decreases, so that it is largest for shallow water conditions. This suggests that to first order, the largest forces are given by high and long waves. Depending on the ratio of structure length to wavelength, however, the sign of this term can become negative, because of the nature of the sinc function, which has a maximum of unity for zero argument, and, then, for increasing argument (here, relative length of structure) oscillates about zero, with decreasing magnitude. The first order contribution would have a maximum positive value for standing waves ($m = 0$) or for waves much longer than the structure (in which case the initial assumption of the theory, that the wall is infinitely long, is not true, but the theory provides an approximate answer). The first-order contribution would have a maximum negative contribution for $mW/L = 1.5$, so that the wall would be just long enough to contain three "half-waves," one crest between two troughs, the net effect being a negative off-shore force. Because of the ability of the sinc functions to change sign, almost any conclusion is, in general, possible. For a practical problem, it would be necessary to investigate all parameter values—it might even be possible to "tune" the structure length to the greatest wave length expected so that the overall force is close to zero.

For the purposes of the present paper, in all of the subsequent analysis, the effect of the overall length of the structure will be ignored, and the maximum force per unit length, P_*, will be considered to examine the effects of the other parameters of the problem. It is a rather unexpected result that the first order term, δF_{11}, is independent of the angle of incidence. Calculations at second order show that the magnitude of F_{20} is relatively small. However, F_{22} contains the integral of the possibly slowly-decaying pressure term, $C_{220}(z)$, and can become quite large, especially for deep water conditions. Fig. 1 shows the dependence of the ratio of F_{22}/F_{11} on the relative wavelength and on the angle of incidence. Clearly, this quantity, expressing the relative importance of second-order terms, can vary quite strongly with the angle of incidence. Typically for deep water, and for waves which have a low angle of incidence, the second-order terms can become large. When this happens, the ratio has a negative value, so that the first and second contributions tend to cancel at the crest and to reinforce at the trough.

This has the effect shown in Figs. 2 and 3, which show the loads predicted by Eqs. 35–38 for P_* as a function of the angle of incidence. Both figures are for the same value of wave height/length of 0.05. Fig. 2 is for a wave length to depth ratio of 1. For small angles of incidence, the onshore force under the crest is significantly smaller than the net off-

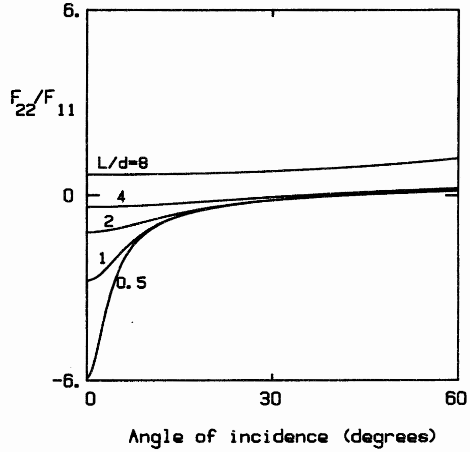

FIG. 1.—Relative Importance of Second-Order Coefficients, Expressed by Ratio F_{22}/F_{11}, and its Variation with Angle of Incidence and Relative Wavelength

shore force under the wave trough. It is clear for small angles of incidence, that the maximum onshore force does not occur at the crest, but at the intermediate maximum, given by Eqs. 37 and 38. For larger angles of incidence, the force under the crest increases, primarily because F_{22} becomes less negative, until it approaches a flat region for large angles of incidence. This gives the surprising result, mentioned by Kuznetzov [see Silvester (4)], that obliquely-incident waves do exert larger forces on the wall, and for the maximum onshore force, should be considered as the design criterion, in contradiction to what might have been ex-

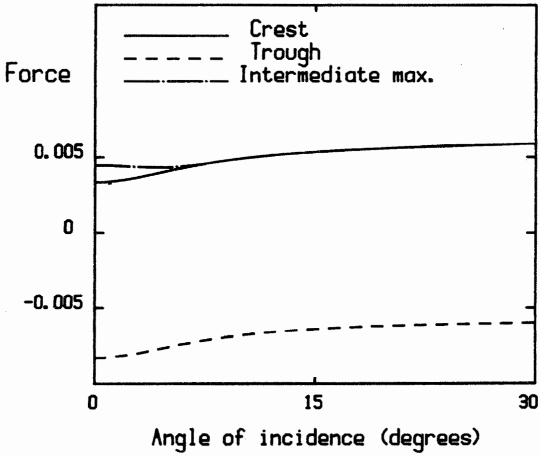

FIG. 2.—Variation of Force per Unit Length with Angle of Incidence for each of Three Parts of Wave, (1) for the Crest $\chi = 0$; (2) for the Trough $\chi = \pi$; and (3) for the Intermediate Maximum χ_m. Values of the Force Plotted $= P_*(\chi)/\rho g d^2$; the Whole Diagram is for the Case $H_s/L = 0.05$; $L/d = 1$

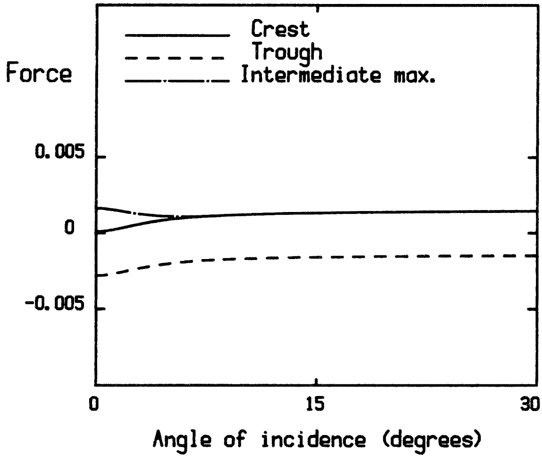

FIG. 3.—See Caption to Fig. 2; this Diagram Is for the Same Relative Wave Height, H_s/L = 0.05, but for Shorter Waves, L/d = 0.5

pected. For the force on a finite length of wall, however, the diminution of the sinc functions with angle of incidence might well outweigh the variation with F_{22} shown here, so that the maximum force on a finite wall might be that for the standing wave, when the entire wall simultaneously experiences the crest, or intermediate maximum.

Fig. 3 shows shorter waves with a ratio of length to depth of 0.5, and a variation with angle of incidence which is relatively more marked. It is important to consider all angles, as well as the possibility of an intermediate maximum in the wave force, which here becomes the criterion for onshore force (for the standing wave). It should be noted, however, that the maximum value of the force in this case is smaller than that for the longer waves in Fig. 2.

Obviously, the behavior of the maximum force as a function of the four variables is a very complicated one. A systematic search for the extreme values of $P(0)$, $P(\pi)$, and $P(\chi_m)$ was made through the three dimensions of kd, θ and δ, by printing out values at equispaced points. Local extrema in angle of incidence were found as well as local extrema in wave height (surprising, that the highest wave need not exert the largest force). Of greater importance was the fact that for all physically-realizable wave heights, the maximum force per unit length for all three, $P(0)$, $P(\pi)$ and $P(\chi_m)$, showed a continuous increase as the wavelength increased (kd decreased), so that the force was greatest for the longest waves. In general, for these long waves, the variation with angle of incidence was relatively small.

The worst case for the onshore force was found to be associated with the crest, $P(0)$, and to be for large angles of incidence. For design, it is recommended that the most extreme value be had for the longest and highest possible wave, approaching the wall very obliquely. Although the theory is not valid in the extreme limit $m \rightarrow 1$, this value would give a slight overestimate of the likely maximum forces over a wide range of incidence angle. The worst case for the offshore force, $P(\pi)$, was found to be also for the longest and highest waves, but in this case, the standing wave $m \rightarrow 0$ gave the greatest loads.

Unfortunately, the Stokes-type of theory on which the present work is based, breaks down in the limit as $kd \rightarrow 0$. Despite the theory revealing a number of unusual phenomena in the variation of the forces on the wall in the limit where the forces are greatest, the theory has limited validity. A critique of the theory in this limit will be given; but, meanwhile, it will be compared with experiment.

Comparison with Experiment

A wide range of experiments on waves reflected by walls are the standing wave experiments of Nagai (10). These were performed over a range of wavelengths ranging from 1.15 times the depth to 15 times the depth, from effectively deep water waves through to shallow water waves. The writer took the values of the force obtained by Nagai from numerical integration of pressure readings, and compared them with the predictions of Eqs. 27, 35, 37 and 38. Some of the experiments were for intermediate maxima, and some were for the crest. In most cases, the actual value of the phase was given by Nagai and could be used in the comparison, so that Eq. 27 could be used.

Results are shown plotted on Fig. 4. It can be seen that the errors are less than 10% over much of the figure. Given the difficulty of conducting such large unsteady experiments, and the supposed limitations of the

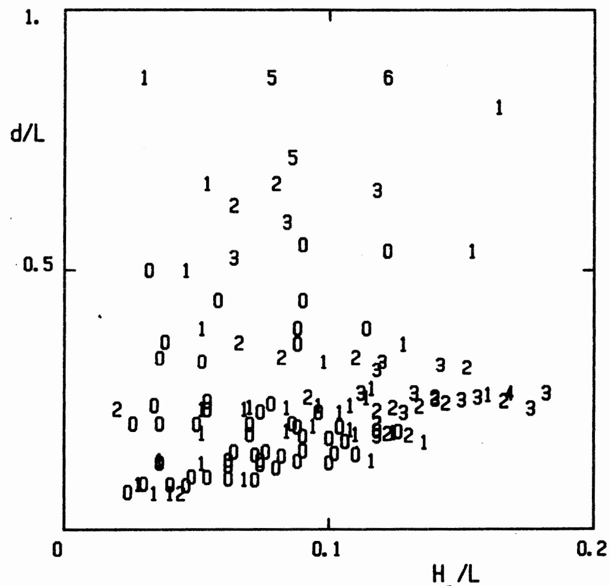

FIG. 4.—Comparison between Present Theory and Experimental Results of Nagai; each Number Shows Decile Difference between Theory and Experiment, thus "0" Means Less than 10% Difference, "1" Means 10–20%, etc.

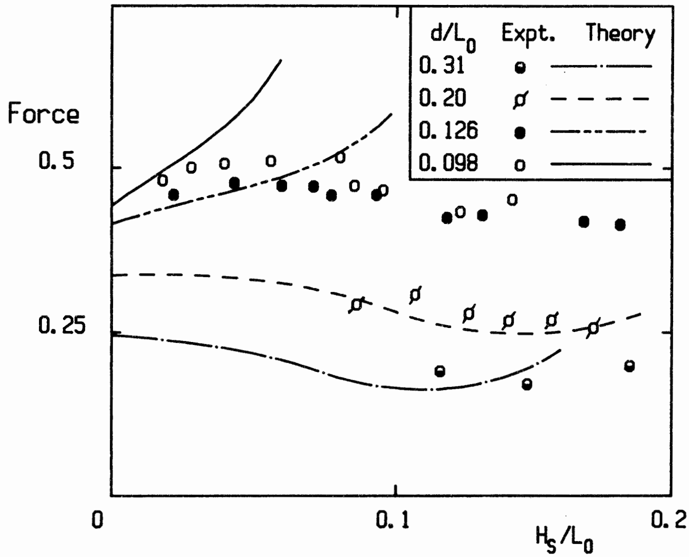

FIG. 5.—Comparison between Present Theory and Experimental Results of Goda; Value of Force Plotted is $P_*/\rho g H_s d$

theory to relatively low short waves, the agreement is quite good. There are three main areas of disagreement. For steep waves, there is a run of experiments with d/L roughly 0.25, which have errors of 20–30% for H_s/L greater than 0.12. Since this limit corresponds to a wave height/water depth ratio of about 0.5, and Nagai reported that most of the waves here were actually breaking, it is not suprising that the theory is not valid. Another area of disagreement is that for waves in shallow water, and in which kd is small. A discussion of this limit is given in the next section. The remaining area of disagreement, where the water is relatively deep, is rather puzzling. The Stokes-type theory should give accurate results for waves in deep water. A possible explanation is that for large depths, the experimental pressures contain a large hydrostatic component, and it may have been difficult to obtain accurate pressure readings after the subtraction of the hydrostatic.

Another set of standing wave experiments, obtained by Goda, are available for comparison (6). In this case, the experimental results were presented as functions of d/L_o, in which L_o = the wavelength of small amplitude waves of the same period. This is an alternative way of specifying the wave period, the two being connected by the first-order version of Eq. 12

$$\frac{d}{L_o}\tanh\left(\frac{2\pi d}{L_o}\right) = \frac{2\pi d}{gT^2} \quad \text{(40)}$$

To make the comparison, the actual wavelengths in the experiments were estimated from the present third order theory by eliminating the period between Eqs. 13 and 40, and solving for kd. Then the forces were calculated, and the results are shown in Fig. 5.

From this figure, it can be seen that for the two sets of results in deeper water, quite good agreement with experiment was obtained. The parts of the curves which are concave up, were obtained from the expressions for the intermediate maximum—Eqs. 37 and 38. For long waves, the theory tends to agree with experiment only for waves which are not too high, as would be expected. However, it is noteworthy that for the upper sets of results, Goda's application of his fourth-order standing wave theory gave considerably better agreement than the third-order theoretical expressions here—Eqs. 35–38. It does seem that the present theory should not be applied to waves which are longer than ten times the water depth. In Ref. 4, it was found that a sudden decrease in the accuracy of Stokes' theory for progressive waves also occurred at this value.

Limitations of Theory

In this section, a review of how and when the theory should and should not be applied is presented.

Long Waves.—If waves are long relative to the depth, it is well-known that Stokes-type theories such as the present, should be applied with great care. For progressing waves, the region in which Stokes theory should not be applied is considered to be $L/d > 10$. Beyond this, for finite wave heights, higher order results tend to be less accurate than those of first order.

From the coefficients given in Eq. 29, it is possible to deduce relatively simply the limiting behavior of the coefficients as far as second order, i.e., as $kd \to 0$

$$F_{11} \sim (kd)^{-1}; \quad F_{20} \sim (kd)^{-2}\left(\frac{3}{8} - \frac{1}{4}m^2\right);$$

$$\text{and} \quad F_{22} \sim \frac{3}{8}(kd)^{-4} + (kd)^{-2}\left[\frac{7}{8} - \frac{m^2}{4} + \frac{4m^2 - 1}{8(1 - m^2)}\right] \quad \text{.................. (41)}$$

Substituting these results into Eq. 35 gives the expression for the force under the crest

$$\frac{P_*(0)}{\rho g d^2} \sim \frac{1H_s}{2d} + \left(\frac{1H_s}{2d}\right)^2\left[\frac{3}{8}(kd)^{-2} + \frac{5}{4} - \frac{m^2}{2} + \frac{4m^2 - 1}{8(1 - m^2)}\right] \quad \text{.............. (42)}$$

It is interesting that in this limit, the result has a strong similarity with long-wave theory for progressing waves—that the wave height parameter appears as H_s/d, and that as $kd \to 0$, the second-order terms are dominated by the term which goes like $(kd)^{-2}$, which becomes large in this limit, and the theory becomes invalid. By taking the ratio of the second-order term to the first in this limit, it can be seen that the *effective* expansion parameter is $(H_s/d)(kd)^{-2}$, which can be rearranged to give a number proportional to H_sL^2/d^3, equivalent to the Ursell parameter for steady waves. It seems that the same limit applies to the present theory for short-crested and standing wave systems as to Stokes' theory for steady waves: for higher-order theory to be valid, the value of $(H_s/d)(kd)^{-2}$ should not be large. With this limitation in mind, the result which should be used for very long waves is the first-order part of Eq. 27

$$\frac{P_*}{\rho g d^2} \approx \frac{1}{2}\frac{H_s}{d} \cos (mkx - \omega t) \quad (43)$$

It is interesting that this result, for the present general case of obliquely-incident waves, is independent of the actual angle of incidence.

Where Incident Wave Details are Specified.—The actual problem of the reflection of a given incident periodic wave train does not correspond precisely to the short-crested wave system at higher orders. If an otherwise steadily-progressing wave approaches the wall, it will encounter already-reflected precursors of itself, which will cause nonlinear interactions between the waves, and the whole problem is in general, unsteady. There is nothing in the present theory which says *how* given incident waves would interact: this short-crested theory presupposes that the incident waves are precisely in the form which when reflected, form an image of themselves. Since the whole system is perfectly periodic in time, all harmonics of the waves are bound to the main motion, and the whole system propagates along the wall without change of form. To first order, the two problems are the same. However, at higher orders, there are few known results to estimate just how equivalent the incident/reflected wave problem and the short-crested wave problem are. Fenton and Rienecker (5), in their study of the normal reflection of solitary waves by a wall, found that the reflected wave differed from the incident one at third order in the wave height. If this result were to hold for shorter waves, there would be little point in using the third-order results of the present theory.

If the crest-to-trough height, H_i, period T_i, current, U, and the angle of incidence, θ, of an incident wave are known, since there is no transmission of energy through the wall, the reflected wave also has these same properties at first order, and as considered in the previous paragraph, possibly to second order. Thus, the reflected wave has a height of H_i, and the height of the short-crested wave system is $2H_i$—a linear superposition of the incident and reflected waves, which is independent of angle of incidence. The period of the combined wave system is still T_i. With this information, the short-crested wave theory can be applied. It is a reasonable approximation to include the second-order terms, in view of their ability to affect the results markedly. However, as considered, the use of third-order theory is rather questionable.

Large Angle of Incidence (Grazing Incidence).—If the waves are glancingly-incident, then regular reflection may not occur. For waves in deeper water, the wave structure near the wall looks more like a periodic wave propagating along the wall, which is little different from the incident wave. The present theory is not strictly applicable to this situation, but as previously shown, the maximum onshore force has very little variation with angle for large angles of incidence, and the use of a design value of say 60° would be very close to the maximum possible for the case of regular reflection. For angles of incidence approaching 90°, the wave near the wall has an amplitude roughly equal to the incident (compared with twice that value as predicted by the present theory), and the force on the wall would be much less than for the regular type of reflection described in this paper.

Wave Breaking.—The possibility of wave breaking at the wall has not been considered. This is probably reasonable for the geometry assumed—a vertical wall and a horizontal bed—because then the waves would be almost completely reflected. Before any wave reaches the wall, it has to pass through a succession of reflected waves. Breaking and dissipation should occur long before the wall is reached, since enough energy would be lost so that breaking at the wall would be unlikely. If, however, the waves were dissipated at the wall through breaking or the use of rubble, then little would be reflected and successive waves could arrive at the wall without having broken before. Thus they would tend to break at the wall, thereby perpetuating the process. There is a real case for installing highly-reflective walls so that when a sea state gradually increases, reflected waves will cause some wave breaking and dissipation before succeeding waves reach the wall!

Shortness of Wall.—The theory on which all the preceding work has been based assumes that the wall is infinitely long, and that the motion is periodic along the wall. In a practical situation where the wall is of finite length, there would be some effects due to diffraction around the end of the wall. This is another problem which is too complicated to consider in the present work. However, all the formulas presented here can be used as a first approximation.

Conclusions

A third-order expansion in wave height for the short-crested wave system adjacent to a vertical wall has been developed, and has been shown numerically to be correct to third order. From this solution, explicit third-order expressions for the force and moment on the wall have been obtained. These formulas contain some unusual and important features at second order, i.e., that the second-order contribution to force may dominate the solution; that maximum forces can occur under wave crests or troughs or intermediate points, and formulas are given for each; and that the worst case for design may be for obliquely-incident waves, rather than standing waves. The formulas presented can be used for design, and are shown to agree with experiment over a range of conditions. The problem of determining the most adverse loading for design becomes a problem of optimization, and determining the maximum of a given function in the space of all the variables of the problem. It is shown that the largest loads are given by long and high waves, despite a number of other subsidiary maximá. The worst case for onshore loading is under the wave crest of waves, which are more glancingly oblique. The worst case for offshore loading—the greatest of all—occurs under the wave trough for standing waves. For the often encountered design situation in which only the incident wave train is specified, there is justification for using the results to second order, but no higher. In some practical problems, there is little justification for using the full theory: for grazing incidence, the theory does not describe the problem well. For the case of very long waves, only first-order theory should be used, for which a simple expression is presented.

Appendix I.—References

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APPENDIX II.—NOTATION

The following symbols are used in this paper:

A_{ijl} = dimensionless coefficients in series for ϕ given in this work;
a_1, a_2 = constants in expression for error term;
B_{ijl} = dimensionless coefficients in series for η given in this work;
b_i, b_{ij} = coefficients in series for η given by HT&S;
$C_{ijl}(z)$ = dimensionless functions in series for pressure in this work;
d = mean depth of water;
e = error in any of the free surface boundary conditions;
e_i = error for an expansion parameter of ϵ_i or δ_i;
F_{ij} = dimensionless coefficients in series for P;
G_{ij} = dimensionless coefficients in series for M;
g = gravitational acceleration;
H_i = crest-to-trough wave height of incident wave train;
H_s = crest-to-trough wave height of short-crested waves;

I_ν = $\int_{-d}^{0} (z + d) \cosh \nu k(z + d)\, dz = (kd \sinh \nu kd)/\nu + (1 - \cosh \nu kd)/\nu^2$;
i = integer;
j = integer;
k = wave number = $2\pi/L$;
L = wavelength of each component wave train;
L_o = wavelength of small amplitude waves, defined by period, Eq. 40;
l = integer;
M = total moment on wall about base of wall;
M_* = moment on wall about base of wall, per unit length;
m = $\sin \theta$;
n = $\cos \theta$;
P = total horizontal force on wall;
$P(0)$ = value of P when crest is at center of wall;
$P(\pi)$ = value of P when trough is at center of wall;
$P(\chi_m)$ = value of P when force has intermediate maximum;
P_* = horizontal force on wall, per unit length;
$P_*(0)$ = value of P_* under wave crest;
$P_*(\pi)$ = value of P_* under wave trough;
$P_*(\chi_m)$ = value of P_* at intermediate maximum;
p = pressure in fluid;
q_j = for $j = 1, 2, 3$—coefficients of quadratic, Eq. 37;
s_j = $\text{sinc}\,(jmW/L) = \sin\,(j\pi mW/L)/(j\pi mW/L)$;
T = period of short-crested waves;
T_i = incident wave period = T;
t = time;
U = current parallel to wall;
$\underline{u}$ = fluid velocity;
W = finite length of wall for force calculations;
x = coordinate along wall; for a finite wall, origin is at center;
$\hat{x}$ = phase variable, equivalent to x: $mk\hat{x} = (mkx - \omega t)$;
y = coordinate normal to wall, into fluid;
z = vertical coordinate;
α_{jl} = $\sqrt{j^2m^2 + l^2n^2}$;
β_i, β_{ij} = dimensionless coefficients in series for ϕ used by HT&S;
γ_i = dimensionless coefficients used by HT&S;
δ = $kH_s/2$ = dimensionless wave amplitude;
ϵ = expansion parameter used by HT&S;
ϵ_i = particular value of ϵ;
η = elevation of free surface relative to mean level;
θ = angle of incidence and reflection of waves, measured from normal to wall;
μ = order of error terms;
ν = dummy variable, see I_ν;
ρ = fluid density;
σ = angular frequency of wave motion in frame with zero current;
ϕ = velocity potential, such that fluid velocity $\underline{u} = \nabla\phi$;
χ = phase, which is 0 at wave crest and π at trough;

χ_m = phase at which force takes a maximum between crest and trough;

ω = angular frequency of wave motion = $2\pi/T$;

ω_i = dimensionless coefficients in series for ω; and

0 = Landau order symbol, used as in $0(\epsilon^4)$ meaning that neglected terms are of the order of the fourth power of ϵ.

WAVE FORCES ON VERTICAL WALLS[a]

Discussion by Eugene H. Harlow[2]

It is stated in the paper, after much review of mathematical theory about the force of waves against vertical walls, that "the possibility of wave breaking at the wall has not been considered. This is probably reasonable for the geometry assumed (a vertical wall and and a horizontal bed) because then the waves would be almost completely reflected."

However, if the waves reaching the wall are multi-directional, then the maximum force will occur when two crests intersect and break at the wall. In the laboratory world of monodirectional flumes, this cannot occur. But in the real world it occurs randomly, but often. Along a length of wall, it rarely happens at the same spot, but since there are an infinite number of positions, breakers can be observed against a wall at frequent intervals and at different positions.

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[a]July, 1985, Vol. 111, No. 4, by John D. Fenton (Paper 19899).

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The peaking of converging crests happens in either deep water or shallow and is commonly observed by anyone who spends much time watching the water surface during strong winds. Waves break in the open sea, without walls, when crests intersect by spilling and curling. Many boats have been rolled over, or upended, by such breakers. The likelihood of such crests breaking against a wall is increased by the fact that few walls are built on a flat seabed. Certainly none are built on a flat seabed of great depth. The nearest approach to that situation is a floating structure with a deep, vertical face such as a large ship nearly broadside to the wave movement. The effect of intersecting crests is often experienced by such ships, even though much of the force of the breaker passes beneath the hull. Damage to superstructures sometimes results.

In most cases, walls exist in shoaling water where steep shoaling sometimes occurs immediately in front of the wall. A beautiful example of how this triggers the breaking of interesting deep-water wave crests can be seen at some of the cliffs along the shore of Guam, west of Apra Harbor, where during or after a storm, breakers at different points along the 200-foot cliffs send great geysers of water and spray far above the land. Although the sea offshore drops off steeply to the Marianas trench, the rubble at the base of the cliffs slows the base of waves, tripping converging crests enough to cause frequent breaking.

The same is true of most breakwaters, and particularly those with walls atop mounds. Storm water level rise caused by barometric effects, astronomical tide, and wind set up can, during a storm, cause sea level to be substantially above normal level. This submerges much of the mound and lifts the breaking crests to wall level. The effect of such shock pressures is not well documented, but it can be disastrous.

For the case addressed in the paper, waves against a wall on a flat seabed, the incidence of local peaking breakers needs to be considered in designing the wall and will probably be the dominant force to be resisted. In addition, the instantaneous pressures can be extremely great at places where the kinetic energy of a curling crest is stopped suddenly.

Closure by John D. Fenton[3]

The writer thanks E. H. Harlow for his detailed discussion. He is quite correct, that in the case where waves are obliquely-incident, it is possible that wave breaking will occur at the wall. The paper under discussion did not include this fact. This, indeed, can give large local pressures, but as noted by Harlow, "the effect of such shock pressures is not well documented." This would seem to be an important but neglected area of research. The writer is not yet convinced that these should provide the design load for a structure as a whole. As Harlow also wrote "local peaking breaker(s) . . . will probably be the dominant force to be resisted." That "probably" is an important, open question. The original paper showed that the maximum force per unit length of wall may not

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be caused by waves which approach with crests parallel to the wall; however, the maximum overall force on a structure will tend to be caused by such waves. In this case, the writer's suggestion may be valid: if at all possible, abrupt shoaling in front of a wall should be avoided, and walls should be made reflective to cause breaking away from the wall.