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Initial movement of grains on a stream bed: the effect of relative protrusion

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Shields (1936) found that the dimensionless shear stress necessary to move a cohesionless grain on a stream bed depended only on the grain Reynolds number. He ignored the degree of exposure of individual grains as a separate parameter. This report describes experiments to measure the dimensionless threshold stress and its dependence on grain protrusion, which was found to be very marked. The threshold stress for grains resting on the top of an otherwise flat bed in a turbulent stream was measured and found to be 0.01 – considerably less than previously-reported values of 0.03–0.06 for beds where all grains were at the same level. It is suggested that the new lower value be used in all turbulent flow situations where the bed is of natural sediments or unlevelled material. An hypothesis is proposed that the conventional Shields diagram implicitly contains variation with protrusion between the two extremes of (i) large grains and large Reynolds numbers, with small relative protrusion, and (ii) small grains, low Reynolds numbers, and protrusion of almost a complete grain diameter. In view of this, the extent of the dip in the Shields plot is explicable in that it represents a transition between two different standards of levelling as well as the transition between laminar and turbulent flow past the grains, the range of which it overlaps considerably.

NOTATION

C_d	drag coefficient
D	nominal grain diameter: that of a sphere of equal volume
f_1, f_2	functions of the variables shown in argument when used
g	gravitational acceleration
g'	$g(\rho_s/\rho - 1)$
h	depth of flow
K	lift factor
p	protrusion of particle above others
R_*	grain Reynolds number $= u_* D/\nu$
Re	$\bar{u}_{\frac{1}{2}} D/\nu$
u	horizontal fluid velocity

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$\overline{u}$ time mean of u
u_* shear velocity $= (\tau/\rho)^{\frac{1}{2}}$
$\overline{u}_{\frac{1}{2}}$ time mean velocity at level of grain centre
y elevation
θ dimensionless shear stress $= u_*^2/g'D$
θ_0 dimensionless threshold shear stress
ν kinematic viscosity
ρ density of fluid
ρ_s density of solid
τ shear stress on bed
τ_0 threshold shear stress on bed

1. Introduction

In his classical work, Shields (1936) examined the forces acting on a cohesionless grain resting on the bed of a water stream. He considered the horizontal resisting force to be proportional to $(\rho_s - \rho)\,gD^3$, where ρ_s is the density of the grain immersed in a fluid of density ρ, g is gravitational acceleration, and D is a representative mean diameter of the grain. The drag force on the grain exerted by the flow was taken to be proportional to $\rho D^2 u^2$, where u is the fluid velocity acting on the grain. This velocity was assumed to be the time mean velocity $\overline{u}$ given by the logarithmic law

$$\overline{u}/u_* = 5.75\lg y/D + f_1(R_*), \tag{1.1}$$

where f_1 is a function of the grain Reynolds number $R_* = u_*D/\nu$, in which the shear velocity $u_* = (\tau/\rho)^{\frac{1}{2}}$, where τ is the bed shear stress, and y is the elevation above a datum somewhere below the bed surface. Shields assumed that y/D is a constant, giving $\overline{u}/u_* = f_2(R_*)$. Substituting this into the expression for drag force, equating drag and resisting forces as corresponding to initiation of motion, and re-arranging,

$$\frac{u_{*0}^2 D^2}{(\rho_s - \rho)\,gD^3} \propto \frac{1}{f_2^2(R_*)}, \tag{1.2}$$

where the subscript 0 refers to initiation (threshold) conditions. Clearly the term on the left side is proportional to the ratio of fluid force on a bed surface grain to the gravity force on that grain. Denoting it by θ_0, the dimensionless threshold stress, (1.2), can be written as

$$\theta_0 = u_{*0}^2/g'D = \theta_0(R_*), \tag{1.3}$$

where $g' = g(\rho_s/\rho - 1)$.

To determine the variation of θ_0 with R_*, Shields used results from almost equidimensional grains laid in flat beds. For the same hydraulic conditions he found that rounded and sharp-edged grains could resist a slightly higher shear stress than angular grains, but when plotted on a θ_0–R_* diagram, shown in figure 1, the scatter of results was relatively small.

Considering a stream bed of similarly-shaped grains, the more one grain projects above the general bed level the more it is exposed to the fluid flow and the greater are the disturbing forces while the resisting forces are less. It may be expected that protrusion of individual bed grains is an important parameter in determining their stability. Previous work, including that of Shields, has assumed that the value of y to be used in equation (1.1) is a constant for all bed and hydraulic conditions. This is because all beds laid were nominally flat – they contained no gross surface features and were flattened so that the tops of the grains were level to within the accuracy which could be practically achieved.

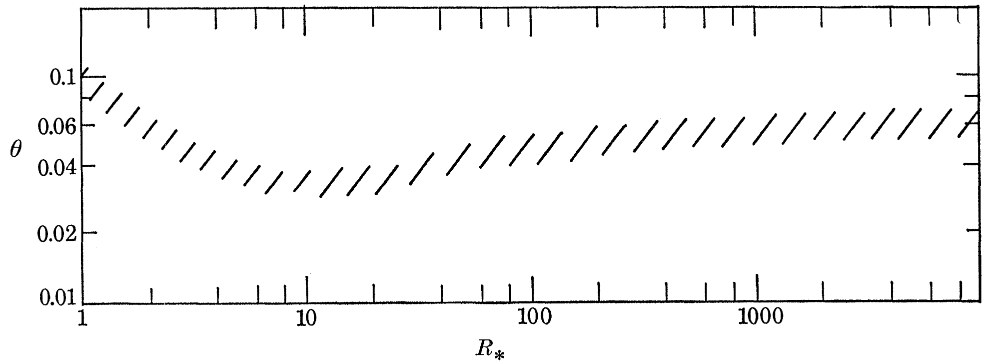

FIGURE 1. The Shields diagram: the shaded region defines the threshold, θ_0, above which grains move.

Comparing different types of bed grains and different methods of levelling, it is obvious that the term 'flat bed' is ambiguous. Beds composed of large rounded stones can be placed grain by grain, each being set to the desired level so that a cross section of the bed will be as in figure 2*a*. On the other hand, if coarse sand is placed, the grains are not packed individually but are tamped down and levelled with some kind of levelling bar. Natural variation of the grains and different geometries of arrangement will ensure differing degrees of exposure of individual grains throughout the bed, as shown in figure 2*b*. Beds of fine sand, even though levelled, tamped and nominally flat, must have many grains, as shown in the enlargement of figure 2*c*, which protrude above the bed to almost a complete grain diameter. Past experiments on different grain sizes, and hence different grain Reynolds numbers have not been considered in interpreting results. Clearly, results from a test situation as in figure 2*a* are not applicable to a bed as shown in figure 2*d* which may be the case for naturally-occurring gravel or material simply dumped in position.

The omission of protrusion as a separate parameter would not matter if all experiments used to build up the Shields diagram and all field conditions to which it were applied were for the same protrusion. For fine materials there is no way, either in the laboratory or in the field, of ensuring that all bed surface grains are co-planar; thus that part of the Shields diagram corresponding to fines (small R_*) is a satisfactory criterion for both conditions. On the other hand, coarse grains may be placed in different ways, from tipping into position randomly (figure 2*d*) to levelling carefully, either in the laboratory or the field (figure 2a). For the fully-turbulent case, θ_0 for randomly packed beds may be much less than the values normally used for co-planar beds. These are $\theta_0 = 0.03$ for a motion criterion of first displacement (Neill 1967, 1968) and $\theta_0 = 0.06$ for a criterion of a small finite solids flow rate (Shields 1936; Taylor & Vanoni 1972).

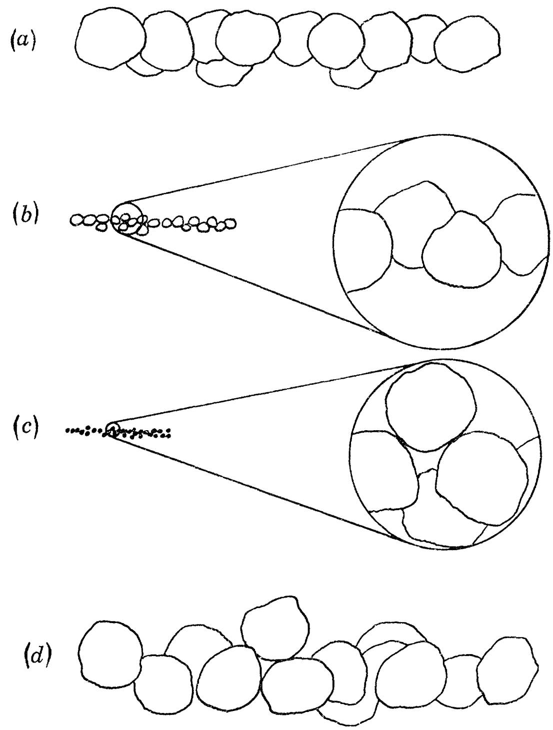

FIGURE 2. (a)–(c) Cross sections of 'flat' bed surfaces of different sized grains, showing different protrusion of bed grains in each case: (a) gravel, laid in a co-planar bed; (b) coarse sand, with individual grains over-riding others as shown enlarged; (c) fine sand, some grains almost completely exposed.
(d) Naturally-occurring gravel or dumped material, geometrically similar to (c).

The object of the present work was to measure the effect on θ_0 of particles protruding above the general bed level, and to relate this to the Shields diagram. Experiments were performed to measure θ_0 and its variation with protrusion. These are described in § 2, and the results given in § 3; some calculations based on previous experimental work are performed and discussed in § 4. In § 5 the ramifications of these results for the Shields diagram are discussed, and final conclusions drawn in § 6.

2. EXPERIMENTS

The most reliable experiments to determine the dimensionless threshold stress have been those in which different fluid stresses have been applied to a bed and the corresponding bed-load discharge measured by collecting the grains removed by entrainment – for example Shields (1936), Paintal (1971), Taylor & Vanoni (1972). Then, by extrapolation to zero transport or by arbitrary definition of a small but finite constant transport rate as corresponding to incipient motion conditions, θ_0 was calculated. This procedure is appropriate to a physical situation in which quantities vary randomly in time and space, for by collecting a large number of individual grains removed from the bed over a long period, the statistical population is large enough that the values obtained for transport rates, and subsequently threshold stress, may be considered reliable. A problem arose in the design of a statistically-significant set of experiments to measure the effect of protrusion p on threshold stress. It was impracticable to set a number of grains protruding a given height above a bed and to measure the rate at which they were entrained, so we had to resort to using a single grain at a given test position. By doing this, the sampling technique was deficient, but consistent results were obtained which did show the large effect of protrusion.

Three different experiments were performed, subsequently referred to as series A, B and C. The first two were intended to measure the variation of θ_0 with p for a given grain and test position, and to compare this with the dependence of θ_0 on R_*. Series C was designed to measure θ_0 for bed arrangements as shown in figure 2*d*, where single large grains may over-ride the surrounding bed.

(*a*) *Series A: transition range of* R_*

A 10 cm wide flume, 2.5 m long, was laid with a bed of 2.5 mm mean diameter angular polystyrene grains glued in place. With this type of bed, and the hydraulic conditions available, flows were in the transition range of grain Reynolds numbers – from $R_* \simeq 30$ to $R_* \simeq 90$. A hole was drilled through the rough fixed bed of the flume, this was tapped and a threaded rod inserted. To perform an experiment, a test grain was placed in the hole, on top of the rod, which was then screwed upwards, pushing the grain into the flow gradually until it was swept away: figure 3*a* shows the arrangement. This was repeated some twenty times for the known constant hydraulic conditions. The value of u_* was obtained by measuring the velocity profile on the flume centre-line and fitting equation (1.1). Each time the test grain was entrained, the level of the top of the rod was recorded. The vertical velocity of introduction was slow enough that dynamic effects such as those due to wake formation were negligible. Ideally the grain should have been introduced into the flow at an infinitesimally slow rate; in practice it was pushed upwards at a constant finite rate – it did not sample the flow at a given protrusion for a statistically-significant period. In this way these experiments overestimate, if anything, the shear stress corresponding to a given protrusion. To offset this, the results of all twenty runs at each stress were examined, and the appropriate value of protrusion taken to be the minimum recorded, any wildly-divergent results being ignored.

Two series of tests were made – runs A1 to A8 used angular polystyrene grains identical to the bed material. To vary the conditions one of these was drilled and filled with lead to give a higher density: this was used in runs A9 to A15. Because of the angularity and smallness of the grains, it was very difficult to measure an effective mean level of the bed in the vicinity of the test hole relative to which protrusion could be measured. After some difficulty, the protrusion was not recorded, but instead the level of the top of the threaded rod was measured from an arbitrary datum. Results are given in § 3.

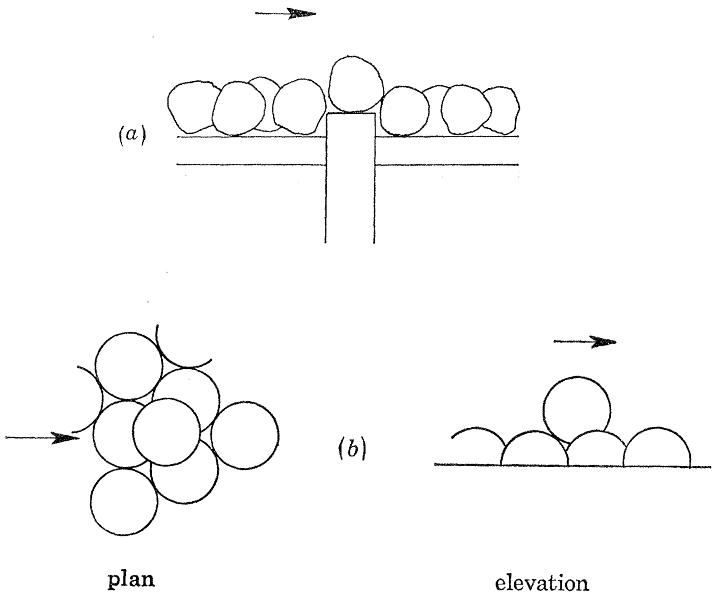

FIGURE 3. Experimental arrangements: (*a*) series A and B, showing rod for pushing test grain up into protruding position; (*b*) series C experiment showing test sphere in position. Arrows indicate flow direction.

(*b*) *Series B: fully turbulent* R_*

This experiment was of the same type as that described in (*a*) and shown in figure 3*a*, but with larger grains so that flow around them was fully turbulent: $R_* = 215$–830. A 10 m long flume, 0.3 m wide was used; the bed material was 0.5 cm–1 cm well-rounded pea gravel glued to a false floor on the flume bottom. Normal flow at a depth of 4.8 cm was used for all tests, the shear stress being varied by changing the bed slope. As in the experiments of series A, a tapped hole was made in the floor of the flume in order to push the single grains into the flow. Test grains used were a porcelain sphere and two selected grains of rounded gravel from the same sample as that used for the bed. These had the following dimensions (cm): grain 1: $1.08 \times 0.79 \times 0.63$ with a nominal diameter (of a sphere having the same volume) of 0.83; grain 2: $1.40 \times 0.74 \times 0.58$, nominal diameter 0.88; and grain 3 was a sphere of diameter 0.64 cm. A plaster of paris mould was used to make models of the three grains with araldite moulding resin and varying proportions of lead filings. These models were the same size and shape as the originals, but with different densities so as to extend the experimental range. The experiments were carried out in the same way as series A, but a smaller scatter of protrusions was recorded, for given hydraulic conditions, so that occasionally only 10 runs were made to define the minimum protrusion at which initiation occurred. Results are given in § 3.

In both series A and B although support by the rod is not precisely the manner of support in a real bed, the fluid forces and the resisting forces of adjacent grains, which are the important quantities governing stability, are representative of real conditions.

(c) *Series C: entrainment of over-riding grain*

These experiments were devised to measure the dimensionless threshold stress for grains which sit on top of an otherwise plane bed, in which case the bed geometry is important. To obtain the lowest bound on the threshold stress for over-riding grains it was reasoned that the situation offering least resistance to motion would be that of a spherical grain sitting above one of the triangular interstices formed in a planar bed of spherical grains packed in a regular hexagonal array. It is easily shown that this gives a relative protrusion of $p/D = 0.82$.

A hexagonal array of hemispheres was set up in the 10 m long × 0.30 m wide flume of series B, using wooden hemispheres of 3.8 cm diameter with their bases glued to the floor of the flume as shown in figure 3*b*. In this situation the bed was non-porous; it was considered that seepage effects on exposed grains would be negligible compared with other fluid forces. Two test grains were made by filling table tennis balls (of the same diameter as the wooden hemispheres) with homogeneous mixtures of lead shot, polystyrene grains and sand. For a given flow, the test sphere was placed over an interstice of the hexagonal array and carefully freed from restraint. A series of runs was made, adjusting the slope of the flume until flow conditions were such that the sphere rolled out of the test position only once in a total period of 5 min. Thus, the criterion of initiation was one of 'first displacement' rather than a small finite rate of transport. This arbitrary but consistent definition of motion was found to be precise in defining threshold of movement. The range of stresses, for a particular series, over which the grain exhibited successively no motion, occasional fluttering, and repeated fluttering with occasional dislodgement, was found to be small. Results are given in the following § 3.

3. Results

(a) *Series A: transition R_*: variation of θ_0 with p/D*

The results are shown numerically in table 1 and are plotted in figure 4. Values of protrusion p were calculated by assuming a reasonable arbitrary bed level: the curve drawn in figure 4 shows the minimum measured threshold stress for a given relative protrusion. Although the results were taken from one particular shape of grain in a particular experimental location, the magnitude of the effect of relative protrusion can be seen. From Shields' diagram (figure 1), a hundredfold increase in R_* from 10 to 1000 gives a corresponding increase in θ_0 from 0.03 to 0.06. Allowing

Table 1. Results of experimental series A, giving the relative protrusion (p/D) at which particles were entrained, and the corresponding dimensionless shear stress θ_0

run number	R_*	θ_0	p/D
A1	66	0.248	−0.20
A2	54	0.164	−0.06
A3	39	0.085	0.30
A4	30	0.050	0.30
A5	27	0.040	0.34
A6	51	0.149	0.04
A7	46	0.119	0.16
A8	35	0.070	0.25
A9	72	0.072	0.34
A10	48	0.032	0.41
A11	78	0.083	0.08
A12	88	0.107	0.05
A13	61	0.053	0.44
A14	70	0.071	0.35
A15	73	0.076	0.16

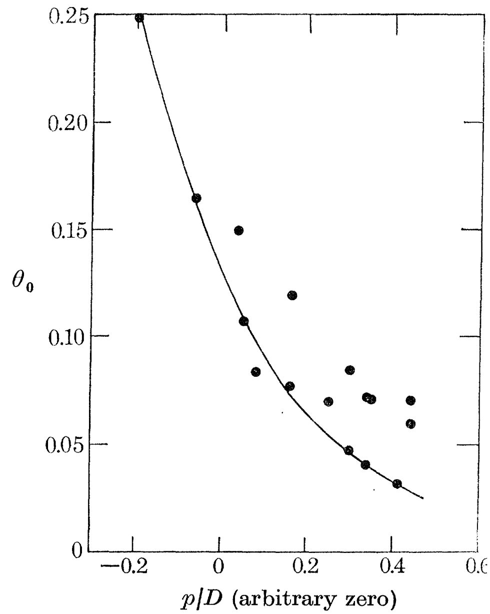

Figure 4. Results from experimental series A, giving the dimensionless stress θ_0 necessary to entrain a grain of relative protrusion p/D. ●, Experimental values; the curve is an approximation to the minimum values.

a particular grain to protrude, this same increase in resistance to motion would be effected by a relative masking of the grain of only $0.2D$. Obviously the co-planarity achieved during laying of a bed is crucial in determining θ_0.

(b) *Series B: fully-turbulent R_*: variation of θ_0 with p/D*

Results are shown in table 2 and figure 5. In calculating θ_0 and R_*, the diameter D was taken to be the nominal diameter – that of a sphere of the same volume. The minimum diameter was used in calculating p/D, as in each experiment the grains were orientated so that this minimum dimension was aligned vertically. Using this diameter correlated the results from the three differently-shaped grains quite well, as can be seen on figure 5, which has been plotted with differently-shaped points representing the different grains and the different degrees of shading representing different Reynolds numbers as described in the caption. While there is apparently

TABLE 2. RESULTS OF EXPERIMENTAL SERIES B, GIVING THE RELATIVE PROTRUSION (p/D) AT WHICH PARTICLES WERE ENTRAINED, AND THE CORRESPONDING DIMENSIONLESS SHEAR STRESS θ_0

run number	grain type	R_*	θ_0	p/D	run number	grain type	R_*	θ_0	p/D
B1	2	830	0.072	0.30	B31	2	525	0.029	0.46
B2	2	820	0.062	0.24	B32	2	525	0.025	0.53
B3	2	820	0.151	0.04	B33	2	525	0.062	0.25
B4	2	820	0.213	−0.17	B34	2	525	0.087	0.18
B5	2	820	0.413	−0.30	B35	3	380	0.044	0.41
B6	3	600	0.085	0.09	B36	3	380	0.035	0.47
B7	3	600	0.107	0.10	B37	3	380	0.061	0.31
B8	3	600	0.149	0.05	B38	3	380	0.152	0.04
B9	1	770	0.289	−0.28	B39	3	315	0.030	0.40
B10	2	690	0.051	0.38	B40	3	315	0.023	0.53
B11	2	690	0.044	0.36	B41	3	315	0.041	0.08
B12	2	690	0.107	0.21	B42	3	315	0.103	0.03
B13	2	690	0.150	−0.09	B43	3	315	0.167	−0.04
B14	1	650	0.203	−0.08	B44	2	430	0.020	0.45
B15	3	500	0.075	0.22	B45	2	430	0.017	0.43
B16	3	500	0.059	0.29	B46	2	430	0.042	0.30
B17	3	500	0.104	0.08	B47	2	430	0.058	0.24
B18	3	500	0.262	−0.03	B48	2	430	0.113	−0.01
B19	3	460	0.062	0.23	B49	1	405	0.079	−0.01
B20	3	460	0.049	0.34	B50	1	405	0.141	−0.05
B21	3	460	0.087	0.16	B51	2	290	0.019	0.48
B22	3	460	0.218	−0.02	B52	2	290	0.027	0.36
B23	3	460	0.354	−0.08	B53	2	290	0.052	0.21
B24	1	590	0.169	−0.07	B54	1	275	0.037	0.24
B25	1	590	0.301	−0.12	B55	1	275	0.065	0.16
B26	2	630	0.042	0.25	B56	3	215	0.048	0.30
B27	2	630	0.036	0.46	B57	3	215	0.100	0.01
B28	2	630	0.089	0.12	B58	3	215	0.144	0.05
B29	2	630	0.125	0.03	B59	3	215	0.077	0.10
B30	2	630	0.241	0.00					

little dependence on grain shape, the grain Reynolds number does affect results slightly.

Considering the variation of θ_0 with p/D, the results are qualitatively similar to those from series A, but in this case the curve through the minima has a steeper gradient so that θ_0 is even more strongly dependent on p/D. As it was possible to measure local bed levels around the test site, the p/D values are absolute. It must be remembered that these results apply only to the site where the test was made. Around the hole the other grains were situated randomly in plan, giving a unique geometrical arrangement and the presumably unique results shown. From figure 5,

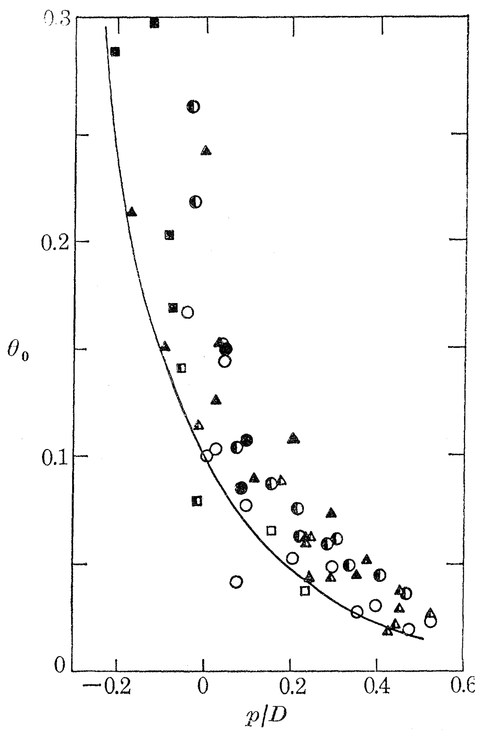

FIGURE 5. Results from experimental series B: □, grain 1; △, grain 2; ○, grain 3. Unshaded, $200 < R_* < 350$; half shaded, $350 < R_* < 550$; shaded, $550 < R_* < 830$.

for a co-planar bed ($p = 0$), $\theta_0 \simeq 0.10$. The fact that the dimensionless stress found by Shields for fully-turbulent flow was 0.06 does not conflict with this result, because of the uniqueness of the present geometrical arrangement, and the deficient sampling technique leading one to expect higher recorded values of θ_0.

The Shields range of θ_0 is from 0.05 to 0.06 with a scatter of ± 0.005, over the experimental range of R_* in the present work. Figure 5 shows that this same variation in θ_0 would be accomplished by varying the protrusion of a grain by a mere 1/20 of its diameter. Thus, under turbulent flow conditions, as with transition range flows, protrusion is an important parameter governing grain stability.

(c) *Series C: measurement of θ_0 for over-riding grains*

Results from the experiments are given in table 3. Taking the mean of the five values of θ_0, the dimensionless threshold stress corresponding to a maximum possible protrusion is 0.0105, or stated more in keeping with the accuracy of the results, is 0.01 ± 0.001. Previously accepted values are 0.03–0.06 for co-planar beds under turbulent flows, depending on the criterion of movement.

TABLE 3. RESULTS OF EXPERIMENTAL SERIES C, GIVING THE DIMENSIONLESS THRESHOLD STRESS FOR A PROTRUSION OF 0.82

run number	h/D	R_*	θ_0
C1	4.4	1690	0.0094
C2	4.3	1700	0.0097
C3	2.8	1760	0.0104
C4	2.8	3200	0.0113
C5	4.2	3280	0.0119

4. PREVIOUS EXPERIMENTS ON MOVEMENT OF OVER-RIDING GRAINS

Coleman (1967) performed a number of experiments in a water tunnel to measure the drag and lift forces on an over-riding sphere placed at an interstice of an hexagonal array of spheres – a geometrical situation identical to that of our series *C*. He measured the drag force directly, then released the sphere from restraint and determined the conditions at which it rolled out of its original position. In this way he was able to calculate the lift force. All his results were in terms of the mean velocity at the centre of the over-riding grain. However, by using some well-established results for flow over rough beds his results can be converted so that they are in terms of the shear velocity and can be related to the present work and to the Shields diagram.

The following data were scaled off Coleman's figure 4: lift factor at motion initiation K, and maximum drag coefficient C_d corresponding to a Reynolds number $Re = \bar{u}_{\frac{1}{2}}D/\nu$, where $\bar{u}_{\frac{1}{2}}$ is the mean velocity at the centre of the over-riding grain. These results are shown in table 4. It is a simple matter to show that

$$\theta_0 = \frac{0.471(1-K)}{C_d}\left(\frac{u_*}{\bar{u}_{\frac{1}{2}}}\right)^2, \tag{4.1}$$

and

$$R_* = Re\left(\frac{u_*}{\bar{u}_{\frac{1}{2}}}\right). \tag{4.2}$$

Using the logarithmic velocity distribution

$$\bar{u}/u_* = 5.75\lg y/D + f_1(R_*),$$

and assuming that the origin is $0.2D$ below the mean tops of the bed grains.

$$\begin{aligned}\bar{u}_{\frac{1}{2}}/u_* &= 5.75 - \lg 0.5 + f_1(R_*)\\ &= f_1(R_*) - 1.7.\end{aligned} \tag{4.3}$$

TABLE 4. RESULTS FOR $Re = \bar{u}_{\frac{1}{2}}D/\nu$, C_d AND K TAKEN FROM FIGURE 4 OF COLEMAN (1967), AND CONVERTED TO R_* AND θ_0 ASSUMING A LOGARITHMIC VELOCITY DISTRIBUTION, AND A LAMINAR PROFILE FOR LOW VALUES OF R_*

			logarithmic profile			laminar profile		
Re	C_d	K	R_*	$\bar{u}_{\frac{1}{2}}/u_*$	θ_0	R_*	$\bar{u}_{\frac{1}{2}}/u_*$	θ_0
47	2.8	−0.58	6.2	7.6	0.0046	9.7	4.8	0.0113
56	2.6	−0.56	7.3	7.7	0.0048	11	5.3	0.0100
56	2.6	−0.36	7.3	7.7	0.0042	11	5.3	0.0088
79	2.1	−0.18	10	7.9	0.0042	13	6.3	0.0067
97	1.9	−0.08	13	7.9	0.0043	14	7.0	0.0055
96	1.9	0.03	12	7.9	0.0039	14	7.0	0.0052
110	1.7	0.11	15	7.8	0.0040	15	7.5	0.0044
140	1.5	0.22	19	7.6	0.0042	17	8.4	0.0035
140	1.5	0.35	19	7.6	0.0035	—	—	—
170	1.3	0.37	23	7.5	0.0040	—	—	—
220	1.2	0.36	31	7.2	0.0050	—	—	—
300	0.99	0.53	43	7.1	0.0044	—	—	—
390	0.87	0.48	58	7.0	0.0058	—	—	—
540	0.76	0.48	82	6.8	0.0070	—	—	—
860	0.65	0.41	130	6.8	0.0093	—	—	—
950	0.63	0.41	140	6.8	0.0095	—	—	—
1060	0.62	0.36	160	6.8	0.0105	—	—	—
1300	0.60	0.33	190	6.8	0.0114	—	—	—
2400	0.55	0.33	360	6.8	0.0124	—	—	—
2550	0.55	0.25	380	6.8	0.0139	—	—	—
3400	0.54	0.44	500	6.8	0.0106	—	—	—
4200	0.54	0.37	610	6.8	0.0119	—	—	—
9100	0.54	0.44	1340	6.8	0.0106	—	—	—

This equation is not applicable for flows in which the grain is immersed in the laminar sublayer. For this case,

$$\bar{u}/u_* = u_* y/\nu,$$

giving

$$\bar{u}_{\frac{1}{2}}/u_* = 0.5R_*. \tag{4.4}$$

Using (4.1)–(4.4), taking values of $f_1(R_*)$ from Schlichting (1968, p. 583) and with Re, K and C_d values as shown in table 4, the results for θ_0 and R_* were obtained, given in table 4 and plotted on figure 6.

The assumption of $\bar{u}_{\frac{1}{2}}/u_*$ plays a crucial rôle in determining θ_0. In view of the uncertainty of the velocity law in the transition region ($5 < R_* < 70$), no importance should be attached to results obtained for this range. As shown on figure 6, the two assumptions (4.3) and (4.4) indicate a marked dip in the variation of θ_0 with R_*, however there is unlikely to be a sudden jump from one law to the other as the experimental points show. A transition curve will join the two, possibly showing a dip between $R_* \simeq 10$ and $R_* \simeq 100$.

Coleman's results agree well with results from series C of the present work for fully-turbulent flow, giving $\theta_0 \simeq 0.01$. It must be emphasized, however, that the analysis of this section based upon previous experimental results and assumed velocity distributions, is an estimate only.

5. Relative protrusion and the Shields diagram

On figure 6 are shown the results of §§ 3 and 4, together with the band of results obtained by Shields which is shown shaded: important features of the latter are the horizontal asymptote of $\theta_0 = 0.06$ for large R_*, the tendency towards a straight line of finite slope for small R_*, and a large dip between R_* of 1–1000. Superimposed on this are the results of experimental series B and C and the results of the calculations based on Coleman's work. Series B results are taken from the minimum curve of figure 5 – the horizontal lines show the threshold stress corresponding to the protrusion indicated. As described in § 3 (*b*), for these results it is not the magnitude of

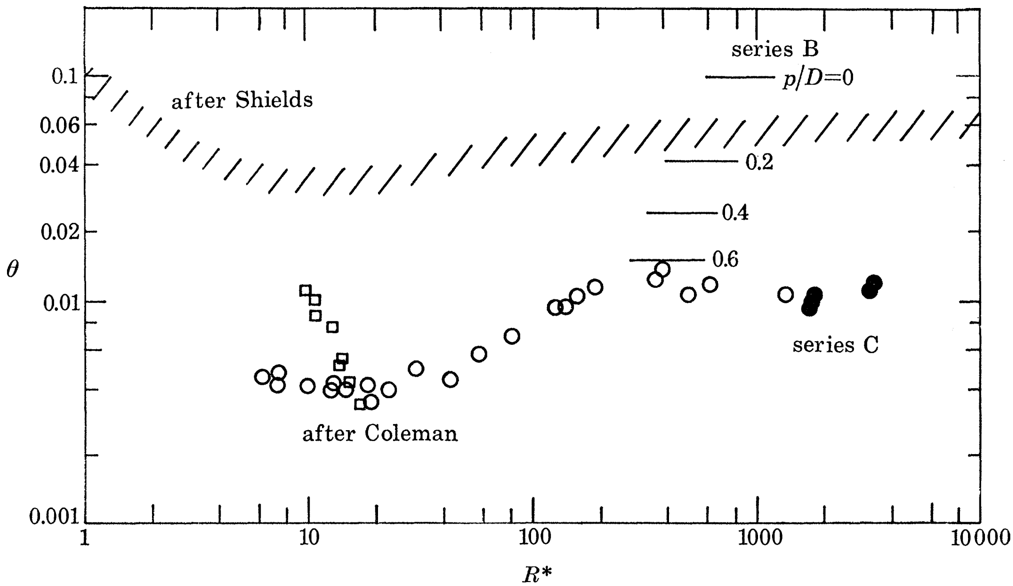

FIGURE 6. Shields plot of dimensionless shear stress against grain Reynolds number. The shaded area defines the threshold stress θ_0 for flat beds as given by Shields. Lines identified as series B are taken from the minimum curve of figure 5, showing the variation of θ_0 with protrusion for a given location. ●, Results from series C for over-riding grains; ○, re-analysed results from Coleman (1967), assuming logarithmic velocity profile; □, the same, with a laminar profile.

θ_0 from series B that is of interest, but rather the magnitude of its variation with p/D. From the diagram it is clear that relative to variation with R_*, which is the whole basis of the Shields plot, p/D is an important parameter indeed – particularly for small values of protrusion.

This is supported by the results plotted from series C and from § 4, showing that experiments for large p/D give values of θ_0 well below the Shields range, asymptoting to 0.01 for large R_*.

In view of these results an hypothesis for the extent of the dip in the Shields diagram may be proposed. Other hypotheses have been put forward by Egiazaroff

(1967) and Taylor & Vanoni (1972), both assuming the dip to be due to variation of velocity with R_*. Egiazaroff assumed that the velocities causing motion are the steady velocities at the bed given by the logarithmic law or the laminar law, obtaining a dip in the same way as our calculations based on Coleman's results. This dip had an upper limit at $R_* \simeq 100$. Taylor & Vanoni allowed for turbulence by considering the initiating velocities at the bed to be proportional to the root-mean-square velocity, and showed that the dip should have a minimum at $R_* \simeq 35$. Ignoring the mean velocity at the bed, on to which the turbulent fluctuations are superimposed, seems rather unjustified and it comes as little surprise that the dip minimum does not occur at the predicted value. The analysis described in § 4, and as plotted on figure 6, shows that the dip has an upper limit of $R_* \simeq 150$ *for constant protrusion*. However, the dip in the Shields plot exists for values of R_* greater than 1000. The following geometrical explanation is proposed.

Consider the nature of all the initiation experiments used to build up the Shields diagram: large grains have been used to obtain results at large R_* and small grains at the other end of the scale. All these experiments have been on 'flat' beds. Using large material, each grain can be laid so that its top is level with all the others, giving a truly co-planar bed of $p/D = 0$. With grains smaller than a certain size, however, it is physically impossible to lay a truly co-planar bed so that some finite protrusion will occur, and a lower θ_0 will be measured. As an example, consider experiments for $R_* \simeq 1000$, assuming $\nu = 0.01\,\mathrm{cm^2\,s^{-1}}$, relative density 2.65 and $\theta_0 \simeq 0.06$, giving $u_* \simeq 10\,\mathrm{cm\,s^{-1}}$, and $D \simeq 1$ cm. Unless an experiment was done with spheres, natural variation and laying of grains this size could not guarantee a protrusion of less than 1 mm ($p/D = 0.1$) – quite large enough to explain the value of θ_0 at $R_* = 1000$ being somewhat less than the value at 10000, which is far outside the range of fluid dynamic variation with R_* itself.

For smaller grains, standards of levelling become worse, higher protrusions occur, and values of θ_0 become smaller, varying with p/D as well as R_*. When working materials are so fine (e.g. sand) that no individual grain levelling is possible, all experiments must be on beds where grains will project to almost their full size. An estimate for R_*, below which this must occur, can be made by assuming that total protrusion is inevitable for grains smaller than $D = 0.5$ mm. Then taking $\nu = 0.01\,\mathrm{cm^2\,s^{-1}}$ and $\theta_0 = 0.03$ (corresponding roughly to the range $R_* = 5$–100), this gives $u_* \simeq 1.4\,\mathrm{cm\,s^{-1}}$, $R_* \simeq 14$, and in keeping with the approximate nature of this analysis, assume that this limit is $R^* = 10$. For values of R_* smaller than this, p/D having reached its maximum possible, all variation of θ_0 is with R_* alone. Thus, in this range, all experimental geometrical conditions will correspond to all field conditions.

6. Conclusions

The Shields diagram, showing the supposedly unique dependence of dimensionless threshold stress θ_0 on grain Reynolds number R_*, implicitly contains variation with relative protrusion p/D. For values of R_* less than 10 all experimental and field situations have grains which are so small as to make individual levelling impossible. At the other end of the scale, for R_* greater than 1000, grains used in experiments to set up the Shields diagram have been large enough that a careful levelling of each has been possible. Between these two extremes the diagram represents a transition in two quantities – (i) between total immersion of the bed in the laminar sublayer and the destruction of this layer ($10 < R_* < 150$), and (ii) between no levelling and careful levelling ($10 < R_* < 1000+$).

For bed material that is placed in position so that individual grains may override others, the Shields diagram is applicable for R_* up to about 10, and after that becomes less and less relevant as the geometry of the bed surface becomes more and more unlike those used to build up the Shields diagram. Experiments performed on spheres resting on a hexagonal array of hemispheres yielded $\theta_0 = 0.01$ for a criterion of first displacement. This contrasts with accepted values of 0.03 and 0.06 for co-planar beds for, respectively, first displacement and a small finite rate of transport. The value of 0.01 should be used in studies of the erosion of coarse natural sediments and for the scour design criterion when coarse-grained materials are used in construction work without any levelling. In each of these situations individual grains may over-ride others by almost a complete diameter. However, the criterion applies to the whole bed, for if the surface grains are removed then those underneath will be exposed, some by almost a complete diameter, and the process repeated continually.

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References

Coleman, N. L. 1967 A theoretical and experimental study of drag and lift forces acting on a sphere resting on a hypothetical stream bed. *Proc. 12th Congress, IAHR* **3**, 185–192. Fort Collins.

Egiazaroff, I. V. 1967 Discussion of 'Sediment transportation mechanics: initiation of motion'. *J. Hyd. Div. ASCE* **93**, HY4, 281–287.

Neill, C. R. 1967 Mean-velocity criterion for scour of coarse uniform bed-material. *Proc. 12th Congress IAHR* **3**, 46–54. Fort Collins.

Neill, C. R. 1968 Note on initial movement of coarse uniform bed-material. *J. Hyd. Res.* **6**, 173–176.

Paintal, A. S. 1971 Concept of critical shear stress in loose boundary open channels. *J. Hyd. Res.* **9**, 91–114.

Schlichting, H. 1968 *Boundary layer theory*, 6th ed. New York: McGraw-Hill.

Shields, A. 1936 Anwendung der Ähnlichkeitsmechanik und Turbulenzforschung auf die Geschiebebewegung, *Mitteil. Preuss. Versuchsanstalt Wasserbau Schiffbau*, Berlin, no. 26, Translation by W. P. Ott & J. C. van Uchelen, Soil Conservation Service, California Institute of Technology, Pasadena.

Taylor, B. D. & Vanoni, V. A. 1972 Temperature effects in low-transport, flat-bed flows. *J. Hyd. Div. ASCE* **98**, 1427–1445.