SHOALING WAVES : NUMERICAL SOLUTION OF EXACT EQUATIONS

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Recently Longuet-Higgins and Cokelet have developed a method for the study of time-dependent two-dimensional irrotational free surface problems, which they applied to spatially-periodic waves in water of infinite depth.

Their approach may be extended to the generation and propagation of disturbances over a bed of arbitrary shape and motion. Kinematic and dynamic equations at the free surface and the bed may be expressed as Lagrangian differential equations for the motion of particles in terms of velocity potential and its derivatives. These are obtained from the numerical solution of a mixed boundary value problem formulated as an integral equation. For any free surface configuration and velocity potential distribution, the integral equation may be solved to give the velocities on the bed and surface, which may be integrated to give the subsequent form of the surface, and the process repeated.

Large amplitude motions on boundaries of irregular shape may be studied, as the method is analytically exact, provided the initial conditions are known. In the present study we examine the motion of disturbances as they approach a coastline. The method may be applied to many two-dimensional problems where magnitude of disturbance and time-dependence have rendered solution previously difficult.

1. INTRODUCTION

This work is based upon research carried out by M.S. Longuet-Higgins and E.D. Cokelet, who obtained a method for time-dependent, irrotational, free-surface problems which they applied to periodic waves in deep water. They used a combination of Eulerian and Lagrangian equations to follow the motion of surface particles, requiring the prior knowledge of initial velocity potential on the free surface of initially-known shape. A set of Lagrangian differential equations were obtained for the co-ordinates and velocity potential of surface particles in terms of surface pressure, elevation and fluid velocities. Although the tangential component of velocity was obtained by differention of velocity potential along the surface, the normal component had to be found by other means. An integral equation was set up, which could be solved for the normal velocity in terms of the shape of the surface and the potential distribution. Then, using the differential equations, subsequent motion of the surface particles was followed. This procedure was repeated for a number of time steps and remarkable pictures of plunging breakers obtained, this breaking caused by a brief application of pressure, antisymmetrically.

In the present work we extend Longuet-Higgins and Cokelet's method to the generation and propagation of disturbances in water of finite depth. Section 2 describes the generalization of their method to allow for the presence and possible motion of a solid boundary of arbitrary shape, the sea bed. Unlike most work on surface gravity waves in water, no essential approximations are made at the analytical stage. In §3, some of the numerical techniques are described, wherein approximations are first made, while application of the method is described in §4.

2. DEFORMATION OF A FREE WATER SURFACE

Consider a layer of liquid, subject to gravity, bounded below by a curve of arbitrary shape as shown in Fig. 1.

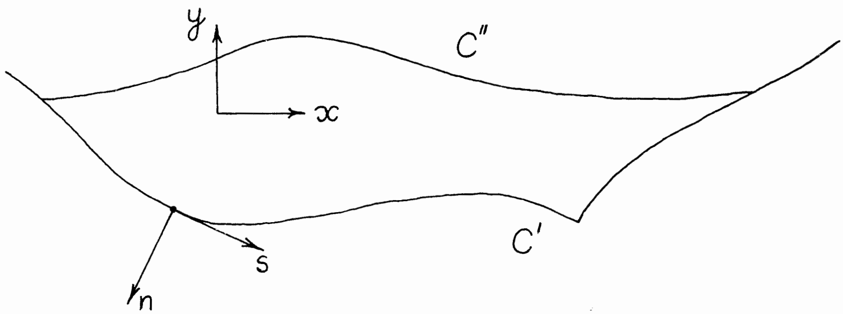


Figure 1

The curve C' denotes the solid boundary which may be given an arbitrary motion, while C" is the free surface on which the pressure distribution may be specified. A rectangular co-ordinate system (x,y) is introduced. Motion of the liquid is taken to be irrotational and incompressible such that the velocity potential ϕ exists and satisfies Laplace's equation. That is, the velocity $\underline{u}$ is given by

$$\underline{u} = \nabla\phi$$

and $$\nabla . \underline{u} = \nabla^2\phi = 0 \tag{1}$$

throughout the area bounded by the curves C', C".

At the free surface $y = y_s(x,t)$ we have the kinematic conditions

$$\frac{Dx}{Dt} = \frac{\partial\phi}{\partial x} \tag{2}$$

and $$\frac{Dy}{Dt} = \frac{\partial\phi}{\partial y}, \tag{3}$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla\phi . \nabla, \tag{4}$$

which denotes differentiation following a given particle. Also, at the free surface, we have the dynamic boundary condition derived from Bernoulli's equation:

$$\frac{\partial\phi}{\partial t} + \frac{p_s}{\rho} + g y_s + \frac{1}{2}(\nabla\phi)^2 = 0, \tag{5}$$

where p_s is the pressure applied at the surface. Combining (4) and (5) we have an expression for the rate of change of ϕ following a particle:

$$\frac{D\phi}{Dt} = -\frac{p_s}{\rho} - g y_s + \frac{1}{2}(\nabla\phi)^2. \tag{6}$$

On the solid boundary C', we have to satisfy the kinematic boundary condition

$$\frac{Dn}{Dt} = V_n \tag{7}$$

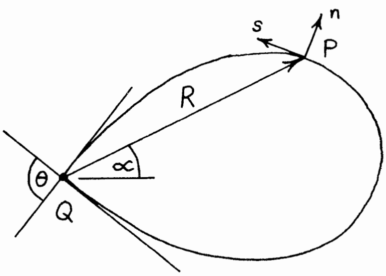
and $$\frac{Ds}{Dt} = \frac{\partial\phi}{\partial s}, \tag{8}$$

where n is the outwardly-directed normal as shown in Fig. 1, and where Vn is the component of the velocity of the boundary normal to itself.

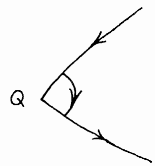
Equations (2), (3), (7) and (8) give us expressions for the time rate of change of the co-ordinates of particles on the boundary of the fluid region, which effectively define the boundary for us, while (6) gives an expression for the rate of change of ϕ for surface particles. These expressions are in terms of the applied surface pressure distribution, motion of the solid boundary, elevation y, and the derivatives of ϕ .

The problem is now to obtain the space derivatives of ϕ for a given geometry. If at some initial instant the derivatives of ϕ are known at all points on the bounding curve C', C", then equations (2), (3), (7) and (8) determine the position of each point a short time later, and similar integration of (6) would give the corresponding ϕ on the new contour. By differentiating along the contour we can obtain the tangential component of velocity $\partial\phi/\partial s$, however,we also need the normal component $\partial\phi/\partial n$.

To obtain this normal derivative, the problem may be formulated as an integral equation using Green's Theorem as follows. Consider a point P on the boundary of a closed contour C and let (R,α) be the polar co-ordinates of P with respect to a point Q also on the contour. For generality we assume that the contour is discontinuous at Q, with interior angle θ , as shown in Fig. 2.



(a) Relationship between points P & Q.



(b) Indented contour at Q.

Figure 2.

Consider the contour indented at Q as shown in Fig.2. If ϕ and G are analytic functions throughout the enclosed region and on the contour, we may write

$$\int_C \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right) ds = 0,$$

provided that no singularities are enclosed. If we let $G = \log R$, such that $\nabla^2 G = 0$, then by familiar manipulations we obtain

$$⨍_C \left(\phi \frac{\partial}{\partial n}(\log R) - \log R \frac{\partial \phi}{\partial n}\right) ds - \phi(Q).\,\theta(Q) = 0 , \tag{9}$$

where the $⨍$ indicates that the principal value of the integral is to be taken at Q. This bar notation will be discontinued, it being understood that all integrals are to be evaluated in the principal sense. We may simplify one of the terms by noting that $\log R$ and α are conjugate functions: $\partial(\log R)/\partial n = \partial\alpha/\partial s$, giving

$$\int_C \frac{\partial \phi}{\partial n}(P).\log R(P,Q).\,ds(P) - \int_C \phi(P).\,d\alpha(P,Q) + \phi(Q).\,\theta(Q) = 0 . \tag{10}$$

This is valid at all points Q on the contour.

Equation (10), in conjunction with the differential equations developed previously, may be used to describe motion subsequent to an initially known disturbance or applied surface pressures and ground motions. At any time, provided ϕ is known on the surface, and $\partial\phi/\partial n$ can be specified on the solid boundary, we can solve for the unknown $\partial\phi/\partial n$ on the surface and ϕ on the solid boundary, from which $\partial\phi/\partial s$ can be found. Thus, particle velocities are known, and the differential equations can be used to describe a small movement of the contour and the changes in ϕ on it, before the integral equation (10) has to be solved for the new geometry, and the whole process continually repeated.

3. NUMERICAL METHODS

In this section, we describe numerical methods which have been developed for the system of equations obtained in §2.

(a) Solution of the integral equation

This step is not unduly complicated, for the I.E. (10) is linear, with simple kernel functions. Perhaps the worst part is the singularity of the logarithmic kernel as P → Q: the other singularity has been removed with the change from s to α as the variable of integration. We now wish to approximate the I.E. by a matrix equation

$$\begin{bmatrix} a_{QP} \end{bmatrix} \begin{bmatrix} (\partial\phi/\partial n)_{P(C'')} \\ \hdashline \phi_{P(C')} \end{bmatrix} = \begin{bmatrix} b_Q \end{bmatrix}. \tag{11}$$

For the first term in (10) we consider three adjacent particles, denoted by -1, 0, +1, respectively, as shown in Fig. 3.

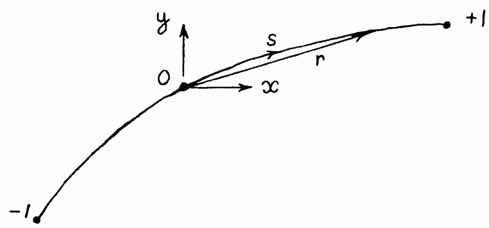


Figure 3 Computational module for $\int_{j-1}^{j+1} \frac{\partial\phi}{\partial n} \cdot \log R \,.\, ds$

Initially, the problem was to obtain expressions for the arc length s, with origin at the centre particle. To do this it was found convenient to introduce the co-ordinate r, the distance between 0 and a point on the curve. Let $\partial\phi/\partial n$ on the curve be:

$$\frac{\partial\phi}{\partial n} = v_0 + v_0' r + \tfrac{1}{2} v_0'' r^2 + O(r^3), \tag{12}$$

and adopting temporary (x,y) with origin at 0,

$$x = x_0' r + \frac{1}{2!} x_0'' r^2 + \frac{1}{3!} x_0''' r^3 + O(r^4),$$
$$y = y_0' r + \frac{1}{2!} y_0'' r^2 + \frac{1}{3!} y_0''' r^3 + O(r^4). \tag{13}$$

Clearly, $x^2 + y^2 = r^2$; substituting (13) into this and equating powers of r, we obtain equations connecting x_0', y_0', etc. Then, using $(ds/dr)^2 = (dx/dr)^2 + (dy/dr)^2$, we obtain

$$ds = (1 + r^2(x_0''^2 + y_0''^2)/8 + O(r^3))\,dr$$
$$= (1 + c)\,dr + O(r^3 . dr),$$

which, multiplied by (12), gives

$$\frac{\partial\phi}{\partial n} . ds = \left(v_0 + r v_0' + r^2\left(\tfrac{1}{2} v_0'' + \tfrac{v_0}{8}(x_0''^2 + y_0''^2)\right)\right) dr + O(r^3 dr). \tag{14}$$

To evaluate the derivatives, we make use of the Taylor expansions in r evaluated at -1, +1 to give

$$\begin{Bmatrix} x_o' \\ y_o' \\ v_o' \end{Bmatrix} = \frac{1}{r_1 - r_{-1}} \left(\frac{r_1}{r_{-1}} \begin{Bmatrix} x_{-1} \\ y_{-1} \\ v_{-1} \end{Bmatrix} + \left(\frac{r_{-1}}{r_1} - \frac{r_1}{r_{-1}} \right) \begin{Bmatrix} x_o \\ y_o \\ v_o \end{Bmatrix} - \frac{r_{-1}}{r_1} \begin{Bmatrix} x_1 \\ y_1 \\ v_1 \end{Bmatrix} \right) + O(r^2), \tag{15}$$

$$\frac{1}{2} \begin{Bmatrix} x_o'' \\ y_o'' \\ v_o'' \end{Bmatrix} = \frac{1}{r_1 - r_{-1}} \left(\frac{1}{r_1} \begin{Bmatrix} x_1 \\ y_1 \\ v_1 \end{Bmatrix} + \left(\frac{1}{r_{-1}} - \frac{1}{r_1} \right) \begin{Bmatrix} x_o \\ y_o \\ v_o \end{Bmatrix} - \frac{1}{r_{-1}} \begin{Bmatrix} x_{-1} \\ y_{-1} \\ v_{-1} \end{Bmatrix} \right) + O(r). \tag{16}$$

These may be substituted into (14), giving $(\partial\phi/\partial n).ds$ in terms of the known nodal values of x and y and the as-yet unknown values of $\partial\phi/\partial n$.

Now we have to deal with the log R term: we write

$$R = R_o + r R_o' + \frac{1}{2!} r^2 R_o'' + O(r^3),$$

where the central derivatives are evaluated using (15) and (16) with nodal values of R. Combining the equations of this section, we obtain

$$\begin{aligned} \int_{-1}^{1} \frac{\partial\phi}{\partial n}.\log R.\,ds = {} & v_o \left[I_o + C I_2 + \frac{1}{r_1 - r_{-1}} \left(I_1 \left(\frac{r_{-1}}{r_1} - \frac{r_1}{r_{-1}} \right) + I_2 \left(\frac{1}{r_{-1}} - \frac{1}{r_1} \right) \right) \right] \\ & + v_{-1} \left[\frac{1}{r_1 - r_{-1}} \left(\frac{r_1}{r_{-1}} I_1 - \frac{1}{r_{-1}} I_2 \right) \right] \\ & + v_1 \left[\frac{1}{r_1 - r_{-1}} \left(\frac{-r_{-1}}{r_1} I_1 + \frac{1}{r_1} I_2 \right) \right] \quad + O(r_1^4, r_{-1}^4). \end{aligned} \tag{17}$$

The terms v_o, v_{-1}, v_1 are the unknowns $\partial\phi/\partial n(0,-1,1)$ the terms inside the square brackets are the contributions to the matrix coefficients. In these,

$$I_o = \int_{r_{-1}}^{r_1} \log\left(R_o + r R_o' + \frac{1}{2!} r^2 R_o''\right) dr,$$

$$I_1 = \int_{r_{-1}}^{r_1} r.\log(R_o + \ldots)\,dr,$$

$$I_2 = \int_{r_{-1}}^{r_1} r^2 \log(R_o + \ldots)\,dr.$$

Initially, we developed analytical expressions for these integrals, which were subsequently found to be ill-suited, often the integral being the difference of two very large numbers, one obtained at +1, the other at -1. Eventually, we resorted to a 5 point integration formula which was found to give good results, the error terms being of order r_1^6, r_{-1}^6. For computational modules incorporating the singularity, this was subtracted and integrated analytically.

The second term in the integral equation, $\int \phi.d\alpha$, could be treated more simply. Initially, for a given point Q, all the values of α were calculated from

$$\alpha(P,Q) = \tan^{-1} \frac{y_P - y_Q}{x_P - x_Q}, \tag{18}$$

except for the point P = Q, which case is described further below.

For this term, the basic computational module was composed of five particles, numbered from -2 to +2 in the direction of integration. Writing a Taylor expansion for ϕ and α in terms of the particle number j, we have

$$\phi = \phi_o + j\phi_o' + \frac{1}{2!} j^2 \phi_o'' + \frac{1}{3!} j^3 \phi_o''' + \frac{1}{4!} j^4 \phi_o^{IV} + \text{5th order terms}, \tag{19a}$$

$$\alpha = \alpha_o + j\alpha_o' + \frac{1}{2!} j^2 \alpha_o'' + \frac{1}{3!} j^3 \alpha_o''' + \frac{1}{4!} j^4 \alpha_o^{IV} + \text{5th order terms}, \tag{19b}$$

where the primes represent differentiation with respect to j. By evaluating these at each of the points -2, -1, +1 and +2, and solving the four equations, we obtain

$$\phi_o' = \frac{2}{3}(\phi_1 - \phi_{-1}) - \frac{1}{12}(\phi_2 - \phi_{-2}), \tag{20a}$$

$$\phi_o'' = \frac{4}{3}(\phi_1 + \phi_{-1}) - \frac{1}{12}(\phi_2 + \phi_{-2}) - \frac{5}{2}\phi_o, \tag{20b}$$

$$\phi_0''' = \tfrac{1}{2}(\phi_2 - \phi_{-2}) - (\phi_1 - \phi_{-1}) \ , \tag{20c}$$

$$\phi_0^{IV} = \phi_2 + \phi_{-2} - 4(\phi_1 + \phi_{-1}) + 6\phi_0 \ . \tag{20d}$$

Four identical equations were developed for α, with the derivatives at 0 in terms of nodal values. Now, differentiating (19b) with respect to j, multiplying by (19a) and integrating

$$\begin{aligned}\int_{\alpha(-2)}^{\alpha(2)} \phi \, . \, d\alpha &= \int_{-2}^{2} \phi \frac{d\alpha}{dj}\, dj \\ &= \phi_{-2}\left(-\tfrac{2}{9}\alpha_0' + \tfrac{4}{9}\alpha_0''\right) + \phi_{-1}\left(\tfrac{32}{9}\alpha_0' - \tfrac{38}{9}\alpha_0''\right) + \phi_0\left(-\tfrac{8}{3}\alpha_0' + \tfrac{8}{3}\alpha_0'''\right) \\ &\quad + \phi_1\left(\tfrac{32}{9}\alpha_0' + \tfrac{38}{9}\alpha_0''\right) + \phi_2\left(-\tfrac{2}{9}\alpha_0' - \tfrac{4}{9}\alpha_0''\right) + \text{4th order terms.}\end{aligned} \tag{21}$$

The terms inside the brackets may be evaluated from the α - equivalent of (20), giving the contributions to the matrix coefficients. Equations (20) are of one higher degree than necessary to produce (21). This is when (18) cannot be used at the singularity P = Q. For this case, we simply assume α_0^{IV} is zero, which from the equivalent of (20d) gives

$$\alpha_2 + \alpha_{-2} - 4(\alpha_1 + \alpha_{-1}) + 6\alpha_0 = 0 \ . \tag{22}$$

Depending on which j corresponds to the singularity, the appropriate α_j is found from (22), and substituted into the equations (20) for α and then into (21).

Now, having developed equations for the influence at Q of both $\partial\phi/\partial n$ and ϕ, the matrix equation (11) may be set up, using expressions (17) and (21). At each point P on the contour, if the appropriate ϕ or $\partial\phi/\partial n$ is unknown, then the matrix coefficient a_{QP} is augmented, while if ϕ or $\partial\phi/\partial n$ is known, it is multiplied by its calculated influence coefficient and taken to the other side of the equation to give b_Q .

Solution of the matrix may be by standard means. In a problem such as the present, when the solution vector changes slowly as the surface disturbance progresses, it would be particularly economical to use an iteration method such as that due to Gauss and Seidel. Unfortunately, due to the logarithmic nature of one of the kernels, coefficients far from the leading diagonal are dominant, and the iteration method does not converge. In work done for the present study, a method of Gaussian elimination was adopted.

(b) Solution of differential equations

For a numerical solution, the system of equations (2), (3), (7) and (8) may be considered as first-order ordinary differential equations in t. As the successive values of Dx/Dt, Dy/Dt and $D\phi/Dt$ come from a solution of the integral equation, which requires considerable computational effort, it is desirable to choose a method for stepping forward in time that requires relatively few I.E. solutions. Such a technique is the Adams-Bashforth-Moulton scheme which, applied to an equation of the form

$$\frac{dy}{dt} = f(t) , \tag{23}$$

is:

$$y_{1p} = y_0 + \frac{\Delta t}{24}\left(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right) , \tag{24}$$

$$y_{1c} = y_0 + \frac{\Delta t}{24}\left(9 f_{1p} + 19 f_0 - 5 f_{-1} + f_{-2}\right) , \tag{25}$$

where Δt is the time step, $f_n = f(t + n\Delta t)$, and y_{1p} , y_{1c} are the predicted and corrected values of y_1 . The errors of this method are $O(\Delta t)^5$. Applied to the present system of D.E.'s, we have to solve the integral equation twice for each time step.

As this method requires information from three previous time steps, a fourth-order Runge-Kutta technique was used to make the first three time-steps from the initial conditions. This uses four evaluations of the time-derivative for each step.

4. APPLICATION TO SHOALING WAVES

The first application of the method described above has been to the deformation of a solitary wave progressing from water of constant finite depth onto a uniformly shelving beach. This was chosen because (1) it is a good approximation to periodic waves approaching a beach (Munk, 1949), (2) initial conditions for the solitary wave are known accurately (Fenton, 1972; Longuet-Higgins and Fenton, 1974; Byatt-Smith and Longuet-Higgins, 1976), (3) a rigorous test of the method is that in the constant depth region it should be able to show that the wave translates without deformation, and (4) there are approximate solutions for this type of geometry and/or wave against which our results could be tested. These solutions include: Carrier and Greenspan (1958) for long disturbances on a beach of constant slope, Biesel (1952) for linear periodic waves on such a beach, Grimshaw (1971) for slow modification of a solitary wave over water of varying depth, and Byatt-Smith (1971) for the reflection of a solitary wave by a vertical wall.

This choice of problem has been a particularly severe test of the method. Most importantly, the length scale of problem is much larger than that used by Longuet-Higgins and Cokelet. By a transform of the deep wave geometry, their integral equation was solved on a closed curve of radius O(1). Solitary waves have an effective horizontal extent of some 10 times the depth: this must be described by computational particles on the solid bottom as well as on the free surface. Eventually, it has been found that at least 100 points are needed: throughout the work described 117 were used, about twice the number used by Longuet-Higgins and Cokelet. Theoretically, however, the influence of a solitary wave extends to infinity, decreasing exponentially. The influence of the exponential tail was obtained analytically, using linear wave theory. This gave contributions to the matrix coefficients in terms of exponential integrals.

A further complication arose when solutions of the integral equation were obtained: values of ϕ for a typical solitary wave were of O(1), while the $\partial\phi/\partial n$ were of O(10^{-2}). To obtain values of $\partial\phi/\partial n$ correct to within 0.1%, a reasonable target where timestepping is to be used, the initial position of the solitary wave had to be so far from the beach that changes of ϕ in the vicinity of the beach were O(10^{-5}), necessitating large step lengths for solution of the integral equation, and many time steps to follow the wave to the beach.

Longuet-Higgins and Cokelet found that the free-surface of their computed deep-water waves developed a saw-tooth instability. This was regular, and they were able to smooth it out by application of a smoothing formula every few time steps. We have encountered a similar instability, how-

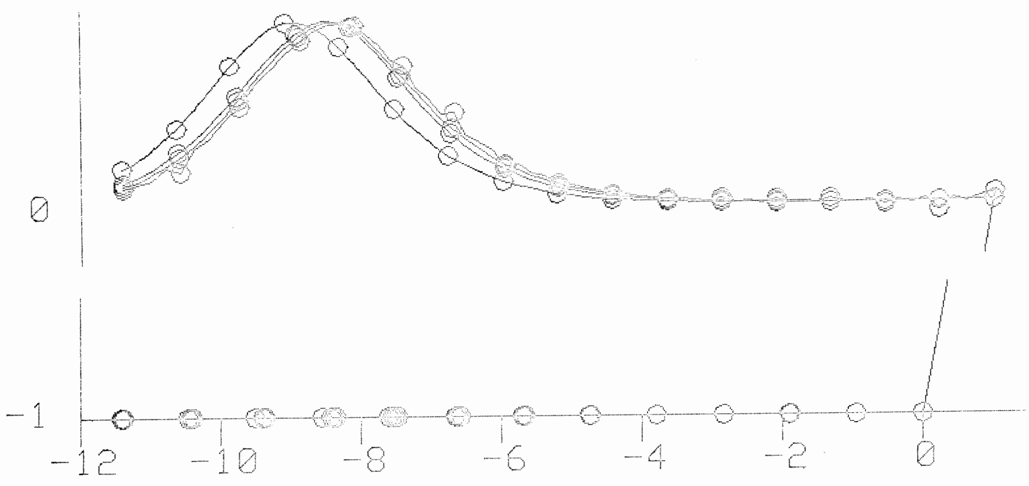


Figure 4 Results for the first few time steps of a solitary wave of relative amplitude 0.5 approaching a beach of slope 1:1, showing the development of numerical instability.

ever it has been rather less regular, and we have not yet eliminated it from our calculations. Figure 4 shows a typical result to date: initially the wave remains smooth, however after several time steps, some small instabilities have begun to develop which grow catastrophically. A rational method for smoothing these is an essential next step in our programme.

5. CONCLUSIONS

We have applied an exact integro-differential set of equations to the problem of a solitary wave approaching a beach. In the numerical approximation of these equations we have not been able to solve them after a certain number of time steps, because of the development of numerical instabilities. This problem has been overcome for the case of deep water waves, and is not expected to impede the application of this new method to waves in water of finite depth.

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