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Some Results for Surface Gravity Waves on Shear Flows

J. D. FENTON

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

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This work attempts to fill some gaps in the subject of steady surface gravity waves on two-dimensional flows in which the velocity varies with depth, as is the case for waves propagating on a flowing stream. Following most previous work the theory is basically inviscid, for the shear is assumed to be produced by external effects: the theory examines the non-viscous interaction between wave disturbances and the shear flow. In particular, some results are obtained for the dispersion relationship for small waves on a flow of arbitrary velocity distribution, and this is generalized to include the decay from finite disturbances into supercritical flows. An exact operator equation is developed for all surface gravity waves for the particular case of flow with constant vorticity; this is solved to give first-order equations for solitary and cnoidal waves in terms of channel flow invariants.

Exact numerical solutions are obtained for small waves on some typical shear flows, and it is shown how the theory can predict the growth of periodic waves upon a stream by the development of a fully-turbulent velocity profile in flow which was originally irrotational and supercritical.

Results from all sections of this work show that shear is an important quantity in determining the propagation behaviour of waves and disturbances. Small changes in the primary flow may alter the nature of the surface waves considerably. They may in fact transform the waves from one type to another, corresponding to changes in the flow between super- and sub-critical states directly caused by changes in the velocity profile.

1. Introduction

THE PROBLEM of surface gravity waves on a flow in which the velocity is not uniform has not received much attention. Irrotational wave theory does provide a large mass of material suitable for extension to the rotational case; but while the change in physical thinking required to make this extension is small, the previous mathematical methods are often inappropriate or inapplicable. For this reason, some rather novel techniques have been developed. Certain basic results, for example dispersion relationships for small waves and first-order approximations to finite-amplitude long waves of permanent type, have been obtained; often, however, these are for particular variations of shear, or are in terms of intractable integrals, so that only very few cases can be solved explicitly.

Thompson (1949) and Biésel (1950) obtained the dispersion relation for small waves on a flow with a constant velocity gradient by the use of linearized boundary conditions. Similarly Hunt (1955) considered a velocity distribution given by a 1/7 power law, and found the first term of a series expression connecting wave length and speed. The first result for arbitrary velocity distributions was given by Burns (1953), who obtained an expression for the speed of long waves, and who discussed the physical implications of his results, which were subsequently extended by Velthuizen & van Wijngaarden (1969), who also studied the role that viscosity plays in the interaction between the primary flow and wave disturbances.

Considering the problem of waves of finite length and height, Hunt (1955) gave a solution of first order in velocity variation and in wave height. Then Ter-Krikorov (1961) and Benjamin (1962*a*) obtained general expressions for the solitary wave on a flow of arbitrary velocity distribution; Freeman & Johnson (1970) subsequently generalized this to obtain a Korteweg-de Vries equation with the coefficients modified to include the effect of shear.

The present work attempts to fill some gaps between these analyses. In particular, dispersion relationships are obtained for small waves on an arbitrary shear flow. Exact results are given for the case U''/U constant, where $U(y)$ is the primary velocity distribution, and for stationary disturbances on a developing fully-turbulent flow. The governing equations can be solved numerically: it is shown how this can be done for all flows, and solutions are produced for the particular case of a 1/7 power law distribution. Throughout, it is emphasized that the results are equally applicable to periodic waves and to decay into a uniform supercritical shear flow from finite disturbances such as sluice gates or solitary waves. This almost trivial generalization has been noted by Lamb (1932, pp. 376, 426) and by Benjamin (1962*b*) for the case of irrotational waves, but for disturbances to shear flows it seems to have remained unnoticed.

Lastly, the work of Fenton (1972) is extended to the case of waves on flows of constant vorticity, for which an exact operator equation is developed. For small waves, this gives the results of the first section, and when extended to finite amplitude waves, a first-order solution for cnoidal and solitary waves is obtained. Because of the assumption of constant vorticity, this is a rather particular case of the solutions to these waves given by Benjamin and by Freeman & Johnson; nevertheless the method is useful in that it relates the wave forms to invariants of the flow, following the technique of Benjamin & Lighthill (1954).

2. Linear Theory

2.1. *Formulation of Equations*

We now proceed to obtain the dispersion relation for small disturbances on a stream in which there is an arbitrary distribution of shear. Considering a co-ordinate system x, y in which the waves are stationary, we introduce a stream function ψ in terms of which the vorticity ζ and the velocity components u, v in the directions x, y respectively, are given by

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}$$
$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}. \tag{1}$$

The bottom of the channel is defined to be the streamline $\psi = 0$, and the kinematical condition for the wave to be stationary is that the free surface is also a streamline.

If c is the absolute velocity of propagation on a horizontal stream which has a velocity distribution $U(y)$, then the velocity of undisturbed fluid in the frame of reference in which the wave system is stationary is $W(y) = U(y) - c$, in the x direction. Thus for waves propagating upstream c is negative, and it is positive for the downstream case.

Throughout the flow, the dynamical condition for steady motion is that the vorticity remains constant on any streamline. Thus

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = \zeta(\psi), \tag{2}$$

where $\zeta(\psi)$ is defined by the velocity distribution in the undisturbed stream:

$$\zeta(\psi) = \frac{dU(y)}{dy}.$$

On the free surface of the flow we have to satisfy the dynamic and kinematic boundary conditions:

$$\begin{aligned} \tfrac{1}{2}W^2(h)+gh &= \tfrac{1}{2}(\psi_x^2+\psi_y^2)_s+g\eta(x) \\ \psi_{y_s}\,.\,\eta'(x)+\psi_{x_s} &= 0, \end{aligned} \tag{3}$$

in which $\eta(x)$ is the height of the free surface above the channel bottom, which has an undisturbed value of h, g is the gravitational acceleration, and the subscript s refers to the free surface.

To obtain a linearized solution to the problem defined by (2) and (3), we expand the dependent variables in terms of a small parameter ε, expressing disturbance to the basic flow. Thus we have

$$\begin{aligned} \psi &= \psi_0 + \varepsilon\psi_1 + \dots \\ \eta &= \eta_0 + \varepsilon\eta_1 + \dots . \end{aligned} \tag{4}$$

Substituting into the free surface conditions (3), and satisfying these identically in ε^0 and ε we have the undisturbed solutions

$$\psi_0 = \int_0^y W(y)\,.\,dy$$

$$\eta_0 = h,$$

and the linearized boundary conditions on $y = h$:

$$\begin{aligned} \eta_1(g + W(h)\,.\,W'(h))+W(h)\,.\,\psi_{1y} &= 0 \\ W(h)\,.\,\eta_{1x} + \psi_{1x} &= 0. \end{aligned}$$

We can eliminate η_1 by differentiation and substitution to give the linearized boundary condition for ψ_1:

$$W^2(h)\,.\,\psi_{1xy} = (g + W(h)\,.\,W'(h))\,.\,\psi_{1x}. \tag{5}$$

Substituting the expansions (4) into the vorticity equation (2) gives

$$\begin{aligned} \nabla^2\psi_0 + \varepsilon\nabla^2\psi_1 &= \zeta(\psi_0 + \varepsilon\psi_1) \\ &= \zeta(\psi_0) + \varepsilon\psi_1\zeta'(\psi_0). \end{aligned}$$

That is,

$$\nabla^2\psi_0 = W'(y), \tag{6a}$$

and

$$\nabla^2\psi_1 = \psi_1\zeta'(\psi_0) = \psi_1\frac{d}{d\psi_0}(W'(y)) = \frac{W''(y)}{W(y)}\,.\,\psi_1. \tag{6b}$$

Thus we have the problem

$$\nabla^2\psi_1 = \frac{W''}{W}\,.\,\psi_1,$$

with boundary conditions

$$\psi_1 = 0 \quad \text{on} \quad y = 0,$$

$$W^2(h)\,.\,\psi_{1xy} = (g + W(h)\,.\,W'(h))\psi_{1x} \quad \text{on} \quad y = h.$$

We can obtain a solution by the method of separation of variables: we seek solutions proportional to e^{-mx}, where m may be real or imaginary, giving respectively, exponential decay from some finite disturbance to a supercritical stream, or sinusoidal waves. Thus, let

$$\psi_1 = P(y)\,.\,e^{-mx},$$

and the problem reduces to the linear second-order differential equation

$$P'' + (m^2 - W''/W)P = 0 \tag{7a}$$

$$P(0) = 0 \tag{7b}$$

$$P(h)\,.\,(g + W(h)\,.\,W'(h)) = P'(h)\,.\,W^2(h). \tag{7c}$$

We can now obtain solutions of this equation, and thus relate m (which for periodic waves is $2\pi i/\lambda$, where λ is the wavelength) to the undisturbed flow quantities $W(h)$, $W'(h)$, and $W''(y)/W(y)$, giving the dispersion relationship for each special case.

2.2. *Long Waves*

In this case m is negligibly small, giving, to close approximation

$$WP'' - W''P = 0.$$

It is easily shown that the solution is

$$P(y) = A\,.\,W(y)\,.\int_0^y \frac{dy_*}{W^2(y_*)} + B\,.\,W(y).$$

If $W(0) \neq 0$, then (7b) gives $B = 0$, and the surface condition (7c) gives

$$\frac{1}{g} = \int_0^h \frac{dy}{(U(y)-c)^2}\,. \tag{8}$$

This is the result obtained by Burns (1953), relating the long wave speed of propagation to the primary velocity distribution.

2.3. *Stationary Waves and Disturbances*

In an appendix to Burns' paper, Lighthill (1953) obtained exact solutions of (7) for the particular case $c = 0$, $U(0) = 0$, $U'(0) \to \infty$; that is, for stationary waves on a flow in which the velocity vanishes at the bottom, but where the velocity gradient is infinite there.

Velocity distributions like $U(y) = U_1(y/h)^n$, with n about 1/7 have been found to fit experimental observations of turbulent flow over smooth surfaces rather well, and which satisfy the bottom velocity conditions stated above. Thus Lighthill solved (7) for periodic waves on a flow of this nature, obtaining the critical Froude number above which no upstream propagation is possible. We follow his analysis, generalizing to allow real and imaginary values of m; unlike him, we do not include surface tension effects.

For

$$\begin{aligned} U(y) &= U_1(y/h)^n \\ &= W(y), \\ W''/W &= n(n-1)/y^2; \end{aligned}$$

thus we have to solve

$$P'' + (m^2 - n(n-1)/y^2)P = 0 \tag{9a}$$

$$P(0) = 0 \tag{9b}$$

$$P(h) \, . \, (g/U_1^2 + n/h) = P'(h). \tag{9c}$$

Solutions of (9a) behave like y^n and y^{1-n} near $y = 0$, so that each satisfies (9b). We require that the solution be uniformly valid, and require the disturbance velocity to tend to zero more rapidly than U itself at $y = 0$, so we choose the solution behaving like y^{1-n}. The solution of (9a) is then, where A is a constant,

$$P = Ay^{\frac{1}{2}}J_{\frac{1}{2}-n}(my),$$

where $J_{\frac{1}{2}-n}$ is a Bessel function of the first kind of order $(\frac{1}{2} - n)$. Substituting into the boundary condition (9c) at $y = h$ gives

$$\frac{U_1^2}{g} = \frac{J_{\frac{1}{2}-n}(mh)}{J_{\frac{1}{2}+n}(mh)} \cdot \frac{1}{m}.$$

Now we introduce the Froude Number F which is to be used throughout:

$$F = (U(h) - c)/(gh)^{\frac{1}{2}} \tag{10}$$

and,

$$\beta = mh \tag{11}$$

which is the dimensionless wave number for periodic waves and the dimensionless decay constant for smooth decay.

In this case, where $c = 0$,

$$F = U_1/(gh)^{\frac{1}{2}};$$

therefore

$$F^2 = \frac{1}{\beta} \cdot \frac{J_{\frac{1}{2}-n}(\beta)}{J_{\frac{1}{2}+n}(\beta)}. \tag{12}$$

For periodic waves, where β is imaginary, and is equal to $2\pi i/\lambda$, where λ is the wavelength non-dimensionalized with respect to the depth h,

$$F^2 = \frac{\lambda}{2\pi} \cdot \frac{I_{\frac{1}{2}-n}(2\pi/\lambda)}{I_{\frac{1}{2}+n}(2\pi/\lambda)}, \tag{13}$$

where I_k is a modified Bessel function of the first kind, of order k.

This equation is that obtained by Lighthill, giving the Froude number beyond which upstream propagation of periodic waves is not possible. The new result (12) is of some real physical interest as well, in that the situation to which it applies often occurs in practice; for example when flow is released under a sluice gate with a smooth channel bottom, a turbulent velocity profile develops, giving the assumed profile in the above analysis, with $n = 1/7$, and of course, $c = 0$. Equation (12) gives the rate of decay $e^{-\beta x/h}$ for any stationary finite disturbance on a turbulent flow over a smooth bottom.

The results from (12) and (13) were computed, and are shown in Section 4.1.

2.4. W''/W Constant

In this case (7) can be solved directly to give

$$P(y) = A \sin (m^2 - W''/W)^{\frac{1}{2}}y,$$

and the surface boundary condition requires

$$\frac{W^2(h)}{g+W(h) \,.\, W'(h)} = \frac{\tan (m^2-W''/W)^{\frac{1}{2}} \,.\, h}{(m^2-W''/W)^{\frac{1}{2}}}. \tag{14}$$

We can non-dimensionalize this using the surface Froude number defined in (10) above and the following dimensionless parameters used throughout the work:

$$F' = W'(h) \,.\, (h/g)^{\frac{1}{2}}$$

and,

$$k^2 = m^2h^2 - h^2(W''/W),$$

which is a constant in this section. We can write (14) as

$$F^2/(1 + FF') = \tan k/k. \tag{15}$$

Thus we have the dispersion relationship for waves on any shear flow in which W''/W is a constant. This is, of course, a rather special case, but there is no limitation on the sign or magnitude of W''/W and it appears reasonable that (15) could be used as an approximation to many types of primary velocity distribution.

Considering (15), there are three distinct cases which we can study, depending on the magnitude of $F^2/(1 + FF')$. If this ratio is greater than unity, then all roots of (15) are real and we have exponential decay into a uniform stream. Examples where this may occur are the decay from a finite disturbance such as a sluice gate or the tail of a solitary wave: in either case we say that the flow is supercritical—the relative speed between disturbance and fluid particles is greater than that of long waves.

Conversely, if $F^2/(1 + FF')$ is less than unity, roots of (15) are all imaginary and we have a periodic wave train of dimensionless wave length λ:

$$F^2/(1 + FF') = ((2\pi/\lambda)^2 + h^2W''/W)^{-\frac{1}{2}} \,.\, \tanh((2\pi/\lambda)^2 + h^2W''/W)^{\frac{1}{2}}. \tag{16}$$

For the case of a linear velocity profile ($W'' = 0$) this is the result obtained by Thompson (1949) and Biésel (1950), and for a constant velocity profile ($W' = 0$) we have the irrotational result of Rayleigh, (Lamb, 1932, p. 375), where $\beta = mh$:

$$F^2 = \tan \beta/\beta. \tag{17}$$

Lastly, if $k = 0$, in which the disturbance due to a wave system so balances the primary velocity distribution that $P(y)$ is proportional to y and the motions are mainly longitudinal, we have $F^2 - FF' = 1$, defining the conditions for the occurrence of long waves. For $m = U'' = 0$, the case of long waves on a flow of constant vorticity, this relationship can be established by the integration of (8).

The results of (15) and (16) are plotted and discussed in Section 4.2 for the constant vorticity case.

2.5. *Numerical Solutions*

All the previous solutions have been for special cases in which analytical expressions could be obtained for the dispersion relationship in each case by an exact solution of (7). These solutions have enabled qualitative and quantitative physical interpretations to be made, which are given in Section 4. In general these solutions are for rather special cases of the shear flow. Where an exact solution is required for a general velocity distribution, we have to resort to numerical means.

The problem to solve is the two-point linear boundary value problem (7). Initial value problems are somewhat simpler to solve numerically, so we transform and non-dimensionalize the variables:

$$p(z) = P(y)/(hP'(y))$$
$$z = y/h$$
$$k^2(z) = m^2h^2 - h^2W''/W,$$

to give the non-linear first order Riccati equation

$$\begin{aligned} \frac{dp}{dz} &= 1+k^2(z)\,.\,p^2 \\ p(0) &= 0. \end{aligned} \tag{18}$$

Numerical integration of this proceeds quite simply from $z = 0$ to $z = 1$ for given mh and h^2W''/W; then at $z = 1$ we obtain, from (7c):

$$F^2/(1 + FF') = p(1). \tag{19}$$

To illustrate the application of this method we produce solutions for the case of turbulent flow over a smooth bottom, where

$$W(y) = U_1\,.\,z^{(1/7)} - c,$$

for which exact solutions have been obtained for the special case $c = 0$ in Section 2.3 above. Differentiating, we obtain

$$k^2 = \beta^2 + \frac{6}{49}\cdot\frac{\delta z^{-13/7}}{\delta z^{1/7}-1} \tag{20}$$

where $\beta = mh$, the dimensionless wavenumber/decay constant, and $\delta = U_1/c$, the ratio of undisturbed velocity at the surface to the wave speed, which is positive for downstream propagation. In the integration of (18) we have these two quantities as parameters. When we have performed the integration to obtain $p(1)$, we use (19) with the 1/7 velocity distribution to give the dimensionless absolute wave speed

$$c^2/gh = p(1)/((1 - \delta)(1 - \delta + \tfrac{1}{7}\delta p(1))), \tag{21}$$

and the surface velocity,

$$U_1 = \delta\,.\,c. \tag{22}$$

Thus for a given stream velocity and wave number, we have the wave speed. This integration was carried out by computer using a Runge–Kutta technique, for different values of β and δ. The results are given in Section 4.3.

3. Finite Amplitude Waves on a Flow of Constant Vorticity

3.1. *Exact Operator Equation*

In Section 2 we obtained dispersion relationships for small waves using a linearized surface boundary condition. Now we proceed to a non-linear theory and obtain exact operator expressions for finite waves on a shear flow with the important limitation that vorticity is constant throughout. These are then solved to obtain first-order equations for cnoidal and solitary waves. We follow the procedure of Benjamin & Lighthill, modifying the technique of Fenton (1972), of symbolic manipulation of infinite series of operators, to allow for the presence of vorticity.

We consider a steady wave system on a flow of constant vorticity G, such that the waves are stationary in a co-ordinate system x, y. There is no restriction on the magnitude or sign of G. The case of G positive corresponds to the case of upstream propagation of waves on actual streams, in which the relative velocity of flow is greater at the surface than at the bottom. Similarly, downstream propagation would require that G be negative.

A stream function $\psi(x, y)$ exists such that horizontal and vertical velocities u and v are given by

$$u = \frac{\partial\psi}{\partial y}$$

$$v = -\frac{\partial\psi}{\partial x},$$

and the equation which holds throughout, from (2), is

$$\nabla^2\psi = G. \tag{23}$$

We can write the kinematic condition on the free surface

$$\psi(x, \eta(x)) = Q, \tag{24}$$

where Q is the steady discharge per unit width in the co-ordinate system chosen, and $\eta(x)$ is the height of the free surface above the channel bottom. The bottom condition is

$$\psi(x, 0) = 0. \tag{25}$$

Now we use the following equation (see, for example, Batchelor 1967, p. 160):

$$\nabla R = \mathbf{u} \times \boldsymbol{\omega}, \tag{26}$$

where

$$R = p/\rho + gy + \tfrac{1}{2}|\mathbf{u}|^2, \tag{27}$$

the energy per unit mass of the liquid, and $\boldsymbol{\omega}$ is the vorticity:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u},$$

where $\mathbf{u}$ is the velocity of flow.

For our case, the two-dimensional flow of constant vorticity G, equation (26) becomes

$$\left(\frac{\partial R}{\partial s}, \frac{\partial R}{\partial n}\right) = \left(0, G\frac{\partial\psi}{\partial n}\right),$$

where s is in the direction of the streamline and n is normal to it. Thus we have R as a constant along any streamline but varying across the streamlines:

$$\begin{aligned} R &= R(\psi) \\ &= G\psi + \text{constant}, \end{aligned}$$

and if we denote $R(Q)$, the energy at the free surface, by R_1,

$$R(\psi) = R_1 - G(Q - \psi). \tag{28}$$

The third invariant, after Q and R_1, is S, the momentum flux per unit span divided by density:

$$S = \int_0^\eta (p/\rho + u^2)\, dy. \tag{29}$$

We can substitute for p/ρ by putting (27) in (29):

$$S = \int_0^\eta (R(\psi) - gy + \tfrac{1}{2}(u^2 - v^2))\, dy. \tag{30}$$

Now, we write an expression for the stream function which satisfies (23) and (25), and which is general, in that it can be used to describe all surface waves. It is, however, limited to flows with constant vorticity. Thus we write

$$\psi(x, y) = \tfrac{1}{2}Gy^2 + (\sin yD)(Iu(x, 0)), \tag{31}$$

in which $D = d/dx$, $\sin yD$ is the operator

$$\sin yD = yD - \frac{y^3}{3!}D^3 + \ldots,$$

I is the integral operator: $D^n I = D^{n-1}$, and $u(x, 0)$ is the velocity along the horizontal bottom. Substituting $u = \psi_y$ and $v = -\psi_x$, and putting (31) into (30) we have

$$S = \int_0^\eta \{R(\psi) - gy + \tfrac{1}{2}[(Gy + (\cos yD)u(x, 0))^2 - ((\sin yD)u(x, 0))^2]\}\, dy \tag{32}$$

and similarly from (24)

$$Q = \tfrac{1}{2}G\eta^2 + (\sin \eta D)(Iu(x, 0)). \tag{33}$$

We note that in interpreting operators such as $\sin \eta D$, obtained by replacing y by η in the $\sin yD$ of (31), the differential operator D must not be taken as acting on η itself when expanding the operators in series.

The equations (32) and (33) constitute a pair of infinite-order integro-differential equations in the two unknown functions of $x - \eta(x)$ and $u(x, 0)$. We can eliminate the integral in (32) by substituting (28), into which we have substituted (31) and (33) to give

$$S = \int_0^\eta \{R_1 - \tfrac{1}{2}G^2\eta^2 + G^2y^2 - gy + G((y \cos yD)u + (\sin yD)Iu - (\sin \eta D)Iu) + \tfrac{1}{2}(\cos yD)u^2\}\, dy$$

in which we have substituted

$$(\cos yD)u^2 = ((\cos yD)u)^2 - ((\sin yD)u)^2,$$

which can be established by expansions of the series on either side.

Performing the integration, we obtain

$$S = R_1\eta - \tfrac{1}{6}G^2\eta^3 - \tfrac{1}{2}g\eta^2 + \tfrac{1}{2}(\sin \eta D)Iu^2. \tag{34}$$

Thus we have a pair of infinite order differential equations (33) and (34) in $\eta(x)$ and $u(x, 0)$. We can invert (33) to give

$$u(x, 0) = (\eta D/\sin \eta D)(Q/\eta - \tfrac{1}{2}G\eta). \tag{35}$$

The term $(\eta D/\sin \eta D)$ is a doubly-infinite series, and when expanding it, the operator D must be taken as operating on $\eta(x)$ as well, when inverting the $\sin \eta D$ series. This is because this term was obtained after substituting $y = \eta(x)$ into (31) and performing the inversion. Thus we have

$$\eta D/\sin \eta D \equiv (\eta D)\left(\eta D - \frac{\eta^3 D^3}{3!} + \cdots\right)^{-1}$$
$$\equiv 1 + \frac{\eta^2 D^2}{3!}\left(1 + \frac{\eta^2 D^2}{3!}(1 + \cdots) + \cdots\right)$$
$$- \frac{\eta^4 D^4}{5!}(1 + \cdots)$$
$$+ \cdots .$$

Substituting (35) into (34), we obtain

$$[\sin \eta D] I[(\eta D/\sin \eta D)(Q/\eta - \tfrac{1}{2}G\eta)]^2 = \tfrac{1}{3}G^2\eta^3 + g\eta^2 - 2R_1\eta + 2S. \tag{36}$$

This is the exact infinite order differential equation for $\eta(x)$ in terms of the constants Q, R_1 and S, and the vorticity G. For irrotational flow we have

$$Q^2[\sin \eta D] I[(\eta D/\sin \eta D)(1/\eta)]^2 = g\eta^2 - 2R_1\eta + 2S,$$

in which R_1 is the energy per unit mass throughout the flow.

We now examine some solutions of the exact equation (36).

3.2. *Small Waves*

In this case, where we have small disturbances to uniform flow with constant vorticity, we express the velocity along the bottom as

$$u(x, 0) = U_0 + U_1 e^{-mx}, \tag{37}$$

where U_0 is the undisturbed velocity, U_1 is a small disturbance velocity, and m is either real, for decay from a disturbance to a uniform stream, or imaginary for the case of periodic waves. Similarly we write

$$\eta(x) = h + \eta_1 e^{-mx}, \tag{38}$$

where h is the undisturbed depth and η_1 is a small disturbance of the free surface. For this uniform flow, the channel invariants become

$$\begin{aligned} Q &= U_0 h + \tfrac{1}{2}Gh^2 \\ R_1 &= \tfrac{1}{2}(U_0 + Gh)^2 + gh \\ S &= U_0^2 h + U_0 G h^2 + \tfrac{1}{2}gh^2 + \tfrac{1}{3}G^2 h^3. \end{aligned} \tag{39}$$

Substituting (37), (38) and (39) into the continuity equation (33) we obtain, retaining first order terms only in η_1 and U_1,

$$\eta_1 = \frac{-\sin mh}{m} \cdot \frac{U_1}{U_0 + Gh} \tag{40}$$

which satisfies the equation exactly for η_1 and U_1 very small.

Now it can be shown that the momentum equation (36) is identically satisfied up to and including second order expansions about a uniform flow. Hence it is simpler to use the equation for energy at the free surface obtained from (27) and from the substitution of (31), after differentiation:

$$R_1 = g\eta + \tfrac{1}{2}[(G\eta + (\cos \eta D)u)^2 + ((\sin \eta D)u)^2].$$

Substituting (37), (38), (39) and (40) into this, we have

$$\frac{(U_0+Gh)^2}{gh+Gh(U_0+Gh)} = \frac{\tan mh}{mh}. \tag{41}$$

This is the same equation as (15) for the particular case $U'(y) = G$, giving the dispersion relationship for small waves on this flow.

3.3. *Finite Amplitude Waves*

Now we consider the case of non-linear dispersive waves on a shear flow of constant vorticity by truncating the infinite order differential operators in the exact equations to give equations for waves for which non-linear and dispersive effects are small but are not negligible. We truncate the operators as follows:

$$(\sin \eta D)(I) \equiv \eta - \frac{\eta^3}{3!}D^2 + \ldots$$

$$(\eta D/\sin \eta D) \equiv 1 + \frac{\eta^2}{3!}D^2 + \cdots$$

and substitute into (36), to give, after re-arrangement,

$$\tfrac{1}{3}\eta'^2(Q + \tfrac{1}{2}G\eta^2)^2 + \tfrac{1}{12}G^2\eta^4 + g\eta^3 + \eta^2(QG - 2R_1) + 2S\eta - Q^2 = 0(a^2h^2/e^4), \tag{42}$$

where the symbols in the error term are: h, measure of flow depth; e, measure of horizontal extent of the disturbances; and a, the amplitude of the disturbances.

Thus we have a differential equation for the free surface, provided that the waves are small, measured by (a/h) and that they are long, as measured by (h/e). The error term shows that the requirement that the equation be applicable is that $(a/h)^2(h/e)^4 \ll 1$.

Now it is convenient to introduce an expansion for η so that we can obtain explicit solutions of (42). We let $\eta(x) = h + \eta_1(x)$, where $\eta_1 = 0(a)$, and substitute into (42):

$$\begin{aligned}\tfrac{1}{3}\eta_1'^2(Q+\tfrac{1}{2}Gh^2)^2 = {} & Q^2 - 2Sh + h^2(2R_1 - QG) - gh^3 - \tfrac{1}{12}G^2h^4 + \\ & \eta_1(-2S + 2h(2R_1 - QG) - 3gh^2 - \tfrac{1}{3}G^2h^3) + \\ & \eta_1^2(2R_1 - QG - 3gh - \tfrac{1}{2}G^2h^2) + \\ & \eta_1^3(-g - \tfrac{1}{3}G^2h) + \\ & 0(a^2h^2/e^4),\end{aligned} \tag{43}$$

where we have ignored terms in η_1^4, which is compatible with the overall order of approximation if we assume that, for waves on flows with vorticity, as for irrotational waves, $ae^2/h^3 = 0(1)$.

Rewriting (43),

$$\eta_1'^2 \cdot \frac{(Q+\tfrac{1}{2}Gh^2)^2}{3g+G^2h} = (r_1 - \eta_1)(\eta_1 - r_2)(\eta_1 - r_3), \tag{44}$$

in which $r_1 \geqslant r_2 \geqslant r_3$ are the roots of the cubic in η_1 on the right hand side of (43). The solution of (44) is

$$\eta_1 = r_2 + (r_1 - r_2)cn^2(\alpha x, k)$$

with

$$k^2 = (r_1 - r_2)/(r_1 - r_3)$$

and

$$\alpha^2 = \frac{3g+G^2h}{4(Q+\frac{1}{2}Gh^2)^2}\cdot(r_1-r_3), \tag{45}$$

giving cnoidal waves on a flow of constant vorticity in terms of the invariants Q, R_1 and S.

Equations for cnoidal waves have been developed by other workers; Hunt (1955) studied waves on a stream of small shear variation, while Freeman & Johnson (1970) obtained the equation for cnoidal waves without the limitation that this variation be small. The above theory must be regarded as a special case of these solutions; nevertheless it is of value in that it shows the role played by the physical flow parameters—discharge, energy, and momentum flux. In fact, the physical interpretation applied by Benjamin & Lighthill to irrotational flow can be used to show the effects of changes in energy or momentum of the present shear flow.

Now we examine the case of disturbances to a uniform shear flow, where

$$\begin{aligned} U(y) &= U_0 + Gy \\ Q &= U_0h + \tfrac{1}{2}Gh^2 \\ R_1 &= \tfrac{1}{2}(U_0 + Gh)^2 + gh \\ S &= U_0^2h + U_0Gh^2 + \tfrac{1}{2}gh^2 + \tfrac{1}{3}G^2h^3, \end{aligned}$$

in which case (43) can be written

$$\tfrac{1}{3}\eta_1'^2(U_0h+\tfrac{1}{2}Gh^2)^2 = \eta_1^2(U_0^2+U_0Gh-gh)-\eta_1^3(g+\tfrac{1}{3}G^2h), \tag{46}$$

where the cubic on the right hand side has the roots $\eta_1 = 0$ (repeated) and

$$\eta_1 = \frac{U_0(U_0+Gh)-gh}{g+\frac{1}{3}G^2h}.$$

Now, depending on the sign of this third root, we can distinguish three cases.

(a) $U_0(U_0 + Gh) = gh$

In this case we have $r_1 = r_2 = r_3 = 0$ and hence $\alpha = 0$ in (45), giving the long wave case. If we substitute a flow with constant velocity gradient into Burns' equation (8) for long waves, we obtain this condition $U_0(U_0 + Gh) = gh$.

(b) $U_0(U_0 + Gh) > gh$

Here, $r_1 = [U_0(U_0 + Gh) - gh]/[g + \frac{1}{3}G^2h]$, $r_2 = 0, r_3 = 0$. Thus $k^2 = 1$ and from (45),

$$\eta = h+\frac{U_0(U_0+Gh)-gh}{g+\frac{1}{3}G^2h}.\operatorname{sech}^2\alpha x,$$

where

$$\alpha^2 = \frac{3}{4}\cdot\frac{U_0(U_0+Gh)-gh}{(U_0h+Gh^2)^2}. \tag{47}$$

This is the equation for the solitary wave on a flow of constant vorticity, as obtained by Benjamin (1962*a*).

As $x \to \pm\infty$, $\operatorname{sech}^2 \alpha x \to e^{\mp 2\alpha x}$; we have an exact theory for this case, given in Section 2, and rewrite (15):

$$\frac{(U_0+Gh)^2}{g+G(U_0+Gh)} = \frac{\tan 2\alpha h}{2\alpha}.$$

Expanding the term on the right hand side to first order,

$$\frac{(U_0+Gh)^2}{g+G(U_0+Gh)} = h+\tfrac{4}{3}\alpha^2 h^3 + O(\alpha^4 h^5),$$

and re-arranging gives

$$\alpha^2 = \frac{3}{4}\cdot\frac{1}{h^2}\cdot\frac{U_0(U_0+Gh)-gh}{gh+Gh(U_0+Gh)}.$$

To ignore the error term, we require α^2 to be small and can substitute $U_0(U_0 + Gh) = gh$ into the bottom line to give

$$\alpha^2 = \frac{3}{4}\cdot\frac{U_0(U_0+Gh)-gh}{(U_0 h+Gh^2)^2},$$

the same equation as in (47), showing that the first-order expression for the length scale of the wave can be obtained by considering the first-order expansion of the dispersion relation obtained by linear theory (15).

(c) $U_0(U_0 + Gh) < gh$

Here, $r_1 = r_2 = 0$, $r_3 = [U_0(U_0 + Gh) - gh]/[g + \frac{1}{3}G^2h]$, and substituting in (45) gives $k^2 = 0$, and $cn^2(\alpha x, 0) \sim \cos^2 \alpha x$. Thus $\eta \sim (r_1 - r_2)\cos^2 \alpha x$, giving a sinusoidal variation of zero amplitude demonstrating that sinusoidal waves of finite amplitude are not possible.

4. Results

In this section we present some results of the various solutions given in Section 2, showing how the effects of shear may cause large corrections to results for wave propagation based on an irrotational theory. Simple experiments were performed to check some of the fundamental assumptions and results; these are described in Section 4.4.

4.1. *Stationary Disturbances on a 1/7 Power Velocity Distribution*

We consider the case of a stationary disturbance on a turbulent flow over a smooth bottom, as analysed in Section 2.2. For this we have $U = U_1(y/h)^{1/7}$, obtaining from (12) and (13),

$$\frac{U_1^2}{gh} = \frac{1}{\beta}\cdot\frac{J_{5/14}(\beta)}{J_{9/14}(\beta)},$$

for β real, and

$$\frac{U_1^2}{gh} = \frac{\lambda}{2\pi}\cdot\frac{I_{5/14}(2\pi/\lambda)}{I_{9/14}(2\pi/\lambda)},$$

for $\beta = 2\pi i/\lambda$. These ratios of the Bessel functions were obtained numerically by the methods of Section 2.5, with $\delta \to -\infty$ and the results are shown in Fig. 1, on which we have plotted $F_m = U_m/(gh)^{\frac{1}{2}} = \frac{7}{8}U_1/(gh)^{\frac{1}{2}}$, where U_m is the mean velocity. For comparison we have also plotted the curve for irrotational steady disturbances, in which case $U_m = U_1$, as we have constant velocity throughout the undisturbed flow.

In Fig. 1, the subcritical curves are plotted with λ as abscissa, and these asymptote to the long wave case $\lambda \to \infty$, $\beta \to 0$; for β real, the supercritical case, we plot the curves with β as abscissa. It may have been more correct to plot the curves continuously as functions of β^2; this, however, would not allow the region $\lambda \to 0$ to be shown. The curves, juxtaposed as they are, do at least indicate their continuous nature. We may note that the curve for the 1/7 power law distribution is representative of results for all actual flows in which vorticity effects are included.

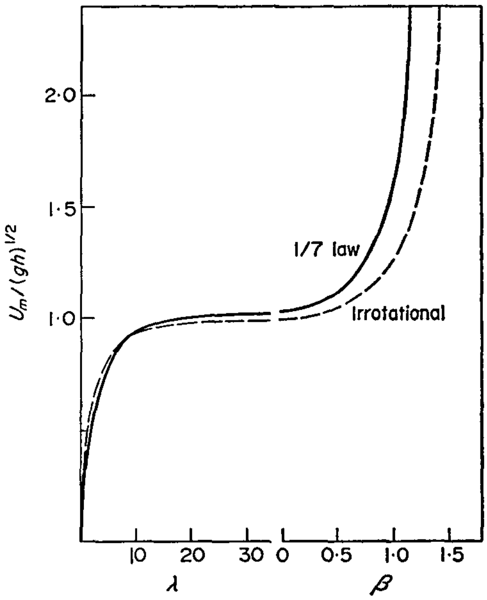

FIG. 1. Dispersion relation for stationary disturbances, comparing irrotational flow with flow having a velocity distribution given by a 1/7 power law. The subcritical part on the left side has wavelength λ for abscissa, while the supercritical half is plotted against β, the decay constant.

In fact, we can show that the presence of shear can play an important part in determining whether or not a flow is super- or sub-critical, and we illustrate this by considering the example of flow released from under a sluice gate. In the region of the gate the flow will be irrotational, but because the disturbance is finite we cannot use any of the theory of this section. As the surface decays to a uniform flow we can use the theory of small disturbances presented in Section 2 because, as the surface decays, a shear profile will develop, and if we consider turbulent flow over a smooth bottom, this will approach a 1/7 power law distribution. We can consider the flow as $U = U_1(y/h)^n$, with U_1, h, and n, all functions of x in the direction of flow.

Considering Fig. 1 we see that the development of a fully-turbulent shear profile corresponds to passing from some point on the irrotational curve to another point on the 1/7 law curve. If we consider flow with a nearly horizontal surface then we can write, for a velocity distribution of exponent $n(x)$, surface velocity $U_1(x)$ and local depth $h(x)$, that Q, the constant discharge, is given by

$$\begin{aligned} Q &= U_1(x)h(x)/(1 + n(x)) \\ &= U_m(x)h(x), \end{aligned}$$

where $U_m(x)$ is the mean velocity. Now for flow which has almost become horizontal, we can consider $U_1(x)$ as decreasing if anything, due to the diffusion of shear from the bottom, and because $n(x)$ is increasing, we see that $U_m(x)$ decreases, requiring that $h(x)$ increases, corresponding to the increase of displacement thickness in a boundary layer. Thus $U_m(x)/(gh(x))^{\frac{1}{2}}$ decreases, and if we consider flow developing from $n = 0$ to $n = 1/7$, it must decrease by 1/8 at least.

From Fig. 1 we see that passing from a point on the irrotational curve to one on the 1/7 curve with an ordinate 7/8 of the original, can result in a marked change in β. In fact it is quite possible to pass into the subcritical region, giving rise to periodic waves. In this case, finite wave theory is more applicable, and we expect the waves to be cnoidal in form. We have shown, however, that small wave theory, when applied to a developing shear profile, can predict a lessening of the decay rate, with the possible growth of periodic waves at some point. This wave growth is well known for irrotational waves and comes about because energy is lost from the flow: a finite amplitude theory can be, and must be, used to describe this situation (Benjamin & Lighthill, 1954). The present case is of interest in that it shows how periodic waves can be generated by the development of a shear profile.

4.2. *Dispersion Relationships for Constant Vorticity*

In Fig. 2 we have plotted the dispersion relationships for a linear velocity profile, obtained from (15) and (16) with $W'' = 0$. The axes are: the dimensionless difference between surface and bottom velocities in the undisturbed stream, $(U_1 - U_0)/(gh)^{\frac{1}{2}}$, for abscissa; and the dimensionless relative velocity between the disturbance and flow on the bottom, $(c - U_0)/(gh)^{\frac{1}{2}}$ for ordinate. Allowing both positive and negative values of the abscissa, corresponding to two possible directions of the streamflow, we may have upstream and downstream propagation on each; thus on Fig. 2 we have duplicated all the information that we would have had by considering flow in one direction only. In this way, starting from the positive abscissa axis, the first and third quadrants correspond to downstream propagation (c and U_1 of the same sign), while the upstream case is represented by the second and fourth quadrants.

Considering one side only of the line of symmetry $c = U_1$, we can see that the long wave curve divides the diagram into two main parts; on one side we have periodic waves, for which the velocity of propagation is less than that of long waves, and we say that the flow is subcritical. On the other side of the long wave curve we have the supercritical region, for which the velocity of propagation is greater than long waves, and on which periodic waves cannot exist; instead we have disturbances such as from solitary waves or sluice gates, in which the free surface decays exponentially to a constant level.

The equation of the long wave curve can be obtained by the integration of (8) with $U(y) = U_0 + (U_1 - U_0)(y/h)$:

$$(c - U_0)^2 - (c - U_0)(U_1 - U_0) = gh$$

and we see that for U_1 large, $c \to U_1$. This limit, $c = U_1$, corresponds to infinitesimally short waves having a relative phase velocity tending towards zero, and which tend to be convected along with the flow. In this region we would have to consider surface tension as well. From the diagram we see that the curves for given β and λ tend to cluster around the long wave curve, such that, for disturbances with a large length scale, periodic or not, the speed is relatively insensitive to the type of disturbance.

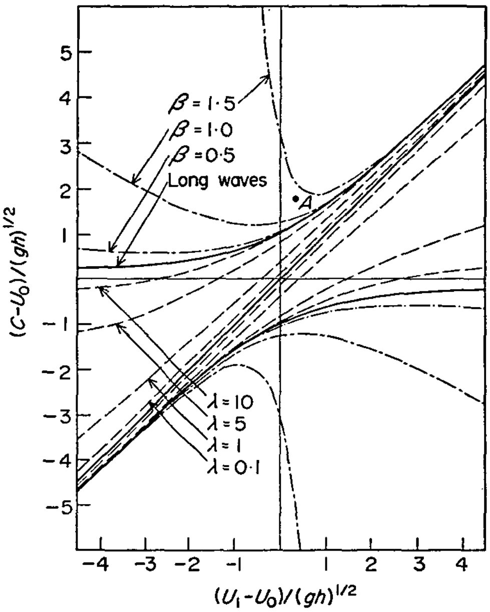

FIG. 2. Dispersion relations for disturbances on a flow of constant vorticity, showing a form of symmetry about the diagonal $c = U_1$. Curves shown — — — are drawn for periodic waves of length shown; those drawn — - — are for exponential decay at the indicated rate. The point A is the experimental result described in Section 4.4.

Finally we note that the two axes correspond to special cases. The vertical axis, on which $U_1 = U_0$, corresponds to waves on irrotational flow, and on which the intercepts of each curve are defined by (17):

$$(c - U_0)^2/gh = \tan \beta/\beta.$$

The horizontal axis corresponds to stationary disturbances (with respect to the flow on the bottom) defined by $c = U_0$.

4.3. *Numerical Solutions for Fully-turbulent Flow over a Smooth Bottom*

The results of Section 2.5 are shown on Fig. 3, plotted on the same axes as Fig. 2, but with $U_0 = 0$, which is more valid physically. Below the abcissa we have shown a scale for $U_m/(gh)^{\frac{1}{2}} = \frac{7}{8}U_1/(gh)^{\frac{1}{2}}$.

Qualitatively the figure is similar to that obtained for constant vorticity, Fig. 2. The two agree exactly, as we may expect, in that the straight line for $\lambda \to 0$ is defined by $c = U_1$ in each, and the intercepts on the vertical axis are the same, corresponding to the irrotational result. There is a similar division into supercritical and subcritical regions, and a similar cluster of curves about the long wave result which is even more marked in the present case, so that for disturbances which have a large length scale, there is a small variation in U_1 or c for large changes in β or λ. Thus we see that very small changes in the stream velocity, when close to critical, may cause very marked changes in the waves propagating upon it.

The curve corresponding to long waves can be obtained by integrating (8) with $U = U_1(y/h).^{1/7}$ We obtain

$$\frac{U_1^2}{gh} = 7\left[\frac{1}{5}+\frac{1}{2}\left(\frac{c}{U_1}\right)+\left(\frac{c}{U_1}\right)^2+2\left(\frac{c}{U_1}\right)^3+5\left(\frac{c}{U_1}\right)^4+\right.$$
$$\left. 6\left(\frac{c}{U_1}\right)^5\log\left(\frac{c-U_1}{U_1}\right)+\left(\frac{c}{U_1}\right)^4\cdot\frac{c}{c-U_1}\right].$$

For stationary disturbances $c = 0$, giving $U_1^2/gh = 7/5$ and $U_m/(gh)^{\frac{1}{2}} = 1{\cdot}035$. These are the values obtained by Lighthill (1953). We note that Fig. 3 contains all the information of Fig. 1, which gives the results for stationary disturbances; the intercepts of all the curves on the horizontal axis of Fig. 3 are defined by the curves in Fig. 1.

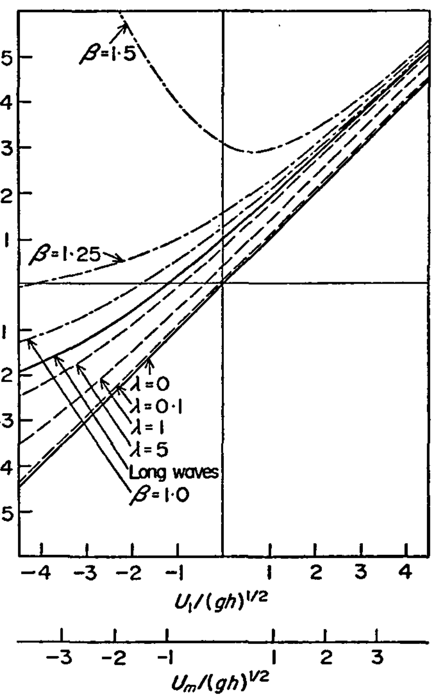

FIG. 3. Dispersion relations for disturbances on a 1/7 power law velocity distribution. The figure is similar to Fig. 2, but for clarity we have omitted curves in the lower half of the diagram. Curves shown — — are drawn for periodic waves of the length shown; those drawn — - - — are for exponential decay at the indicated rate.

4.4. *Experiments*

A basic assumption of the linearized theory is that the free surface of a shear flow follows an exponential law when decaying from a finite disturbance to a supercritical flow. To check this assumption, a series of simple experiments were run; they were not intended to be detailed, but merely to illustrate by example the behaviour of disturbances to supercritical shear flows.

The apparatus consisted of a gauze screen placed transversely across an open channel in which flow was subcritical, while downstream conditions could be varied by a control gate. Effectively the screen removed energy and momentum from the flow (mainly the latter), and produced a non-uniform velocity profile, so that the net effect was to create a shear flow downstream, the regime of which could be controlled independently.

With the downstream gate almost closed, there was little momentum loss at the screen, and the flow remained subcritical, with a depth only slightly less than upstream. This, with its small adjustment of the free surface to the new level, was of little interest.

When the gate was opened to an intermediate position, with a greater momentum loss at the gauze, periodic finite-amplitude waves were created. This is to be expected if we consider the theory of Section 3 and follow a physical reasoning similar to Benjamin & Lighthill: if the momentum loss at the gauze is intermediate between that required to produce uniform sub- and super-critical flow, then we expect a wave train. This is the reverse case of an undular hydraulic jump. In the experiments the waves were quite spectacular and broke at the first two crests, thereby reducing the energy of the flow and the wave amplitude. The amplitudes observed were too large for the quantitative theory of Section 3 to be applied; however they did show strikingly the magnitude of disturbances created by changes in energy or momentum.

When the downstream control was removed, the flow became supercritical, and the sudden spectacular decay from the screen to the shallow fast flow was observed, as shown in Fig. 4. Accurate measurements of the free surface were made with a micrometer needle gauge. The velocity measurements, intended to be approximate, were made with a simple pitot tube device—a glass tube bent at right angles, open at both ends, so that the velocity head was the difference between the water level in the tube and that in the channel.

A typical surface profile is shown in Fig. 4, which profile is also shown on a semi-logarithmic plot. We see that it is closely exponential, even up to the vicinity of the screen, where the theory is not applicable. Several surface profiles were measured for different flow rates, and in each case the decay was close to being exactly exponential, thus justifying this assumption in Section 2.

From velocity measurements for the case of the profile shown in Fig. 4, it was felt justified to assume a linear profile. Using the notation of Section 2 for this case, $F = 2{\cdot}12, F' = 0{\cdot}34$. Solving (15) gives $\beta = 1{\cdot}28$. This compares with the value of β, measured from Fig. 4, as $1{\cdot}35$, and a value of $1{\cdot}38$ obtained from (17) by assuming irrotational flow with the measured discharge. Thus the experimental observation falls between the irrotational and constant vorticity theories.

The assumption of a linear profile for this case may introduce large errors, because of the importance of W''/W as shown by the theory, while in this analysis we have assumed it to be zero. In addition, the point A shown on Fig. 2 is that corresponding to the measured values of F and F'. We see that in the region of A, the accuracy of velocity measurements is not at all compatible with the accuracy of the measurement of β, thus the above result can be taken as no more than a qualitative indication of the validity of the theory.

The experiments were of some value in addition to illustrating the exponential decay: they showed the sensitivity of the rate of decay to the velocity magnitude and distribution. They showed, as does the theory, the important part played by shear when considering the propagation of waves and disturbances, and how small fluctuations in the velocity profile may significantly alter dispersion characteristics of waves. This is also shown by the numerical results for waves on turbulent shear flows: small variations in the primary flow may alter the wave characteristics considerably and indeed may transform one wave type to another.

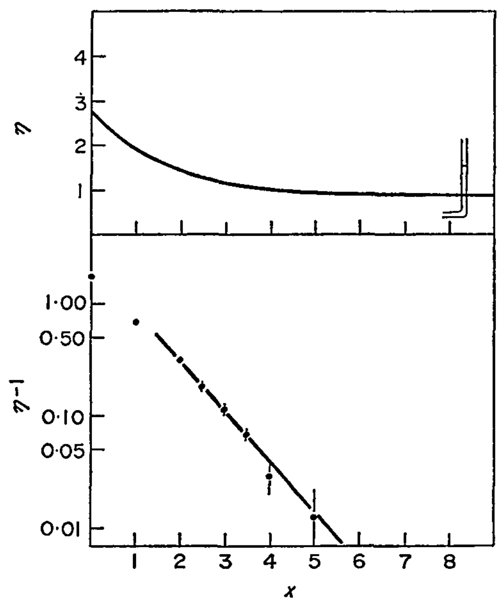

FIG. 4. Decay of free surface from a gauze screen to uniform supercritical flow. In the upper part the profile is plotted, with the screen in the plane of the η axis; the pitot tube used for measuring velocities of flow is shown. Below this diagram, the profile is plotted semi-logarithmically with the same x axis, with the accuracy of measurement and plotting shown. Exponential nature of decay is shown by the tendency of the points to lie on a straight line.

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