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LONG WAVES IN OPEN CHANNELS – THEIR NATURE, EQUATIONS, APPROXIMATIONS, AND NUMERICAL SIMULATION

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ABSTRACT

The long wave equations for waves in rivers and canals are considered and some results obtained that contradict current understanding and practice. Some generalisations of traditional coefficients are described, including a simple approximation for non-prismatic channels. Criticism is made of traditional reliance on the Gauckler-Manning equation. The method of characteristics is also criticised for providing misleading insight into the nature of the equations and the waves they describe. It is shown by linearising the equations that long waves have propagation characteristics that depend on wave period, so that the behaviour is more complicated than often believed. Traditional methods of non-dimensionalising the equations also give a misleading picture of them. Terms that have been previously believed to be inertial terms, of the magnitude of the Froude number squared, are in fact of the magnitude of the time rate of change of boundary conditions such as the inflow hydrograph. Accordingly, the nomenclature and application of some well-known approximations are criticised. Considering computational methods, the simplest forward-time central-space finite difference scheme is shown to be more stable than widely believed, and can be used to develop simple simulations. Finally, the problem of an open downstream boundary is considered and a good treatment shown to be to simply treat the end point as if it were an ordinary point in the stream and numerically solve the equations there also. This is in opposition to current theoretical understanding, but it seems to work well.

Keywords: Wave propagation, low-inertia approximation, kinematic wave, numerical methods, stability

1. THE LONG WAVE EQUATIONS

The long wave equations form the basis of many numerical models for the propagation of disturbances and floods in waterways, as well as for backwater analysis. We consider the equations as obtained and presented by Fenton (2010), here using the formulation in terms of cross-sectional area A and discharge Q as functions of the independent variables, x and t, where x is a horizontal cartesian co-ordinate along the line of the river, making the usual approximation that river curvature can be neglected, and t is time. The mass conservation equation is

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=i, \qquad [1]$$

where i is the rate of inflow volume per unit length from other sources such as rainfall, tributaries, seepage *etc.*. The equation is exact for a straight channel. The momentum equation is

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+\frac{gA}{B}\frac{\partial A}{\partial x}=gA\tilde{S}+\beta_i\, i\, u_i-\frac{\lambda P}{8}\frac{Q|Q|}{A^2}, \qquad [2]$$

where we have chosen for simplicity not to expand the second term, which can be trivially done. We have used the Darcy-Weisbach formulation of resistance, as recommended by the ASCE Task Force on Friction Factors in Open Channels (1963).

The symbols used in Eq. [2]are

β – a Boussinesq momentum coefficient
g – gravitational acceleration
B – surface width of the channel
$\tilde{S}$ – mean bed surface slope at a section
β_i – Boussinesq momentum coefficient of inflow
u_i – x-component of inflow velocity
λ – the Weisbach resistance coefficient
P – wetted perimeter.

The equations are well-known. We now consider those terms appearing here which are not used in other presentations.

$\tilde{S}$ – Local mean downstream slope of the stream bed evaluated across the section

This is defined at a section as

$$\tilde{S}=-\frac{1}{B}\int_B\frac{\partial Z}{\partial x}\mathrm{d}y, \qquad [3]$$

where the bed elevation is $z=Z(x,y)$, with y the transverse co-ordinate. $\tilde{S}$ is defined with a negative sign such that in the usual situation where the bed slopes downwards in the direction of x so that Z decreases, $\tilde{S}$ will be positive. If the bottom geometry is precisely known, this can be precisely evaluated. However the geometry is likely to be only approximately known and a typical bed slope of the stream is often used. On the other hand, if it is known, the term incorporates what in other presentations is referred to as the non-prismatic contribution, which seems never to have been specifically evaluated and presented.

β – Boussinesq momentum coefficient

This allows for non-uniformity of velocity in space and time. It is defined as

$$\beta = \frac{1}{U^2 A}\int_A \left(\overline{u}^2 + \overline{u'^2}\right) dA, \qquad [4]$$

where $U = Q/A$ is the time and area mean velocity over the section, $\overline{u}$ is the time mean of the streamwise velocity u at any point and u' is the fluctuating departure of velocity from the mean at that point, whose mean value is zero. The traditional definition of β does not contain the $\overline{u'^2}$ term (Fenton 2005). In Nezu& Nakagawa (1993, figs 4.9 and 4.10) are shown some values of $\sqrt{\overline{u'^2}}/U$ (note that they use the symbol u' for what we write as $\sqrt{\overline{u'^2}}$ here). Squaring them, to give the quantity as it appears in equation [4], $\overline{u'^2}/U^2$ varies from roughly 0.1 near the bed to 0.01 some 10% of the depth above the bed, to some 0.006 near the surface. Although small, this "buffeting" effect of turbulence may be significant compared to the non-uniformity of $\overline{u}$ over the section traditionally used to evaluate β. For example, using a power law for a wide channel, such that $u \propto z^{\nu}$ for a typical $\nu \approx 1/7$ gives a contribution to β of 0.016, while from Nezu & Nakagawa's results we estimate a mean turbulent contribution of about 0.01, giving $\beta = 1.016 + 0.01 = 1.026$. The correction of 2.6% is not particularly important in such a straight laboratory channel. In practical situations β may be rather larger, such as the familiar situation where secondary currents drive the maximum of the velocity down to a point 60% of the depth above the bed, such as shown by Nezu(2005, fig. 11b). In compound channels where there are overbank flows such as described by Knight (2013, fig. 8) the flow in the shallow part may be considerably slower, leading to a rather larger value of β. In this case, of course, the actual values are poorly known. Fortunately for the dynamics of the problem, that does not matter very much in most cases, for as we will see below, the fluid momentum term in equation [2], identified by the coefficient β, will be shown below to be relatively unimportant in many situations. In our simple integrated model the interfacial stresses between water in the main channel and the overbank flow mutually cancel, and the accuracy of modelling of the velocity distribution and turbulence is not particularly important.

λ – Weisbach dimensionless resistance coefficient

Using the Weisbach form for the forces of the boundary on the flow makes incorporation of the resistance much more rational than the Gauckler-Manning form, in terms of Manning's n. The ASCE Task Force on Friction Factors in Open Channels (1963) recommended its use, but that suggestion has been almost entirely ignored.

Consider the expression for the magnitude of the shear force τ on a pipe wall (*e.g.* §6 of White 2009)

$$\tau = \frac{\lambda}{8}\rho V^2, \qquad [5]$$

where the Weisbach coefficient λ is a dimensionless resistance factor (for which the symbol f is often used, but here we follow the terminology of fundamental researchers in the field in the first half of the twentieth century), and V is the mean velocity in the pipe. The denominator 8 follows from the original introduction of λ in the Darcy-Weisbach formula for head loss in a pipe, with a term $2g$ in the expression for head and a term 4 in the relationship between head loss and τ. It can be seen that the last term in equation [2] has been obtained simply by assuming $V = U = Q/A$ for a small slope and multiplying the stress by the wetted perimeter P to give the force per unit length.

One advantage of the Weisbach formulation, being directly related to stress and force, is that one can linearly superimpose contributions, so that in a more complicated situation, where there may be bed-forms and vegetation contributing to the resistance, the forces can be added and we can write, for contributions from various parts

$$\lambda P = \sum_i \lambda_i P_i .$$

Another example where a formula such as this would be useful is a glass-walled laboratory flume with a rough bed.

An idea of the problems which the empiricism of the Gauckler-Manning form causes is given by the different formulae for the Manning coefficient n, all found in one report on resistance in streams:

$$n = \sum_i n_i \quad \text{or} \quad n = \left(\sum_i n_i^2\right)^{1/2} \quad \text{or} \quad \frac{1}{n} = \left(\sum_i \frac{1}{n_i^2}\right)^{1/2} .$$

There were different recommendations as to when each method is to be preferred, but there was no inclusion of weighting according to what fraction of the perimeter is to be assigned to each contribution.

Discussion

The momentum equation [2] is actually rather simple. Except for the inflow momentum term, which is usually small and poorly known, there are few parameters in the differential equation: the momentum coefficient β, gravitational acceleration g, the resistance coefficient λ, plus the geometric quantities of surface width B, the wetted perimeter P, and the local mean slope $\tilde{S}$. A convenient formula for λ has been given by Yen (2002, eqn 19) in terms of relative roughness and channel Reynolds number. Still, it is usually not well known, and it remains a problem. However, it is not nearly as large a problem as the continued use of the irrational Gauckler-Manning formulation.

In practice, the surface elevation η is more important than the area A, but for the purposes of this paper it is simpler to use area A, as it is more fundamental in the mass conservation equation [1] and occurs throughout the momentum conservation equation [2] with clear physical significance. The only term where it is not quite so fundamental is the driving gravity term $gA/B \times \partial A/\partial x$, which actually comes from a term $\partial \eta/\partial x$ from the pressure gradient in the fluid.

However, there is a possible reason for using A even in computations. If the geometry of the bed of a stream were well-known, then $A(x,\eta)$, $B(x,\eta)$ and $P(x,\eta)$ could be evaluated at computational sections x, probably in discrete tabular form for interpolation, and the use of η seems justified. However, in rivers the geometry is poorly known, and that implied accurate interpolation using presumed data and functional relationships is not in keeping with the reality of the approximately-known problem. If the area A were used as dependent variable, then ideally the corresponding $B(x,A)$ and $P(x,A)$ would have to be known also. However we suggest they are not well-known, and one might use only approximate values of B and P, possibly constant at each computational section, B from surveys or aerial photographs and possibly using $P \approx B$ for wide streams, and in any case the factor λ is only approximately known. One could calculate the initial values of A at computational points along the stream by assuming notional approximate values of A at the points, with the given initial constant flow Q, and perform a simulation until the values of A became steady. These could then be used for the real simulation with varying input discharge at the upstream boundary. Hence, one does not really need accurate bed surveys. This procedure seems simple and in accordance with the often-unknown details of river geometry.

2. THE METHOD OF CHARACTERISTICS

This has been responsible for the widespread misunderstanding of the nature of wave propagation and the wrong belief that long waves travel at a speed of approximately $\sqrt{g \times \text{Depth}}$.

The long wave equations [1] and [2] for no inflow give four ordinary differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta\frac{Q}{A} \pm C, \tag{6}$$

$$\left(-\beta\frac{Q}{A} \pm C\right)\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\mathrm{d}Q}{\mathrm{d}t} = gA\tilde{S} - \frac{\lambda}{8}P\frac{Q|Q|}{A^2}. \tag{7}$$

The first pair, Eq. [6] with + and – signs, describes the paths of two characteristics in (x,t) space, where the gradient (velocity) of the characteristics contain an advection velocity $\beta Q/A$, with the momentum coefficient β surprisingly multiplying the mean water velocity Q/A. Relative to this, information propagates up- and downstream at speed C defined by

$$C = \sqrt{\frac{gA}{B} + \frac{Q^2}{A^2}\left(\beta^2 - \beta\right)}. \tag{8}$$

Using β has provided a surprise, that this so-called "long wave speed" is generally not just $\sqrt{gA/B}$. Equation [6] has led to the widespread belief that long waves travel at a unique speed C. In fact, the other pair of ordinary differential equations Eq. [7], to be satisfied by A and Q, immediately suggest that this is not the case. The right side is only zero in steady uniform flow, in all other cases A and Q are continually changing along the characteristics, and hence so is the surface elevation. Information of a certain type is travelling at speed C relative to the water, but in general, what we might see as a wave with an identifiable crest, is *not* necessarily travelling at C.

Below, in §7 we will suggest that traditional reliance on the method of characteristics to determine the nature of boundary conditions has given a misleading view of downstream open boundary conditions.

3. THE NATURE OF LONG WAVES

To obtain the actual behaviour of waves we linearise the long wave equations about a uniform steady flow so that mathematical solutions can be obtained. This was first done by Lighthill& Whitham (1955)and more recently by Ponce & Simons (1977).

3.1 Linearisation

We consider relatively small disturbances about a uniform flow on a constant slope S, with area A_0 and discharge Q_0 and write

$$A = A_0 + \varepsilon A_1 \quad \text{and} \quad Q = Q_0 + \varepsilon Q_1,$$

where ε is a small quantity. As A_0 and Q_0 are constant, all derivatives of A and Q in Eqs. [1]and[2]are of order ε so that the coefficients need only be written to zeroth order, and the linearisation is simple. The only non-trivial operations are in the remaining resistance and slope terms on the right of Eq. [2]. We take them to the left of the equation and introduce the function ϕ for them

$$\phi = \frac{\lambda}{8}P\frac{Q^2}{A^2} - gAS, \tag{9}$$

where we have written $Q|Q| = Q^2$, as we consider small perturbations about a uniform flow, which is unidirectional. The linearised equations are

$$\frac{\partial A_1}{\partial t} + \frac{\partial Q_1}{\partial x} = 0, \tag{10}$$

$$\frac{\partial Q_1}{\partial t} + \left(C_0^2 - \beta^2 U_0^2\right)\frac{\partial A_1}{\partial x} + 2\beta U_0\frac{\partial Q_1}{\partial x} + \left.\frac{\partial\phi}{\partial Q}\right|_0 Q_1 + \left.\frac{\partial\phi}{\partial A}\right|_0 A_1 = 0, \tag{11}$$

in which the mean fluid velocity $U_0 = Q_0/A_0$ has been used for simplicity, as well as the speed $C_0 = \sqrt{gA_0/B_0 + \left(\beta^2 - \beta\right)U_0^2}$, the linearized value of the speed of characteristics, Eq. [8]. We note that $\phi = 0$ for the uniform flow, such that Eq.[9]gives

$$\frac{Q_0}{A_0} = U_0 = \sqrt{\frac{8g}{\lambda_0}\frac{A_0}{P_0}S_0}, \tag{12}$$

which is just the Weisbach uniform flow formula.

To evaluate $\partial\phi/\partial Q$ we differentiate Eq. [9] and eliminate Q_0 by using the uniform flow formula to give the two forms

$$\left.\frac{\partial\phi}{\partial Q}\right|_0 \approx \sqrt{\frac{gS_0\lambda_0}{2A_0/P_0}} = 2\frac{gS_0}{U_0}. \tag{13}$$

We have neglected the dependence of λ on Q (coming from its dependence on Reynolds number).

For the other derivative we obtain

$$\left.\frac{\partial\phi}{\partial A}\right|_0 = -3gS_0\left(1 - \tfrac{1}{3}\left(\frac{A_0}{P_0}\left.\frac{\partial P}{\partial A}\right|_0 + \frac{A_0}{\lambda_0}\left.\frac{\partial\lambda}{\partial A}\right|_0\right)\right) \tag{14}$$

In this case, both derivative terms in $\partial P/\partial A$ and $\partial\lambda/\partial A$ in general may have finite contributions, the first expressing the effect of finite channel width, and the second because the resistance coefficient has a strong variation with relative roughness.

Equations [10] and [11] are combined into a single equation by introducing a function $v(x,t)$ such that $A_1 = \partial v / \partial x$ and $Q_1 = -\partial v / \partial t$, so that it satisfies the mass conservation Eq. [10] identically. The linearised momentum Eq. [11] becomes the Telegrapher's equation:

$$\sigma_0\left(\frac{\partial v}{\partial t}+c_0\frac{\partial v}{\partial x}\right)+\frac{\partial^2 v}{\partial t^2}+2\beta U_0\frac{\partial^2 v}{\partial x\partial t}-(C_0^2-\beta^2U_0^2)\frac{\partial^2 v}{\partial x^2}=0. \quad [15]$$

We have introduced the symbols σ_0 and c_0, where:

Inverse time scale σ_0: this is simply $\partial\phi / \partial Q|_0$ as given by Eq. [13] in terms of the geometry of the channel and the resistance coefficient:

$$\sigma_0 = \sqrt{\frac{gS_0\lambda_0}{2A_0/P_0}} = 2\frac{gS_0}{U_0}. \quad [16]$$

The σ_0 has dimensions of T^{-1}. It will be found below that it is an important scale for wave behaviour.

A wave speed c_0: this is the ratio of the two derivatives

$$c_0 = -\frac{\partial\phi / \partial A|_0}{\partial\phi / \partial Q|_0},$$

and using Eqs. [13] and [14]:

$$c_0 = \alpha U_0, \quad \text{where} \quad \alpha = \frac{3}{2}\left(1-\frac{1}{3}\left(\frac{A_0}{P_0}\left.\frac{\partial P}{\partial A}\right|_0+\frac{A_0}{\lambda_0}\left.\frac{\partial \lambda}{\partial A}\right|_0\right)\right), \quad [17]$$

such that α is roughly $3/2$. The quantity c_0 is a wave speed, as will be shown below.

3.2 A simple noteworthy result

There is one result from above that is worth noting. If we take the Weisbach uniform flow formula [12] and calculate the Froude number of the uniform flow, we obtain

$$F_0^2 = \frac{U_0^2}{gA_0/B_0} = \frac{8}{\lambda_0}\frac{B_0}{P_0}S_0, \quad [18]$$

and as B_0/P_0 will not vary much with depth of flow, and for many rivers sufficiently wide, in fact $P_0 \approx B_0$, we have the result

$$F_0^2 \approx \frac{S_0}{\lambda_0/8}. \quad [19]$$

At any section in a stream S_0 is independent of flow, and λ_0 does not vary much, showing some variation with depth of flow and hence with discharge, and we have the conclusion that at any point in a stream, the Froude number is almost independent of flow. So, whether in normal flow or flood flow, the Froude number at a section in a river is about the same. This might just be of theoretical interest, but it is worth noting. In passing, we estimate a Froude number for a river of velocity $0.5\,\mathrm{ms}^{-1}$ on a depth of $2\,\mathrm{m}$, $F_0^2 \approx 0.5^2/(10\times 2) = 0.0125$; for such a case, dynamical effects are of the order of 1%.

3.3 Nature of wave propagation

The terms in the Telegrapher's equation [15] can be grouped: the first two terms can be characterised as $\sigma_0\times$ first derivatives, and the last three terms are all second derivatives. We now examine the equation in two limits and then the general case:

Very-long waves: For disturbances that have a long period, such that $\partial^2/\partial t^2 \ll \sigma_0\partial/\partial t$, which we will call "very long waves", the last three terms in Eq. [15] can be neglected, and the equation becomes

$$\frac{\partial v}{\partial t}+c_0\frac{\partial v}{\partial x}=0,$$

with a general solution $v = f_1(x-c_0t)$, where $f_1(.)$ is an arbitrary function given by the upstream conditions. This solution is a wave propagating downstream at speed $c_0 = \alpha U_0$. This has been called the "kinematic wave speed", and the equation has been widely known as the "kinematic wave equation" because the approximation has previously been proposed that terms of order F^2 in the momentum equation have been neglected. The solution here, and the dimensionless analysis below show that no approximation has been made by neglecting dynamical terms and it is actually a very long wave approximation.

Not-so-long waves: In the other limit, for disturbances which are shorter, such that $\partial^2/\partial t^2 \gg \sigma_0\partial/\partial t$, for which we use the term "not-so-long" waves, Eq. [15] becomes

$$\frac{\partial^2 v}{\partial t^2}+2\beta U_0\frac{\partial^2 v}{\partial x\partial t}-(C_0^2-\beta^2U_0^2)\frac{\partial^2 v}{\partial x^2}=0,$$

which is a second-order wave equation with solutions

$$v = f_{21}\left(x-(\beta U_0+C_0)t\right)+f_{22}\left(x-(\beta U_0-C_0)t\right)$$

where $f_{21}(.)$ and $f_{22}(.)$ are arbitrary functions determined by boundary conditions. In this case the solutions are waves propagating upstream and downstream at velocities of $\beta U_0 \pm C_0$, such that in the usual terminology C_0 is the "long wave speed". We have shown here that it is the speed of waves that are not so long, apparently paradoxically: they are long enough that the pressure distribution in the fluid is still hydrostatic, but only at the short wave limit of such waves.

Intermediately-long waves: It is possible to obtain general solutions of the Telegrapher's equation [15] by assuming a solution periodic in time, which we might consider to be a single component of a Fourier series describing a general input hydrograph. We assume $v = \exp(\mathrm{i}(\kappa x-\omega t))$, where i $=\sqrt{-1}$, and where the frequency ω is real, but in general κ is a complex quantity whose nature determines the behaviour of the solutions in space. Substituting this solution into equation[15] gives a quadratic equation for the coefficient κ which in general has real and imaginary parts $\kappa_\mathrm{r}(\omega)$ and $\kappa_\mathrm{i}(\omega)$, both of which are functions of real frequency ω. The solution is

$$v = \exp(-\kappa_\mathrm{i}(\omega)x)\exp\left(\mathrm{i}\kappa_\mathrm{r}(\omega)\left(x-\frac{\omega}{\kappa_\mathrm{r}(\omega)}t\right)\right), \quad [20]$$

so that as waves are input into the channel with a frequency ω, they decay along the channel at a spatial rate given by $\kappa_\mathrm{i}(\omega)$ and progress at a propagation velocity of $\omega/\kappa_\mathrm{r}(\omega)$. We have the important physical result that a wave periodic in time will propagate at a velocity dependent on the wave frequency (*i.e.*it depends also on period) of $c = \omega/\kappa_\mathrm{r}(\omega)$ and will decay in space

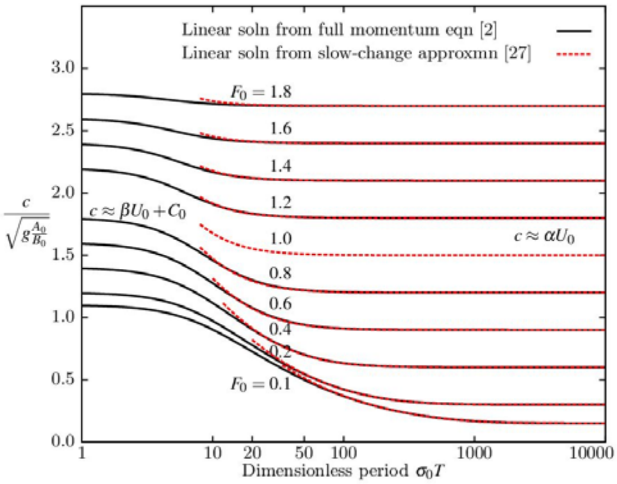

Figure 1Dependence of wave speed on wave period and Froude number, also shown is the slow-change approximation

due to resistance with a decay rate $\kappa_i(\omega)$, also dependent on wave period/frequency. This means that the whole system is, in general, diffusive (rate of decay depends on frequency) and dispersive (propagation velocity depends on frequency), which is much more complicated than the superficial deductions from the method of characteristics.

Results from the linear solutions are plotted on Figure 1, along with an approximation to be obtained below. The figure shows that the wave speed depends on wave period, that shorter waves travel faster than longer waves, that in the “not-so-long” limit, $\sigma_0 T < 2$, the wave speed is $c \approx \beta U_0 + C_0$, corresponding to the result from the method of characteristics, and for “very long” waves, $\sigma_0 T > 500$, $c \approx \alpha U_0$.

4. NON-DIMENSIONAL LONG WAVE EQUATIONS

Consider the full long wave equations [1] and [2]. Here, in contrast to previous work, we assert that the variation with time in the channel is determined by the time variation of the input to the system, of scale T, say, while the length scale of disturbances L is not known *a priori*. We introduce dimensionless quantities denoted by asterisks, such that $t = t_* T$, and $x = x_* L$. For channel width we use the width scale B_0 such that $B = B_* B_0$, and for the perimeter we write $P = P_* P_0$, in terms of a perimeter scale P_0. Now introducing the area scale A_0 and discharge scale $Q_0 = U_0 A_0$, the dependent variables A and Q are scaled as $A = A_* A_0$ aid $Q = Q_* U_0 A_0$. In addition we write the channel slope and friction factors in terms of reference values with a 0 subscript as $\tilde{S} = S_0 S_*$, and $\lambda = \lambda_0 \lambda_*$. The equations we obtain are

$$\frac{\partial A_*}{\partial t_*} + \frac{U_0 T}{L}\frac{\partial Q_*}{\partial x_*} = 0, \qquad [21]$$

$$\frac{U_0}{gS_0 T}\left(\frac{\partial Q_*}{\partial t_*} + \frac{U_0 T}{L}\frac{\partial}{\partial x_*}\left(\beta\frac{Q_*^2}{A_*}\right)\right) + \frac{A_0/B_0}{S_0 L}\frac{A_*}{B_*}\frac{\partial A_*}{\partial x_*} + \frac{\lambda_*}{8}P_*\frac{Q_*^2}{A_*^2} - A_* S_* = 0, \qquad [22]$$

where we have used the uniform flow Eq. [12] to write λ_0 in terms of U_0, and where it has been assumed that flow does not reverse so that we can use Q^2 in the resistance term. It can be seen that in each equation the coefficient $U_0 T / L$ expresses the relative importance of a spatial x derivative to the time derivative.

In the traditional approach (e.g. Woolhiser and Liggett 1967, and subsequent researchers), the length and time scales of disturbances, and hence their velocity of propagation, are assumed related by the mean fluid velocity, such that $T = L / U_0$, the equations become after re-arrangement:

$$\frac{\partial A_*}{\partial t_*} + \frac{\partial Q_*}{\partial x_*} = 0, \qquad [23]$$

$$F_0^2\left(\frac{\partial Q_*}{\partial t_*} + \frac{\partial}{\partial x_*}\left(\beta\frac{Q_*^2}{A_*}\right)\right) + \frac{A_*}{B_*}\frac{\partial A_*}{\partial x_*} + \frac{S_0 L}{A_0/B_0}\left(\frac{\lambda_*}{8}P_*\frac{Q_*^2}{A_*^2} - A_* S_*\right) = 0, \qquad [24]$$

where $F_0^2 = U_0^2 / (gA_0/B_0)$. In this way, in the commonly-used so-called "low inertia approximation", if the two leading terms, the time derivative term and the fluid momentum flux term in β are neglected, it seems to be appealing to F_0^2 being small, hence the appellation "low inertia".

We assert, however, that T is imposed on the system from outside in the form of a boundary condition and the channel has to respond with that value. Also, it is not correct always to assume that L and T are related by $T = L / U_0$. This is shown by the solution of the linearised equations for $\kappa(\omega)$ above, translating into $L(T)$ here, which is a known complicated function of T. In fact from equation [16] we can express the leading coefficient of the momentum equation [22] simply as $2/\sigma_0 T$. This gives the dimensionless long wave equations

$$\frac{\partial A_*}{\partial t_*} + \frac{U_0 T}{L}\frac{\partial Q_*}{\partial x_*} = 0, \qquad [25]$$

$$\frac{2}{\sigma_0 T}\left(\frac{\partial Q_*}{\partial t_*} + \frac{U_0 T}{L}\frac{\partial}{\partial x_*}\left(\beta\frac{Q_*^2}{A_*}\right)\right) + \frac{A_0/B_0}{S_0 L}\frac{A_*}{B_*}\frac{\partial A_*}{\partial x_*} + \frac{\lambda_*}{8}P_*\frac{Q_*^2}{A_*^2} - A_* S_* = 0. \qquad [26]$$

Immediately we see a different result in the momentum equation, that neglecting the time derivative and the fluid momentum term is not a "zero inertia approximation" but is actually valid when $\sigma_0 T$ is large, so it is actually a *very long wave approximation*, or, as "long" has connotations of length, we might prefer the term *slow change approximation*, expressing the behaviour in time. Froude number has not entered the equations. In view of this, the term "kinematic wave", originating with Lighthill & Whitham (1955) seems to be a misnomer.

5. A COMMON APPROXIMATION – NEGLECT OF TIME DERIVATIVE AND FLUID INERTIA TERMS

Now we introduce the common approximation to the momentum equation where both the time derivative and the inertial terms are neglected, commonly referred to as the "low-inertia approximation" but which we have shown to be a slow change approximation for $\sigma_0 T$ large. As equation [26] suggests, we retain the last three terms in the momentum equation [2], giving

$$\frac{gA}{B}\frac{\partial A}{\partial x} = gA\tilde{S} - \frac{\lambda}{8}\frac{Q|Q|P}{A^2}\,. \qquad [27]$$

For the equations with Eq. [1] and this very long wave approximation, Eq. [27], the propagation velocity of the linearised solution are also shown plotted on Figure 1, including an extra curve for critical Froude number $\beta F_0^2 = 1$ (with $\beta = 1$ here), as the solution Eq. [27]is valid for that value also. The results show very surprising agreement with the full solution for longer downstream travelling waves even for large Froude numbers. For these downstream travelling waves the accuracy of the approximation does depend only on the period and is independent of Froude number, supporting our contention that Eq. [27] is not a low-inertia approximation, but is a long period or *slow-change approximation*.

6. NUMERICAL SOLUTION BY EXPLICIT FINITE DIFFERENCES

The simplest scheme for the numerical solution of the long wave equations is that using explicit finite differences. Time derivatives in the differential equations [1] and [2] are approximated by forward-time approximations

$$\left.\frac{\partial f}{\partial t}\right|_{(x,t)} \approx \frac{f(x,t+\Delta) - f(x,t)}{\Delta}, \qquad [28]$$

where f can be Q or the other dependent variable, A, η, or h, and where Δ as shown is a time step. Space derivatives are approximated by the centre-space expression

$$\left.\frac{\partial f}{\partial x}\right|_{(x,t)} \approx \frac{f(x+\delta,t) - f(x-\delta,t)}{2\delta}, \qquad [29]$$

where δ is a space step. Substituting these expressions with f replaced by A and Q into equations [1] and [2] gives a pair of explicit equations for the values at the next time step $A(x,t+\Delta)$ and $Q(x,t+\Delta)$ in terms of the values of A and Q at the three points $(x-\delta,t)$, (x,t), and $(x+\delta,t)$ at the present time step t.

Liggett & Cunge (1975, p111) suggested that such an explicit scheme (**F**orward-**T**ime-**C**entral-**S**pace) was unconditionally unstable and instead of FTCS they named it "The Unstable Scheme". This may have contributed to the extensive use of implicit methods, such as the Preissmann Box scheme. Such schemes are stable and allow large time steps, but they are complicated and require many more calculations, including the solution at each time step of a system of nonlinear equations, the number of equations being equal to the number of space steps. This complexity may have contributed to computational hydraulics, once being a cottage industry with skilled people, similarly to the tendencies of the industrial revolution, becoming dominated by large software houses and the down-skilling of such people.

The author believes that the deductions of Liggett & Cunge concerning the simple explicit method are wrong. A linear stability analysis, unfortunately too long and complicated to present here, shows that the scheme has a quite acceptable stability limitation, and it opens up the possibility of this as a much simpler method for computations of floods and flows in open channels. When compared with computations over a wide range of slopes and roughnesses encountered in practice, the stability limit found gives quite a sharp estimate of the allowable time step in practical calculations.

Now we present the results of the stability analysis, giving a procedure to follow.

Procedure to determine time step for stability:

1. For each of the minimum and maximum flows expected, $Q_1 = Q_{\min}$ and $Q_2 = Q_{\max}$, solve the uniform flow problem to give values in both cases of area $A_{1,2}$, top widths $B_{1,2}$ and wetted perimeters $P_{1,2}$.
2. Calculate Froude numbers $F_{1,2} = Q_{1,2}\sqrt{B_{1,2}/\left(gA_{1,2}^3\right)}$
3. Calculate $\Omega_1 = \dfrac{2\pi}{N}\dfrac{F_1A_1/B_1}{S_0\delta}$ and $\Omega_2 = \dfrac{F_2A_2/B_2}{S_0\delta}$, where δ is the computational step length in x, and there are N steps.
4. For $i = 1$ to 2 do

   Calculate $\sigma_{0i} = \sqrt{\dfrac{gS\lambda}{2A_i/P_i}}$

   If $\Omega_i \le \dfrac{\sqrt{2+F_i}}{1-F_i}$ then $\Delta_i \le \dfrac{2}{\sigma_{0i}}$

   Else $\Delta_i \le \dfrac{2}{\sigma_{0i}}\dfrac{2+F_i}{(1-F_i)^2\Omega_i^2}$
5. The maximum time step for stability is the minimum of the two values Δ_i so calculated.

It is interesting that the time rate constant σ_0 found above to be so important in theoretical studies of the wave propagation behaviour, has re-appeared here as fundamental in determining stability.

To test this procedure we considered a single example stream with a trapezoidal section, 10m wide at the bottom with side slopes 1:1, with $\beta = 1$. The stream was 20km long and was divided into $N = 16$ computational steps. We also considered a range of slopes and resistance coefficients corresponding to the majority of streams in the USA and New Zealand according to the two compendia of resistance coefficients by Barnes (1967) and Hicks& Mason (1991), given by notional Froude numbers in the range $0.1 \le F_0 \le 0.8$ and resistance $0.025 \le \lambda_0 \le 0.25$. From those two quantities, in each case, slope was calculated from the approximation to equation [19] $S_0 = \lambda_0 F_0^2/8$, so that the slope varied between 3×10^{-5} and 2×10^{-2}. The range of parameters is shown by the shadowed grey lines on the base of Figure 2.

The procedure described above was followed to determine the theoretical limits on time step in each case. Next, to compare those predictions with actual computations, for each slope and resistance coefficient, a flood was simulated using the FTCS scheme described above. The minimum initial flow was that given by the conditions in each case from Barnes and Hicks & Mason, the maximum flow10 times that.

The results are shown on Figure 2, where the actual limiting time steps Δ are plotted. Typical values varied between several seconds for steep and fast flows, to something of the order of a hundred seconds over much of the domain. Generally the predictions of the theory here predicted the actual stability limit found by computations to within a few percent, however for some small slopes and Froude numbers it can be seen that there is as much as a 100% disagreement for some points, made less apparent by the logarithmic scale. For practical purposes, this does not matter – an order of magnitude estimate for the limiting time step is often enough. It is surprising that, despite the highly nonlinear nature of the problem, with a ten-fold flow increase, generally the simplified linear stability analysis gave a reasonably accurate value for the allowable time step.

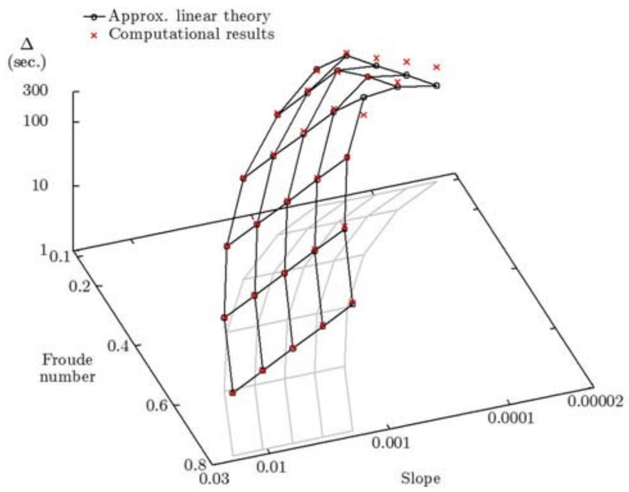

Figure 2Comparison of actual computational time steps required for stability with those predicted by the approximate linear theory.

What is clear is that the Forward-Time-Central-Space method is quite capable of simulating flows and waves in open channels. The time steps allowed might be less than for implicit schemes, but the simplicity is an important advantage. While the time steps obtained can sometimes be small, a typical run time on a personal computer was only second(s). Such small time steps do not seem to be a problem in practice.

That is not a conclusive proof of the applicability of the limits in real streams and floods, of course, but provides a guide to the estimation procedure that might be followed – and to the sorts of time steps expected: small in the case of streams on large slopes, but finite for the majority of streams.

7. DOWNSTREAM OPEN BOUNDARY CONDITION

Finally, the downstream boundary condition for open channel computations is considered. There are four types which are most common, much as described by HEC-RAS (2010, p2-45):

Stage Hydrograph – such as for a lake, reservoir, or the sea, where the water surface elevation downstream is specified as constant or as a function of time.

Flow Hydrograph– where recorded gauge data is available and the model is being calibrated to a specific flood event, however this seems relatively rare.

Rating Curve –where there is a unique functional relationship between flow and surface elevation. Most commonly this is where there is a control structure such as a spillway, weir, gate, or flume. HEC-RAS (2010) mentions the existence of a looped rating curve and provides a warning. We believe that this is more of a manifestation of a problem which is improperly posed, such as when the stream roughness increases during and after a flood event, or unsteady effects are important, so that the functional relationship varies with time. This could be included in a program.

Open boundary – The remaining boundary condition is where the computational domain is truncated, and no computations are performed downstream of that point.

The first three boundary conditions are capable of rational implementation. For the last, the open boundary, HEC-RAS (2010) calls it the *Normal Depth* boundary and advocates using Manning's equation to give a stage considered to be normal depth if uniform flow conditions existed downstream. However because uniform flow conditions do not usually exist in natural streams, they suggest that this boundary condition should be used far enough downstream from the study area that it does not affect the results in the study area".

We believe that one does not have to use that approach, and that a very simple and rational alternative is at hand for an open boundary. However, it goes in the opposite direction from the conventional understanding of open channel hydraulics, where an important principle is "one boundary condition has to be given for every characteristic entering through the boundaries of the solution domain"(Cunge Holly & Verwey 1980, p31). Szymkiewicz (2010, p170) similarly notes "… the correct classification of considered partial differential equations and the knowledge of their characteristics structure has fundamental meaning for well posedness of solution".

In this way, conventionally a boundary condition is applied even at an open boundary at a truncated position in the stream. However, if one can truncate a computational domain then it must be because downstream the region is unimportant and no significant information is coming back from that region. This means that by applying an arbitrary boundary condition, such as the uniform flow condition, the information entering the computational domain is incorrect, and hence the warning by HEC-RAS (2010) to place such a boundary far from the region of interest.

Here, we advocate simply doing away with the downstream boundary condition if it is wrong or arbitrarily approximated. Instead we suggest simply treating the end point as if it were an ordinary point in the stream and numerically solving the equations there also, but using backward differences for the derivatives. If the boundary condition is open, all important information is coming from upstream, as the waves that are input at the upstream boundary progress downstream. To use upstream or backward differences seems sensible.

Accordingly, instead of using the centre difference equation [29], we use the three-point backward-difference expression (more accurate than the obvious two-point one):

$$\left.\frac{\partial f}{\partial x}\right|_{(L,t)} \approx \frac{f(L-2\delta,t)-4f(L-\delta,t)+3f(L,t)}{2\delta}, \qquad [30]$$

where f can be Q or the other dependent variable A, η, or h. Substituting into equations [1] and [2] at $x=L$ and manipulating, gives two explicit equations for A and Q at $(L,t+\Delta)$.

This was tested in a model case: a stream of slope 0.0001, length $20\mathrm{km}$, with a trapezoidal section of bottom width $20\mathrm{m}$ and sides $1:1$, and with a Manning $n=0.035$. An initial flow of $10\mathrm{m^3s^{-1}}$ was increased smoothly to $100\mathrm{m^3s^{-1}}$ and back down to the original flow. Two calculations were performed, using a uniform flow boundary and no boundary condition, as we have suggested here. Calculations were repeated, considering the same stream but with a length of $40\mathrm{km}$. We were only interested in the results at the $20\mathrm{km}$ point, so that we could test our two boundary conditions applied at the truncation point. Results are shown in Figure 3. It can be seen that three of the results agree closely: where the boundary condition was applied a further $20\mathrm{km}$ downstream, both the uniform flow boundary condition and our no boundary condition agreed closely, as we might expect. Importantly for our purposes, the figure shows that the no boundary condition applied at the actual boundary also agreed closely with the presumably accurate other calculations. However, the widely-used uniform flow boundary condition gave quite different results, in both magnitude and time.

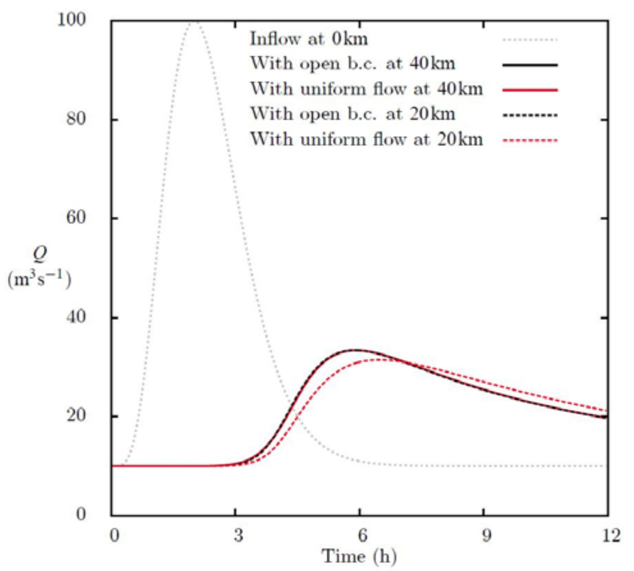

Figure 3 Computed hydrographs at 20 km with two different boundary conditions applied at each of two different places, 40 km (computational region twice the actual), and 20 km (boundary condition applied at the outflow point of interest)

We performed other calculations. For steeper streams, with less diffusion, the conventional uniform flow boundary condition did perform better than in the example. However the example shows that the uniform flow boundary condition must be viewed with some doubt. The ability just to deal simply and rationally with the end point, within the river reach of computational interest, and not to have to arbitrarily add on a uniform flow where uniform flow may not exist, seems to be a desirable one.

The apparent success of this approach, following the essential nature of the physical problem, *but not that dictated by the characteristic formulation* leads us to the conclusion that the characteristic formulation has failed again, in this case of providing adequate boundary conditions. We suggest that the rule stated by other authors above, of requiring one boundary condition for each entering characteristic, is misleading. Instead, a more pragmatic deduction is possible, that boundary conditions should be provided where they make physical sense.

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