ON THE ENERGY AND MOMENTUM PRINCIPLES IN HYDRAULICS

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Abstract

Expressions are obtained for discharge, momentum and energy conservation in pipes and channels that include the effects of boundary layers, secondary flows, and turbulence. The procedure is in the tradition of hydraulics, where the effects are not modelled exactly but approximately by generalised energy (Coriolis) and momentum (Boussinesq) correction factors. They are larger than the traditional definitions of those coefficients, that hydraulics teaching and practice has tended to ignore. It is suggested that they should be included in both the teaching and practice of pipe and channel hydraulics:

(a) They are a better model of the physical situation and there can be a gain in accuracy of calculations of up to 5-10%;

(b) For pedagogical purposes they reveal the real nature of the complexity of hydraulic flows while approximating them simply;

(c) As the numerical values of the correction factors are only approximately known, including them reminds that while one is improving accuracy, the calculations are approximate; and

(d) They give a means of testing the sensitivity of solutions to uncertainty in their values.

The momentum equation is presented in a generalised alternative form. It is suggested that most energy calculations in hydraulics actually use an integrated energy equation rather than Bernoulli's equation, even if they do not acknowledge that. It would be better and simpler to introduce energy conservation to students using the integral form, then as shown here, it can be used to obtain Bernoulli's equation for the cases where it is applicable, along a single streamline.

Several problems of elementary hydraulics are solved using the energy or momentum correction factors, to show how they help accuracy and understanding in teaching. Their fundamental nature is demonstrated by an improved theory for the Venturi flow meter.

Keywords: Energy; Momentum; Coriolis coefficient; Boussinesq coefficient; Pipes; Channels

1. INTRODUCTION

In the application of the energy and momentum principles in pipes and channels, standard hydraulic practice is to assume that the velocity is constant across a section and to ignore the fact that there are boundary layers on all solid surfaces. This seems unnecessary and unsatisfactory. A simple and approximate method of simulating the real flow is to introduce energy (Coriolis) and momentum (Boussinesq) correction factors as fundamental to the process of describing real fluid flow. They are usually presented in textbooks, however, as a brief afterthought rather than as fundamental, and thereafter are almost completely ignored. To quote Shakespeare's *Hamlet*, they are "… a custom more honoured in the breach than the observance".

Most presentations of the energy principle in hydraulics use Bernoulli's equation, valid along a streamline. In general the Bernoulli "constant" varies across streamlines, a point that is not always emphasised in lectures, books, or understood by students; and, for real fluids it varies along the streamline, but by different amounts for different streamlines. It would be more satisfactory to use the energy principle in hydraulics in an integrated sense. Many textbooks do this, but usually as an afterthought. Similarly, the momentum principle has been treated inadequately, although it has always been presented as an integral principle. Although the *Coriolis and Boussinesq coefficients* or *energy and momentum correction factors* are mentioned in textbooks, also as an afterthought, they are rarely applied. This neglect of the proper consideration of energy and momentum in practical flow problems means that there can be up to a 5-10% error even in simple flow calculations ("... a siphon consists of a pipe from a reservoir ... What is the flow in the siphon?"). It would seem to be important to use the integral form of the conservation of energy and momentum principles both in introducing the subject to students and for hydraulic practice.

Some of the more scholarly books on the subject include Jaeger (1956, Chapter II, §7 & 9) who incorporated the effects of non-uniformity of velocity distribution using Coriolis and Boussinesq coefficients in channel flow expressions. Montes (1998, Chapter 2) included both in his fundamental equations for open channel flows; gave numerical values from experiments including the effects on converging and diverging flows; and included the coefficients in all the equations involved with energy and momentum considerations. These seem to be the most satisfactory presentations, and it is surprising that other textbooks of a more fluid mechanical nature give less satisfactory treatments.

Streeter, Wylie, & Bedford (1998, Chapter 3), usually authoritative, gave a brief definition of the coefficients and some examples of their magnitudes, but did not examine the effects of the coefficients, and except for some questions about evaluating the coefficients for specified velocity distributions, they were not incorporated in any examples or problems. White (2003), also usually authoritative, gave a very similar presentation, but did include two non-trivial examples. However, in nearly 200 exercise problems set on mass, momentum, angular momentum, and energy, there were none incorporating the correction coefficients. There was no mention of the coefficients in the open channel section of either book.

With the exception of Montes (1998, p23), those presentations used the conventional definitions of the correction coefficients, ignoring the effects of secondary flows and of turbulence. Here, an integral approach is adopted to study conservation of mass, momentum, and energy. Hydraulic flows are approximated in terms of generalised Coriolis and Boussinesq correction factors, and the importance of the coefficients is explored.

2. GENERALISED INTEGRAL PRESENTATION

Consider the generalised Reynolds Transport Theorem, the conservation equation in integral form written for a stationary control volume (CV) bounded by a control surface (CS), for mass, momentum, and energy, here written as one matrix equation, taken from three separate equations given in White (2003, §3.2) and Streeter *et al.* (1998, §3):

$$\underbrace{\frac{\partial}{\partial t}\int_{CV}\overline{\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \rho e\end{bmatrix}dV}}_{\text{A. Rate of change inside CV}} + \underbrace{\int_{CS}\overline{\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \rho e+p\end{bmatrix}\mathbf{u}.\hat{\mathbf{n}}\,dS}}_{\text{B. Flux across CS}} = \underbrace{-\int_{CS}\overline{\begin{bmatrix}0\\ p\\ 0\end{bmatrix}\hat{\mathbf{n}}\,dS}}_{\text{C. Force on boundary}} + \underbrace{\overline{\begin{bmatrix}0\\ \mathbf{F}+\mathbf{T}\\ -\dot{E}\end{bmatrix}}}_{\text{D. Body forces, friction \& losses}}, \quad (1)$$

where ρ is the fluid density, $\mathbf{u}$ is the fluid velocity vector with components (u,v,w), e is the internal energy per unit mass $e = gz + (u^2+v^2+w^2)/2$, in which g is gravitational acceleration, z is the vertical co-ordinate, directed upwards, $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards from the control surface at any point such that $\mathbf{u}.\hat{\mathbf{n}}$ is the outwards component of velocity normal to the surface, dS is the elemental area of the control surface, dV is an element of volume, t is time, $\mathbf{F}$ is the force exerted on the fluid in the control volume by body forces such as gravity, $\mathbf{T}$ is the force exerted by friction, and $-\dot{E}$ is the rate of energy dissipation. In the energy equation mechanical energy is converted to heat because of turbulence and viscous decay, however it is irretrievably lost as far as the water motion is concerned and is not considered. The bars over the terms denote a time average, so that the effects of turbulent fluctuations can be included.

3. STEADY INCOMPRESSIBLE FLOW FOR AN ARBITRARY CONTROL VOLUME

3.1 TERM A

If the motion is steady, $\partial/\partial t = 0$, Term A in equation (1) is zero. This more general case will be considered in a later paper.

3.2 TERM B

This is the contribution due to transport across the control surface. On solid boundaries the normal component of velocity $\mathbf{u}.\hat{\mathbf{n}} = 0$, there is no contribution to transport, and the integral has contributions only over parts of the boundary through which fluid flows. Usually there are a discrete number of such flow boundaries, and the control surface is chosen such that on each, the local surface element is planar and the main component of velocity crosses it perpendicularly. Let this dominant velocity component be $u\hat{\mathbf{n}}$ and let the other components $v\mathbf{j}$ and $w\mathbf{k}$, composed only of turbulent fluctuations, be in the plane of each local flow boundary. Then $\mathbf{u} = u\hat{\mathbf{n}} + v\mathbf{j} + w\mathbf{k}$, such that $\mathbf{u}.\hat{\mathbf{n}} = u,$ where u can be positive or negative and can vary over that part of the control surface. As incompressible flow is considered, ρ can be taken outside all integrals, giving

$$\text{Term B} = \rho \sum_j \overline{\int_{A_j} \begin{bmatrix} 1 \\ u\hat{\mathbf{n}} + v\mathbf{j} + w\mathbf{k} \\ p/\rho + gz + \left(u^2 + v^2 + w^2\right)/2 \end{bmatrix} u\, dA}, \qquad (2)$$

where the summation in j is over all flow boundaries. Now each contribution is considered.

Mass/volume flux: For a particular boundary j, the contribution using the first element in equation (2) is $\rho\overline{\int_{A_j} u\, dA}$, including the mean of the integral of the normal component of velocity over the area A_j, giving Q_j, the time-mean discharge across face j. The contribution will be positive if fluid leaves the control volume and negative if it enters. As the top row in all other terms A, C, and D in equation (1) is zero for steady flow, this is the only contribution, and the mass conservation equation for incompressible flow becomes, after dividing through by ρ:

$$\sum_j Q_j = 0, \qquad (3)$$

stating that no fluid accumulates inside the control volume.

Momentum flux: Considering the second element of the array in equation (2), it is assumed that the boundary does not fluctuate, such that the time mean over the integral can be replaced by the integral of the time mean of the integrand. For each flow boundary j the contribution is then $\rho\int_{A_j} \overline{\left(u\hat{\mathbf{n}} + v\mathbf{j} + w\mathbf{k}\right)u}\, dA$. The velocity components are written as $u = \overline{u} + u'$, $v = v'$ and $w = w'$, where $\overline{u}$ is the time mean velocity, the time means of the contributions in the plane have been assumed to be zero, and $\left(u', v', w'\right)$ are the fluctuating departures from the mean whose mean values are zero, $\overline{u'} = \overline{v'} = \overline{w'} = 0$. Substituting into the integral gives

$$\rho\int_{A_j} \overline{\left(u\hat{\mathbf{n}} + v\mathbf{j} + w\mathbf{k}\right)u}\, dA = \rho\int_{A_j} \left(\left(\overline{u}^2 + \overline{u'^2}\right)\hat{\mathbf{n}} + \overline{u'v'}\,\mathbf{j} + \overline{u'w'}\,\mathbf{k}\right) dA.$$

The Reynolds stress contributions $\overline{u'v'}$ and $\overline{u'w'}$ in the $\mathbf{j}$ and $\mathbf{k}$ directions are difficult to obtain without solving for the flow field, however they are likely to be small and will be ignored. This leaves the contribution to the momentum flux as being an integral of the square of the mean velocity plus the mean of the square of the fluctuations. Following Boussinesq the integral is written here in terms of a momentum coefficient such that over a plane element of control surface of area A, $\overline{\beta}$ is defined as

$$\overline{\beta} = \frac{1}{U^2 A}\int_A \left(\overline{u}^2 + \overline{u'^2}\right) dA, \tag{4}$$

where $U = Q/A$ is the time and area mean velocity over the section. The traditional Boussinesq momentum correction coefficient β does not contain the $\overline{u'^2}$ term. The addition of this "buffeting" effect of turbulence seems to be necessary theoretically, and may lead to some revision of results. The contribution to momentum flux over a single planar flow boundary is now approximated in terms of this coefficient by

$$\rho\int_A \left(\overline{u}^2 + \overline{u'^2}\right) \hat{\mathbf{n}}\, dA = \rho\overline{\beta}U^2 A\hat{\mathbf{n}} = \rho\overline{\beta}\frac{Q^2}{A}\hat{\mathbf{n}}.$$

The total momentum flux across all such flow boundaries is, summing over all boundaries j:

$$\text{Net momentum flux across control surface} = \rho\sum_j \overline{\beta}_j \frac{Q_j^2}{A_j}\hat{\mathbf{n}}_j. \tag{5}$$

It is interesting that the contribution to momentum flux is always in the direction of the outwardly directed normal, whether flow is entering or leaving, and this makes expressions simpler to set up and evaluate, particularly for teaching purposes.

Energy flux: The third element of the vector in equation (2) gives, over a flow boundary j:

$$\text{Contribution to term B} = \rho\int_{A_j} \left(\frac{p}{\rho} + gz + \frac{1}{2}\left(u^2 + v^2 + w^2\right)\right) u\, dA, \tag{6}$$

the total energy flow rate crossing that section. To evaluate the first two terms in this expression it can be noted that at sections where the fluid enters or leaves the control volume the flow is usually parallel. Hence the pressure distribution, whether in pipes or open channels, is usually very close to hydrostatic so that $p + \rho gz$, the *piezometric head*, is constant over a section through which flow passes. These first two terms of the integral can be taken outside the integral sign, and the result $\int_{A_j} \overline{u}\, dA = Q_j$ used, and so the integral of the first two terms is closely approximated by $\rho\left(p/\rho + gz\right)_j Q_j$. The remaining kinetic energy flux term in equation (6) is, dropping the subscript j for generality here, $\frac{\rho}{2}\int_A \left(u^2 + v^2 + w^2\right) u\, dA$, the details of which are rarely known. As with the momentum coefficient $\overline{\beta}$ the term in u dominates all the contributions, and u is of the magnitude of $U = Q/A$. The kinetic energy coefficient $\overline{\alpha}_1$ is introduced such that

$$\overline{\int_A \left(u^2 + v^2 + w^2\right) u\, dA} = \overline{\alpha}_1 U^3 A = \overline{\alpha}_1 \frac{Q^3}{A^2}, \tag{7}$$

and in practice the time mean over the integral can be replaced by the integral of the time mean. The reason for adopting the subscript 1 is because in the integrand, the square of the speed $V^2 = u^2 + v^2 + w^2$ is weighted with the velocity component u^1. In unsteady applications, which are not treated in this paper, the energy contribution to Term A in equation (1) gives an integral $\int_A \overline{\left(u^2+v^2+w^2\right)}dA$ for which a similar coefficient $\overline{\alpha}_0$ would be introduced, reflecting that in this case the term multiplying the kinetic energy contribution is u^0.

To estimate the effects of secondary currents and turbulence the velocity components can be written in terms of mean and fluctuating components $u=\overline{u}+u'$, where $\overline{u}$ is the time mean velocity and u' is the instantaneous fluctuating part of the velocity, with similar terms for the other velocity components. Recognising that the streamwise velocity components are rather larger than the secondary flow components, the time mean value of the integrand for $\overline{\alpha}_1$ becomes approximately

$$\overline{\left(u^2+v^2+w^2\right)u} \approx \left(\overline{u}^2+\overline{w}^2+\overline{v}^2+3\overline{u'^2}\right)\overline{u} + \text{terms quadratic in secondary velocity fluctuations.}$$

Some of the omitted Reynolds stress terms $\overline{u'v'}$ and $\overline{u'w'}$ might be negative, but the dominant terms are all positive and the factor of 3 means that the contribution will be more significant. It is strange that in calculating the transport of kinetic energy in this manner that previous work has neglected all but the first term and has written the Coriolis coefficient α as

$$\alpha = \frac{\int_A u^3 dA}{U^3 A}. \tag{8}$$

A summary of the coefficients is given in Table 1. Yen (1973) has presented such coefficients in his study of the equations for open channels.

Table 1. Traditional and generalised definitions of Coriolis and Boussinesq coefficients

$\alpha = \frac{1}{U^3 A}\int_A u^3\, dA$	Traditional Coriolis coefficient, equation (8)
$\overline{\alpha}_0 = \frac{1}{U^2 A}\int_A \overline{\left(u^2+v^2+w^2\right)}dA$	Generalised Coriolis coefficient for calculating mean kinetic energy over a section; not used in this paper – see note after equation (7)
$\overline{\alpha}_1 = \frac{1}{U^3 A}\int_A \overline{\left(u^2+v^2+w^2\right)u\,dA}$	Generalised Coriolis coefficient for calculating mean *transport* of kinetic energy over a section, using all velocity components and a time average for turbulent contributions, equation (7)
$\beta = \frac{1}{U^2 A}\int_A u^2\, dA$	Traditional Boussinesq coefficient
$\overline{\beta} = \frac{1}{U^2 A}\int_A \overline{u^2\, dA} = \frac{1}{U^2 A}\int_A \left(\overline{u}^2+\overline{u'^2}\right)dA$	Generalised Boussinesq coefficient with allowance for turbulence, equation (4)

Equation (6) can be written as

$$\text{Contribution to term B} = \rho\left(\frac{p}{\rho}+gz+\frac{\overline{\alpha}_1}{2}\frac{Q^2}{A^2}\right)_j Q_j = \rho g H_j Q_j, \tag{9}$$

in which the *Mean Total Head* H across a section is defined in terms of the kinetic energy coefficient $\overline{\alpha}_1$:

$$H = \frac{p}{\rho g} + z + \frac{\overline{\alpha}_1}{2g}\frac{Q^2}{A^2}. \tag{10}$$

Combining mass, momentum and energy contributions, equations (3), (5), and (9), Term B can be written as sums of mean quantities over the flow boundaries:

$$\text{Term B} = \rho \begin{bmatrix} \sum_j Q_j \\ \sum_j \overline{\beta}_j \frac{Q_j^2}{A_j} \hat{\mathbf{n}}_j \\ g\sum_j (QH)_j \end{bmatrix}. \tag{11}$$

3.3 TERM C

The only contribution in Term C is to the momentum equation, $-\int_{\mathrm{CS}} p\,\hat{\mathbf{n}}\,dS$, which has two contributions. One is over solid surfaces on which, unless all details of the flow field are known, the pressure p is not known. However, the sum of all those contributions is the total force $-\mathbf{P}$ of the fluid on the surrounding structure. The other contribution to the integral is over the flow boundaries. If such a boundary A_j is planar, then $\hat{\mathbf{n}}$ is a constant, and it can be taken out of each of the integrals, leaving $\int_{A_j} p\,dA$ that will be written in shorthand as $\left(\overline{p}A\right)_j$, where $\overline{p}$ is the mean pressure over the section. As p usually varies linearly across such surfaces, this can usually be evaluated, such as for a channel section, or a simple value assumed, such as the value at the centre of a pipe. The result is

$$\text{Term C, momentum contribution} : -\sum_j \left(\overline{p}A\,\hat{\mathbf{n}}\right)_j - \mathbf{P}. \tag{12}$$

3.4 TERM D

The last term in equation (1) is the array $\begin{bmatrix} 0 & \mathbf{F}+\mathbf{T} & -\dot{E} \end{bmatrix}^{\mathrm{T}}$, where $\mathbf{F}$ is the force exerted on the fluid in the control volume by body forces such as gravity, and friction forces, $\mathbf{T}$ is the force exerted by friction forces, and $-\dot{E}$ is the rate of energy dissipation. For an arbitrary-shaped control surface, little can be said about the friction forces and losses. However, for elongated shapes such as pipes and channels, empirical formulae exist and these important cases will be considered below. The body force is

$$\mathbf{F}_{\text{body}} = \rho \mathbf{g} V, \tag{13}$$

where $\mathbf{g}$ is the body force vector per unit mass, which will usually have only one component, that of gravitational acceleration, and V is the volume enclosed by the control surface.

3.5 COLLECTING TERMS

Taking all the contributions from A, B, C, and D from equations (11), (12), and (13), substituting into equation (1), grouping terms, and writing each component of the 3-vectors as a separate equation, after dividing through by density ρ where that is useful, gives the

Integrals for steady flow through a control volume:

Mass: $\sum_j Q_j = 0$, (Continuity equation) (14a)

Momentum: $\sum_j \left(\overline{p}A + \rho\overline{\beta}\frac{Q^2}{A} \right)_j \hat{\mathbf{n}}_j + \mathbf{P} = \rho \mathbf{g} V$, (14b)

Energy: $\sum_j H_j Q_j = \frac{-\dot{E}}{\rho g}$, where $H = \frac{p}{\rho g} + z + \frac{\overline{\alpha}_1}{2g}\frac{Q^2}{A^2}$ (14c)

The summations are over all parts of the control surface through which fluid passes, with the discharge being taken as positive/negative whether the fluid leaves/enters the control volume respectively. Usually the loss term $-\dot{E}$ is assumed zero for arbitrary control volumes, but it might be possible to approximate this based on the nature of the volume. It should be noted that the energy equation is derived from integral expressions, hence the presence of the coefficient $\overline{\alpha}_1$ in the definition of mean total head. It is not Bernoulli's equation but in simple applications it gives a similar result. In the definition of H it has not been necessary to take p and z at any specific point; as the combination of the two terms $p/\rho g + z$ is close to constant anywhere on a particular surface element.

4. APPLICATION OF THE ENERGY PRINCIPLE TO PIPES AND CHANNELS

A specific type of control surface is now considered – a single element such as a length of pipe or channel, with no other inflow or outflow other than the ends. Following usual hydraulic practice, the energy loss is represented in terms of a head loss, ΔH, such that $\dot{E} = \rho Q g \Delta H$. Equation (14c) yields

$$\left(\frac{p}{\rho g} + z + \overline{\alpha}_1 \frac{1}{2g}\frac{Q^2}{A^2} \right)_{\text{in}} Q - \left(\frac{p}{\rho g} + z + \overline{\alpha}_1 \frac{1}{2g}\frac{Q^2}{A^2} \right)_{\text{out}} Q - \Delta H\, Q = 0,$$

and as Q is a constant factor, it is possible to divide through and re-arrange:

$$\left(\frac{p}{\rho g} + z + \overline{\alpha}_1 \frac{1}{2g}\frac{Q^2}{A^2} \right)_{\text{out}} = \left(\frac{p}{\rho g} + z + \overline{\alpha}_1 \frac{1}{2g}\frac{Q^2}{A^2} \right)_{\text{in}} - \Delta H, \quad (15)$$

almost exactly as written in conventional applications of Bernoulli's equation, but with the addition of the $\overline{\alpha}_1$ coefficients as this has been obtained in an integral sense. To calculate the friction losses ΔH_{f}, Weisbach's equation can be used, expressed generally here as

$$\Delta H_{\text{f}} = \overline{\lambda} P \frac{Q^2 L}{8gA^3},$$

where L is the length of pipe or channel, and allowing for variable roughness around a channel boundary, such that $\overline{\lambda} P = \int_P \lambda\, dP$ where λ is the Darcy dimensionless friction coefficient and P is the length of solid perimeter around the cross-section. For a circular pipe of constant roughness, the familiar result is $\overline{\lambda} P / A = 4\lambda / D$, where D is diameter.

Local losses with coefficients K_i can be included such that each head loss $\Delta H_i = K_i U^2/2g$, giving the overall result

$$\Delta H = \Delta H_{\mathrm{f}} + \sum_i \Delta H_i = \frac{Q^2}{2gA^2}\left(\bar{\lambda} P \frac{L}{4A} + \sum_i K_i\right). \tag{16}$$

5. BERNOULLI'S EQUATION FOR STEADY FLOW

Most elementary courses in hydraulics concentrate on Bernoulli's equation that is valid along a streamline, but apply it to situations where the concept of a streamline seems tenuous and where the integral form of the energy equation would be more valid. A simple example is, for example, between the surface of a water supply reservoir and the flow from a garden hose. However, there are some problems where it is useful to use Bernoulli's equation along a single streamline, such as in the theory of the Pitot tube, or calculating the velocity of the surface of a flow by putting a finger in it and measuring how high the water rises.

Bernoulli's equation is usually derived in a rather difficult manner in textbooks. For example Streeter *et al.* (2003, §4.5) integrate Euler's equation of motion along a streamline, while White (2003, §3.7) considers a streamtube and rather difficult linear momentum considerations. It is simpler to use some of the earlier results here. Consider a control volume that is a length of streamtube, which by definition has no fluid crossing its sides, so that equation (15) applies. For no losses, $\Delta H = 0$, and then if the limiting case of a streamtube of zero cross-section is considered, the velocity distribution is constant over the infinitesimal section, such that $\bar{\alpha}_1 = 1$, giving

Bernoulli's equation:

$$\frac{p}{\rho g} + z + \frac{1}{2g}\frac{Q^2}{A^2} = \text{Constant along a streamline}\,. \tag{17}$$

6. VALUES OF MOMENTUM AND ENERGY CORRECTION FACTORS

Effects of boundary layers and primary velocity distribution: For a 1/7 power velocity law, $\alpha = 1.045$ (Montes, 1998, p27), and $\beta = 1.016$. For a turbulent logarithmic velocity distribution for moderately rough pipes and channels, values of α from $1.05-1.1$ are typical, while β varies from about 1.015 to 1.04. Montes quoted laboratory measurements of α over a smooth concrete bed giving 1.035-1.064. For earth channels, larger values were found, such as 1.25 for irrigation canals in southern Chile and 1.35 in the Rhine River. These results were all for open channels. For circular pipes White (2003) gave values for different power laws, with 1.058 for a 1/7 law velocity distribution and 1.106 for a 1/5 law.

Effects of secondary flow: The contributions from v^2 and w^2 to $\bar{\alpha}_1$ are usually smaller than the effects of the primary velocity distribution, while they do not affect $\bar{\beta}$. If a secondary flow velocity were 10% of the streamwise velocity its contribution to the magnitude of $\bar{\alpha}_1$ would be 0.01 and if 20%, the contribution would be 0.04, larger values occurring downstream of a pipe bend or in a meandering river.

Effects of turbulence: Montes (1998, p23) quoted experiments on the effects of turbulence, with the contribution to α being 0.013 for smooth channels and 0.025 for rough channels at

5% of the depth above the bottom. They decreased rapidly with distance from the wall, so at 50% of the depth they were about half. Higher levels exist inside hydraulic jumps.

7. EXAMPLE SOLUTIONS FOR COMMON PROBLEMS

Some elementary hydraulics problems will now be considered. The solution of each is enhanced by including momentum or energy coefficients where appropriate. Routine inclusion of the coefficients in the derivations, such as above, and in problem solving as exemplified below, might be considered in the future for undergraduate teaching.

7.1 THE FORCE DUE TO A JET OF WATER

Finding the force of a jet is a common early problem in the application of the momentum principle. Consider a horizontal jet of water in air travelling in the $+x$ direction with discharge Q and cross-sectional area A, that strikes a vertical plate, after which the fluid is deflected and flows parallel to the plate. Consider a control surface along the sides of the jet, with a left vertical boundary through which the jet enters, on which $\hat{\mathbf{n}} = -\mathbf{i}$, where $\mathbf{i}$ is a unit vector in the $+x$ direction, and a right boundary that of the plate through which no fluid passes. The integral momentum equation (14b) written for the horizontal x direction gives, where P is the magnitude of the force of the jet on the plate,

$$\rho\bar{\beta}\frac{Q^2}{A}(-\mathbf{i}) + P\mathbf{i} = 0\mathbf{i}, \text{ hence } P = \rho\bar{\beta}\frac{Q^2}{A}.$$

If the fluid is deflected back in the direction from whence it came, it is necessary to invoke Bernoulli's equation and mass conservation to establish that the cross-sectional area is still A. The fluid now leaves the control volume travelling in the $-x$ direction but where $\hat{\mathbf{n}} = -\mathbf{i}$ again, giving the well-known result that the force is twice that of the first problem.

For pedagogical purposes, (a) the coefficient $\bar{\beta}$ appears with a simple physical significance; (b) it modifies the results by up to 5%, or at least establishes that the results are not exact; and (c) the expression of the momentum integral equation (14b) in terms of the outwardly directed unit normal vectors seems simple to apply. Textbooks use an expression like $\rho Q(\mathbf{u}_{\text{out}} - \mathbf{u}_{\text{in}})$, that has a plausible simple significance as the mass rate of flow times the change in the velocity vector, however it too should be modified by a momentum coefficient $\bar{\beta}$.

7.2 THE SIPHON

One of the first examples given to students in the application of energy principles, in the form of Bernoulli's equation, is that of a simple siphon, consisting of a pipe immersed in a pool, with the other end discharging at some point below, with no losses. If the reservoir is at a height h above the open end of the pipe, then the energy equation (15) quickly gives

$$Q = \frac{A\sqrt{2gh}}{\sqrt{\bar{\alpha}_1}}.$$

Bernoulli's equation would use $\bar{\alpha}_1 = 1$, but the result here is that $Q \sim \bar{\alpha}_1^{-1/2}$, so that if $\bar{\alpha}_1$ were 1.1, the calculated discharge would be reduced by something like 5%.

At the next level of sophistication in solving this problem, friction and local losses might be included. Equations (15) and (16) give, where the pipe is of length L and diameter D,

$$Q = \frac{A\sqrt{2gh}}{\sqrt{\bar{\alpha}_1 + \sum_i K_i + \lambda L/D}}.$$

Clearly $\bar{\alpha}_1$ (≈ 1.1) acts in the same manner as a local loss, for which common values are 0.5 or 1, or more for valves. There may be few or many of these, and the friction loss might be small or large so that the effect of $\bar{\alpha}_1$ might be large or small, but this recognises its role.

Often in such problems, the case is considered where the pipe passes over a hill with higher elevation $z_{\max}$, say, where pressure will be $p_{\min}$, so that the possibility of cavitation can be explored. Having calculated Q as above, the pressure at the crest of the hill becomes

$$p_{\min} = \rho g \left(h - z_{\max} - \frac{1}{2g}\left(\frac{Q}{A}\right)^2 \left(\bar{\alpha}_1 + \sum_{\text{inlet}}^{\text{hill}} K_i \right) \right),$$

and the effect of $\bar{\alpha}_1$ is to make cavitation more likely if vapour pressure is approached.

7.3 THE VENTURI METER

Consider a Venturi constriction in a pipe, where there are pressure-tapping points in the side of the pipe at section 1 before the constriction and section 2 at the side of the constriction. Using a control surface along the pipe and that crosses it at sections at 1 and 2, the integral energy equation (15) is more conveniently written using the *piezometric* head $h = p/\rho g + z$, conveniently measured by a manometer and independent of the orientation of the pipe, so that for no friction loss between the two points:

$$h_1 + \frac{\bar{\alpha}_1}{2g}\frac{Q^2}{A_1^2} = h_2 + \frac{\bar{\alpha}_1}{2g}\frac{Q^2}{A_2^2},$$

so that $$Q = \frac{1}{\sqrt{\bar{\alpha}_1}} A_1 A_2 \sqrt{\frac{2g(h_1 - h_2)}{A_1^2 - A_2^2}}. \quad (18)$$

Usual practice (Streeter *et al.* 1998, §10.7), is to write the equation for the Venturi meter as

$$Q = C A_1 A_2 \sqrt{\frac{2g(h_1 - h_2)}{A_1^2 - A_2^2}},$$

where $C \approx 0.97 - 0.99$ is termed a "velocity coefficient" due to "losses in the energy equation". However comparison with equation (18) shows that $C = 1/\sqrt{\bar{\alpha}_1}$ and so those measured values of C could be theoretically explained by values of $\bar{\alpha}_1 = 1.02 - 1.06$, without having to invoke energy losses in the short section between tapping points. This simple example shows the utility of using the integral form rather than Bernoulli's equation.

8. CONCLUSIONS

For flows in pipes and channels, it is preferable to use an integral form of the conservation of energy principle, rather than Bernoulli's theorem that is valid only along a streamline. Equations for integral momentum and energy principles have been formulated. Traditional Coriolis and Boussinesq coefficients have been found to be defective, as they neglect the effects of turbulence and secondary currents. Generalised values of the correction factors have been obtained for which typical values are $1.05-1.15$, and it is asserted that these factors and the integral forms of the momentum and energy principles should be included both in the teaching of hydraulics and in most applications in hydraulic practice.

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