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The application of numerical methods and mathematics to hydrography

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Abstract

Traditional simple formulae and methods for the everyday practice of flow measurement have been believed to work well for many years. However practitioners have occasionally expressed doubt about those formulae as well as the desire to have more accurate methods, should they be necessary. This paper presents some applications of mathematical and computational methods to the practice of flow measurement, resulting in more-accurate and possibly simpler hydrographic procedures. Also, existing procedures for ultrasonic velocimetry are criticised and the more-accurate methods are recommended for that too. Finally a correction method is presented for the effects of rising or falling stage on rating curves.

1. Introduction

Traditional formulae and methods for the measurement of streamflow are simple, and given the complexity of the problem they are solving, are surprisingly accurate. One of the most common problems is to find the mean horizontal velocity on a vertical line in a stream flow. This is usually done with only two measurements, giving acceptable results. This is quite remarkable, given how rapidly the velocity varies over the vertical. The streamflows so calculated are accurate enough for many practical purposes. However, such methods might be used to calibrate rather more sophisticated measuring equipment, when greater accuracy would be desirable. Also, in the irrigation industry generally better accuracy might be necessary. In addition, the traditional formulae make no rational allowance for when the velocity profile bends back or forward, which is often found to be the case. It would seem that, given that computing equipment is generally available and used, that rather more sophisticated methods for the analysis of flow data might be implemented. In this, the Australian Standard 3778 *Measurement of water flow in open channels* is not particularly helpful.

This paper addresses some traditional problems of flow measurement and proposes some formulae and methods which are more general and accurate and which might simplify the measurement of discharge in streams. Some defects and inaccuracies of traditional formulae are noted. It is then shown that in Australian and International Standards for ultrasonic velocimetry the method for calculating the mean velocity on a beam path is wrong, even though the result is right. A correct derivation is presented. Then the problem of vertical integration of beam data for the discharge is considered. Conventional practice is asserted to be quite defective, and it is suggested that flow results from ultrasonic meters are not as accurate as they might be. Some suggestions are made for applying the abovementioned methods developed in this work. Finally methods for correcting points on rating curves for rising and falling stage are presented.

2. Measurement of discharge by the velocity-area method

The velocity-area method is widely used to calculate the discharge in streams. It requires integrating the velocity over the cross-sectional area A, $Q = \int_A u\, dA$, where Q is the discharge and u is the velocity. This can be expressed as a double integral

$$Q = \int_B \int_0^{h(y)} u\, dz\, dy. \tag{2.1}$$

The velocity is integrated from the bed $z = 0$ to the surface $z = h(y)$, where h is the local depth, z is a local vertical co-ordinate based on the bed and y is the co-ordinate across the waterway, then these contributions are integrated across the channel, for values of the transverse co-ordinate y over the breadth B.

The first step is to compute the integral of velocity with depth, or as it is often expressed, the mean velocity over the depth. An example of a common formula in hydrography is where the mean velocity over a vertical is approximated by the two-point formula

$$\bar{u} = \tfrac{1}{2}\left(u_{0.2h} + u_{0.8h}\right), \tag{2.2}$$

that is, the mean of the readings at 0.2 of the depth and 0.8 of the depth. Here we produce some theory to examine the accuracy of this equation, and to propose rather more general formulae which should be more accurate.

3. Calculation of mean velocity on a vertical

3.1 A general two-point formula

Consider the law for turbulent flow over a rough bed, which can be obtained from the expressions on p582 of Schlichting (1968):

$$u = \frac{u_*}{\kappa} \ln \frac{z}{z_0}, \tag{3.1}$$

where u_* is the shear velocity, $\kappa = 0.4$, $\ln()$ is the natural logarithm to the base e, z is the elevation above the bed, and z_0 is the elevation at which the velocity is zero. (It is a mathematical artifact that below this point the velocity is actually negative and indeed infinite when $z = 0$ – this does not usually matter in practice). If we integrate equation (3.1) over the depth h we obtain the expression for the mean velocity:

$$\bar{u} = \frac{1}{h} \int_0^h u\, dz = \frac{u_*}{\kappa}\left(\ln \frac{h}{z_0} - 1\right). \tag{3.2}$$

Now it is assumed that two velocity readings are made, obtaining u_1 at z_1 and u_2 at z_2. This gives enough information to obtain the two quantities u_*/κ and z_0. Substituting the values for point 1 into equation (3.1) gives us one equation and the values for point 2 gives us another equation. Both can be solved to give the solution

$$\frac{u_*}{\kappa} = \frac{u_2 - u_1}{\ln(z_2/z_1)} \quad \text{and} \quad z_0 = \left(\frac{z_1^{u_2}}{z_2^{u_1}}\right)^{\frac{1}{u_2 - u_1}}. \tag{3.3}$$

It is not necessary to evaluate these, for substituting into equation (3.2) gives a simple formula for the mean velocity in terms of the readings at the two points:

$$\bar{u} = \frac{u_1\left(\ln(z_2/h)+1\right) - u_2\left(\ln(z_1/h)+1\right)}{\ln(z_2/z_1)}. \tag{3.4}$$

As it is probably more convenient to measure and record depths rather than elevations above the bottom, let $h_1 = h - z_1$ and $h_2 = h - z_2$ be the depths of the two points, when equation (3.4) becomes

$$\bar{u} = \frac{u_1 \left(\ln(1 - h_2/h) + 1\right) - u_2 \left(\ln(1 - h_1/h) + 1\right)}{\ln\left((h - h_2) / (h - h_1)\right)}. \tag{3.5}$$

This expression gives the freedom to take the velocity readings at any two points, and not necessarily at points such as $0.2h$ and $0.8h$. This might simplify streamgauging operations, for it means that the hydrographer, after measuring the depth h, does not have to calculate the values of $0.2h$ and $0.8h$ and then set the meter at those points. Instead, the meter can be set at any two points, within reason, the depth and the velocity simply recorded for each, and equation (3.5) applied. This could be done either *in situ* or later when the results are being processed. This has the potential to speed up hydrographic measurements.

If the hydrographer were to use the traditional two points, then setting $h_1 = 0.2h$ and $h_2 = 0.8h$ in equation (3.5) gives the result

$$\bar{u} = 0.4396\, u_{0.2h} + 0.5604\, u_{0.8h} \approx 0.44 u_{0.2h} + 0.56 u_{0.8h}\,, \tag{3.6}$$

whereas the conventional hydrographic expression is (see *e.g.* #7.1.5.3 of Australian Standard 3778.3.1 2001):

$$\bar{u} = 0.5\, u_{0.2h} + 0.5\, u_{0.8h}\,. \tag{3.7}$$

The nominally more accurate expression, equation (3.6), gives less weight to the upper measurement and more to the lower. It might be useful, as it is just as simple as the traditional expression, yet is based on an exact analytical integration of the equation for a turbulent boundary layer.

The author has tested this by taking a set of gauging results. A canal had a maximum depth of 2.6m and was 28m wide, and a number of verticals were used. The conventional formula (2.2), the mean of the two velocities, was accurate to within 2% of equation (3.6) over the whole range of the readings, with a mean difference of 1%. That error was always an overestimate. The more accurate formula (3.6) is hardly more complicated than the traditional one, and it should in general be preferred. Although the gain in accuracy is slight, in principle it is desirable to use an expression which makes no numerical approximations to that which it is purporting to evaluate. This does not necessarily mean that either (2.2) or (3.6) gives an accurate integration of the velocities which were encountered in the field. In fact, one complication is where, as often happens in practice, the velocity distribution near the surface actually bends back such that the maximum velocity is below the surface. This will be considered below.

3.2 Theoretical comparison of traditional formulae for a pure logarithmic profile

Now we compare several different expressions for the mean velocity. Some of these are set out in Boiten (2000, p82) and some in Australian Standard 3778.3.1 (2001, #7.1.5):

One-point method

$$\bar{u} = u_{0.6h}\,. \tag{3.8}$$

O'Neill's improved one-point method Dr I. C. O'Neill (personal communication) has suggested, based on a rational approach, that instead of sampling at 0.6 of the depth it is more accurate to sample at 0.625, giving

$$\bar{u} = u_{0.625h}\,. \tag{3.9}$$

Three-point method (1)

$$\bar{u} = \tfrac{1}{3}\left(u_{0.2h} + u_{0.6h} + u_{0.8h}\right)\,. \tag{3.10}$$

Three-point method (2)

$$\bar{u} = 0.25\, u_{0.2h} + 0.5\, u_{0.6h} + 0.25\, u_{0.8h}\,. \tag{3.11}$$

Four-point method

$$\bar{u} = 0.25\, \left(u_{0.2h} + u_{0.4h} + u_{0.7h} + u_{0.9h}\right)\,. \tag{3.12}$$

Analytical expressions for the errors of each of these methods were calculated. It is possible to show that they are simple functions of the relative roughness z_0/h as shown in Table 3-1, and in Figure 3-1.In the figure it can be seen that for increasing roughness, the errors increase. It is interesting that several of the traditional one, two, and three-point formulae, have the same accuracy. The four-point formula gains little in accuracy from including an extra point, while the new two-point formula seems quite accurate. This does not necessarily mean that any of these gives an accurate integration of the velocities which might be encountered in the field. If the velocity distribution is not strictly logarithmic, then these results for accuracy do not hold.

Method	Error
Traditional one-point method, equation (3.8)	$-\frac{0.084}{\log_e(z_0/h)+1}$
Traditional two-point formula, (3.7)	$-\frac{0.084}{\log_e(z_0/h)+1}$
Three-point method (1), (3.10)	$-\frac{0.084}{\log_e(z_0/h)+1}$
Three-point method (2), (3.11)	$-\frac{0.084}{\log_e(z_0/h)+1}$
Four-point method, equation (3.12)	$+\frac{0.060}{\log_e(z_0/h)+1}$
O'Neill's improved one-point method (3.9)	$-\frac{0.019}{\log_e(z_0/h)+1}$
New two-point formula, equation (3.6)	$-\frac{0.00053}{\log_e(z_0/h)+1}$

Table 3-1. Errors of various methods – the dependencies on relative roughness are shown in Figure 1.

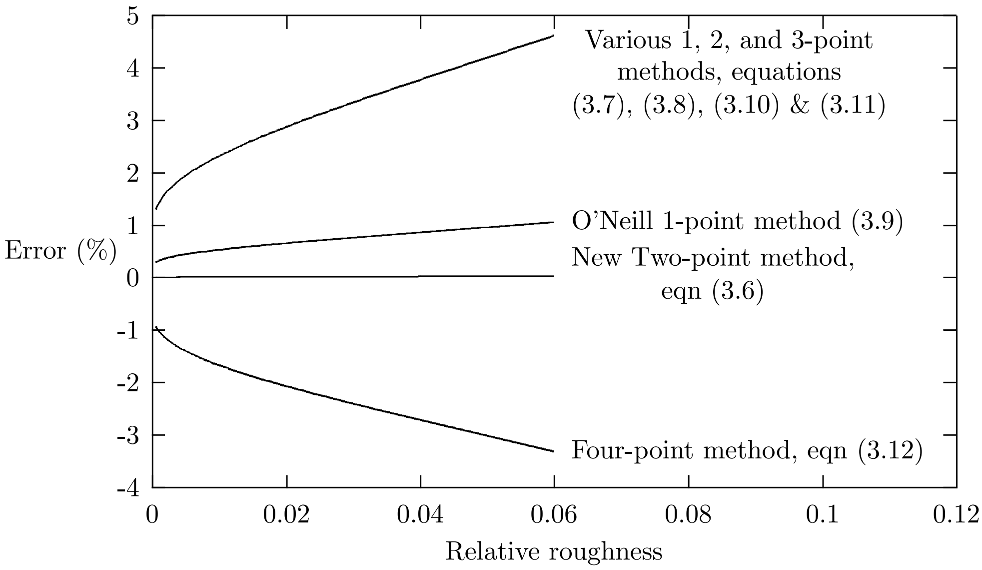

Figure 3-1. Errors of various methods and their variation with relative roughness

Now we consider a more general problem where measurements are taken at three or more points, to obtain more information about the velocity field, and hence to allow for it deviating from a precisely logarithmic distribution.

3.3 A general three-point method

If greater accuracy is required, possibly where it is clear that the velocity profile is deviating from a logarithmic form and where it may have a maximum at some point beneath the surface, more points can be taken, and probably in general, should be. The law for turbulent flow over a rough bed, equation (3.1) is generalised to allow for a more general variation of velocity, such that we write:

$$u = \frac{u_*}{\kappa} \ln \frac{z}{z_0} + a_1 \frac{z}{h}, \tag{3.13}$$

where a_1 is an as-yet unknown coefficient to be found by measurement, with units of velocity. If three velocity readings are taken at three depths h_1, h_2, and h_3, then using the corresponding three velocity measurements u_1, u_2, and u_3, equation (3.13) gives three simultaneous equations in the three unknowns u_*/κ, z_0, and a_1. It is possible to solve these equations for the unknowns u_*/κ, z_0 and a_1, and for the solutions to be written down, and then equation (3.13) can be integrated, giving

$$\bar{u} = \frac{u_*}{\kappa} \left(\ln \frac{h}{z_0} - 1 \right) + \frac{a_1}{2}.$$

The results can be presented most economically as the pseudo-code, given in Table 3-2, where it is assumed that it is the depth of the measurements h_1, h_2, and h_3 which is recorded, as well as the overall depth h. The accuracy of this method will be examined further below.

```
for i from 1 to 3 do
    Z_i = 1 − h_i/h
for i from 1 to 2
    for j from i + 1 to 3 do
        r_ij = ln(Z_i/Z_j)
        Δ_ij = Z_i − Z_j
δ = r_12 Δ_23 − r_23 Δ_12
a_1 = (−u_1 r_23 + u_2 r_13 − u_3 r_12) / δ
u_*/κ = (u_1 Δ_23 − u_2 Δ_13 + u_3 Δ_12)/δ
ln Z_0 = ln Z_1 + (a_1 Z_1 − u_1) / (u_*/κ)
ū = −u_*/κ × (ln Z_0 + 1) + a_1/2
```

Table 3-2. Procedure to obtain mean velocity from three measurements

3.4 A general four-point method

A similar procedure can be followed, assuming an additional quadratic term in the velocity profile:

$$u = \frac{u_*}{\kappa} \ln \frac{z}{z_0} + a_1 \frac{z}{h} + a_2 \left(\frac{z}{h} \right)^2, \tag{3.14}$$

and by taking readings at four depths, enough information is obtained to obtain the solution for $\bar{u}$. Once again this is best presented as pseudocode, given in Table 3-3. The accuracy of this method too will be examined further below.

```
for i from 1 to 4 do
    Z_i = 1 - h_i/h
for i from 1 to 3 do
    for j from i+1 to 4 do
        r_{i,j} = ln(Z_i/Z_j)
for i from 1 to 2 do
    b_i = u_i r_{i+1,i+2} - u_{i+1} r_{i,i+2} + u_{i+2} r_{i,i+1}
    for j from 1 to 2 do
        c_{ij} = Z_i^j r_{i+1,i+2} - Z_{i+1}^j r_{i,i+2} + Z_{i+2}^j r_{i,i+1}
δ = c_11 c_22 - c_12 c_21
a_1 = (c_22 b_1 - c_12 b_2)/δ
a_2 = (c_11 b_2 - c_21 b_1)/δ
u_*/κ = (a_1(Z_2 - Z_1) + a_2(Z_2^2 - Z_1^2) + u_1 - u_2)/r_{1,2}
ln Z_0 = ln Z_1 + (a_1 Z_1 + a_2 Z_1^2 - u_1)/(u_*/κ)
ū = -u_*/κ × (ln Z_0 + 1) + a_1/2 + a_2/3
```

Table 3-3. Procedure to obtain mean velocity from four measurements

3.5 Least-squares approximation methods

The above methods have assumed that the approximating function, whether equation (3.1), (3.13) or (3.14) actually passes through each of the data points, such that they *interpolate* the data points. In many physical situations the process of interpolation, of ensuring that some mathematical function actually passes through all data points, is unreasonable, because the points show scatter, as is usually the case in hydrography. In such cases it is more reasonable to use some form of lower-order approximation, such that the function does not pass through each of the points, but passes close to them, such that the sum of the squares of the errors is minimised. For the case of hydrographic measurements then, a function of the form

$$u = b_0 \ln z + b_1 + b_2\, z + b_3\, z^2 + \ldots, \tag{3.15}$$

can be assumed, suggested by equation (3.14), where the coefficients b_i are to be determined. This is done by computing the sum of the errors in the measured velocities U_i for each experimental measurement i:

$$\varepsilon = \sum_i \left(b_0 \ln z + b_1 + b_2\, z + b_3\, z^2 - U_i\right)^2,$$

and then finding the coefficients such that the total error ε is minimised. One possible problem in all of this is that the logarithm function actually goes to $-\infty$ at $z = 0$, but for practical hydrography this is usually not a problem. One way of overcoming this, and satisfying the boundary condition on the bed that on the bed $z = 0$ the velocity $u = 0$ is to use a modified version of the power law velocity distribution. It has been sometimes used in turbulent boundary layer research that the velocity distribution varies like z^ν, where ν is a small number, typically $1/7$. That is a result which has been used in fluid mechanics for flow over flat plates. For flows of large Reynolds number, such as in open channel flow, a value of $1/10$ has been suggested (see p565 of Schlichting, 1968). To generalise for practical hydrography the more general expression could be written:

$$u = z^\nu \left(c_0 + c_1 z + c_2 z^2 + \ldots\right), \tag{3.16}$$

and the procedure described above used to find the coefficients c_i.

Both these approaches were used to obtain some of the results in the following section which took some real velocity distribution data, and applied the various methods.

3.6 Some field measurements

	Case 1	Case 2	Case 3
The four-point logarithmic interpolation formula, equation (3.14)	0.307	0.315	0.294
Least-squares fit of a logarithmic function, (3.15) truncated after b_2 term	0.304	0.314	0.312
Least-squares fit of a power-law function, (3.16), $\nu = 1/7$, truncated after c_2 term	0.308	0.322	0.307
Traditional four-point rule, equation (3.12)	0.310	0.322	0.294
New two-point rule, equation (3.6)	0.306	0.319	0.295
Traditional two-point rule, equation (2.2)	0.303	0.324	0.304

Table 3-4. Results for mean velocity on three verticals obtained by several different formulae and methods.

Some measurements in a large irrigation canal of depth about $2.5\,\mathrm{m}$ were taken by professional hydrographers. In the middle of the canal, on each of three verticals $1\,\mathrm{m}$ apart five velocity readings were taken, at depths $0.2h$, $0.4h$, $0.7h$, $0.8h$, and $0.9h$ such that different standard formulae for mean velocity could be applied. The various methods which were applied to the results, in descending order of nominal accuracy are given in Table 3-4. These should be best interpreted in association with the results shown in Figure 3-2, which show the measured values by filled circles. Considering the first profile, Case 1, it can be seen that the first three formulae, the nominally most accurate methods, agree quite closely, to within 1% with each other, the four-point rule equation (3.12) is 1% high, the new two-point rule equation (3.6) agrees well with the accurate methods, and the traditional two-point rule equation (2.2) is 1% low. Examining the figure, however, the arbitrariness of even the nominally-accurate methods becomes obvious, for it can be seen that they do not all agree near the bed nor near the surface, even though when integrated they give results which agree quite closely, as shown in the Table. Further evidence for this is provided in Case 2, where the logarithmic interpolation and approximation results agree closely, the four-point rule equation (3.12) is some 3% high, as is the traditional two-point rule equation (2.2), while the new two-point rule equation (3.6) again agrees well with what we might believe to be accurate results.

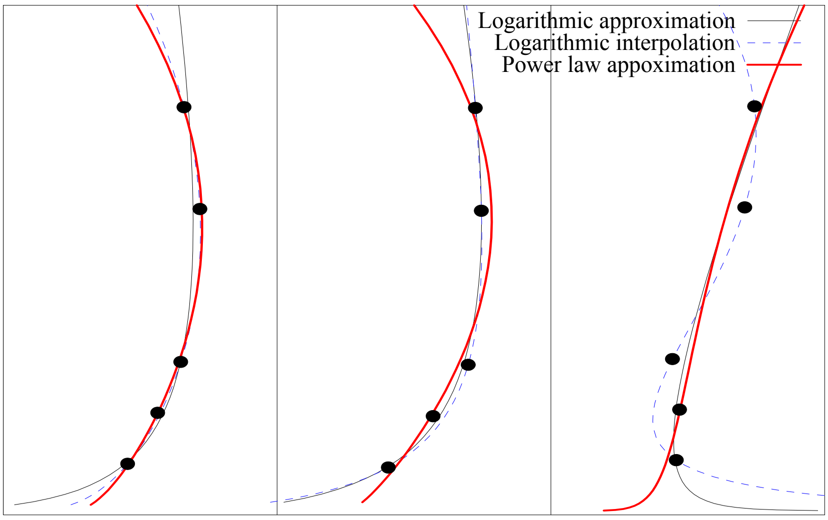

Figure 3-2. Gauging results for three verticals and the approximations used to obtain values for the mean velocities

All of this comes undone in the last Case 3, where Figure 3-2 shows that the streamgauging results near the bed are quite irregular, and this has caused problems for the logarithmic methods in particular. It can be seen that both predict a velocity of $+\infty$ at the bed rather than the $-\infty$ which the traditional approximation implies. Given the obviously incorrect distributions for the two methods, one cannot really believe either of the results from logarithmic laws for the mean velocity in the last column of the Table, but the value of $0.307\,\mathrm{m\,s^{-1}}$ obtained from the power law approximation seems plausible. In this case both the four-point rule equation (3.12) and the new two-point rule equation (3.6) agree but are lower than the rest, while the traditional two-point rule equation (2.2) is more consistent with the others. What is true is that none of the methods are particularly good for this case where the measured values show considerable irregularity.

These examples have shown that when good-quality results are obtained, consistent results for mean velocity are obtained, even though the underlying assumed velocity distributions can be noticeably different. On the other hand, if poor-quality results are obtained, we cannot be certain of any of the results for mean velocity. This seems to suggest the importance of determining the velocity profile at the extremities, near the bed, and near the surface, to reduce the arbitrariness of the results, as noted in Australian Standard 3778.3.1 (2001).

4. Global approximation methods

This suggests a couple of procedures for more accurate determination of the mean velocity. One procedure would be to take a reading close to the surface and one close to the bed, and to fill in with one or two in the middle, and then to use any of the multi-point methods described above. This might gain a few percent in accuracy, desirable in some situations, but unimportant in others. However, if only two points were measured, it would seem to be advisable to use the general formula given above as equation (3.5) rather than the conventional two-point formula.

There is another procedure which could lead to rather quicker and yet more accurate streamgauging. This would be to install some form of constant velocity lifting mechanism which would traverse the flow meter through the water column such that it samples the flow velocity at all points equally, effectively doing the integration itself. Such a procedure is already recommended in #7.1.5.4 of Australian Standard 3778.3.1 (2001), but without the emphasis on constant velocity of traverse to give an accurate result. The conventional multi-point procedure is to set the meter at each of several depths, and let it count revolutions at each depth for a finite interval. The results described above show how that can occasionally be inaccurate. The traversing procedure might be to set the meter near the bottom, start the lifting mechanism, and count the number of revolutions that the meter takes as it samples at a constant vertical rate the velocity at all depths until it appears at the surface. One need only record the total depth, the time taken for the traverse and the total number of revolutions. This procedure could conceivably take half or even a quarter of the time of the existing approach, yet would be much less susceptible to error as it is doing the procedure of determining the mean velocity itself. There might be some practical considerations, such as when starting the traverse having to very quickly reach the full velocity of traverse, maintaining the angle of the meter, stopping the count the instant that the meter breaks the surface, ensuring that the rise velocity does not affect the meter calibration, and so on. Most of these effects could be minimised by having a relatively slow velocity of traverse. If this procedure were followed, then, of course, none of the formulae presented above would be necessary.

5. Integration of the mean velocities across the channel

Having obtained the mean velocity on each vertical, the problem now is to integrate across the width of the channel. Here traditional practice seems to be in error – often the *Mean-Section* method is used. In this the mean velocity between two verticals is calculated and then multiplied by the area between them, so that, given two verticals i and $i+1$ separated by b_i the expression for the contribution to discharge is assumed to be

$$\delta Q_i = \tfrac{1}{4} b_i \, (d_i + d_{i+1}) \, (\overline{u}_i + \overline{u}_{i+1}) \,. \tag{5.1}$$

This is actually not correct. From equation (2.1), the task is actually to integrate across the channel the quantity which is the mean velocity times the depth, and equation (5.1) is not a consistent approximation to that task, however plausible the procedure might sound. The simplest expression which is a consistent approximation, and which is as accurate as is reasonable under the circumstances of irregularly varying depth and mean velocity, is the Trapezoidal rule:

$$\delta Q_i = \tfrac{1}{2} b_i \, (\overline{u}_{i+1} \, d_{i+1} + \overline{u}_i \, d_i) \,.$$

In fact, the well-known *Mid-Section* Method takes as the elemental contribution

$$\delta Q_i = \overline{u}_i \, d_i \times \tfrac{1}{2} \, (b_i + b_{i+1}) \,,$$

which when the individual contributions are summed, is the same as the Trapezoidal Rule, although the latter is slightly simpler to apply.

To examine where the Mean-Section Method is most obviously inaccurate, we consider the case at one side of the channel, where the area is a triangle. We let the water's edge be $i = 0$ and the first internal point be $i = 1$, then the Mean-Section Method gives

$$\delta Q_0 = \tfrac{1}{4} b_0 \overline{u}_1 d_1,$$

while the Trapezoidal rule gives

$$\delta Q_0 = \tfrac{1}{2} b_0 \overline{u}_1 d_1,$$

which is correct, and we see that the Mean-Section Method computes only half of the actual contribution. The same happens at the other side. Contributions at these edges are not large, and in the middle of the channel the formula is not so much in error, but in principle the Mean-Section Method is wrong and should not be used. Australian Standard 3778.3.1 (2001) still presents the Mean-Section Method, but with a warning and correction procedure at the edges. We are of the opinion it should never be used.

Rather, the Trapezoidal rule should be used, which is just as easily implemented. In a gauging in which the author participated, a flow of 1693 Ml/d was calculated using the Mean-Section Method. Using the Trapezoidal rule, the flow calculated was 1721 Ml/d, a difference of 1.6% in this case. Although the difference was not great, practitioners should be discouraged from using a formula which is wrong.

6. Ultrasonic velocimetry

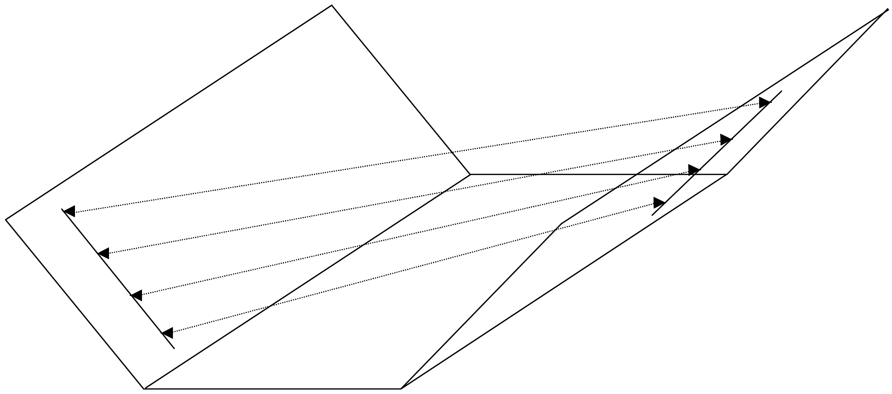

Figure 6-1. Array of four ultrasonic beams in a channel

This is a method primarily used in the irrigation industry in Australia. Consider the situation shown in the figure, where some three or four beams of ultrasonic sound are propagated diagonally across a stream at different levels. The time of travel of sound in one direction is measured, as is the time in the other. The two are different because the velocity of propagation is increased in the downstream direction and decreased when the sound propagates back up against the current. The difference can be used to compute the mean velocity along that path, *i.e.* at that level. These values then have to be integrated in the vertical.

6.1 Mean velocity on a path

The first problem is to calculate the mean velocity along a beam path. In all textbooks and the International and Australian Standards (#2.8 of Australian Standard 3778.3.7 2001), a constant velocity is assumed - precisely what is being sought to measure, and ignoring the fact that velocity actually varies along the path and indeed is zero at the ends, as suggested in Figure 6-2. Here we include the variability of velocity in our analysis.

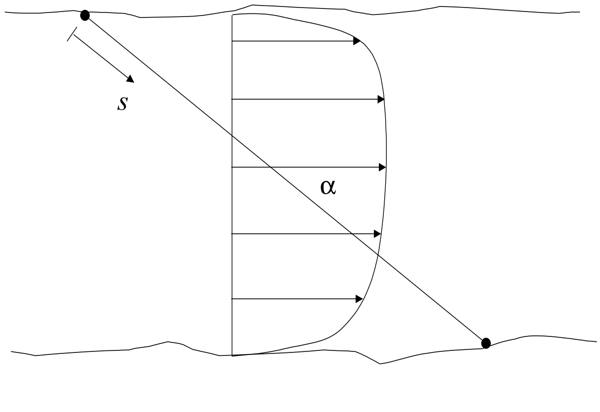

Figure 6-2. Layout of ultrasonic beam path in the turbulent shear flow of an open channel

Consider the fluid velocity vector down the channel u be inclined to the beam path at an angle α, as shown in Figure 6-2. The streamwise velocity component is written as $u(s)$, where s is distance along the path, and which shows that the velocity does, in general, depend on position along the beam, then the component along the path is $u(s)\cos\alpha$. Let c be the speed of sound in water. The time dt taken for a sound wave to travel a distance ds along the path with the general direction of flow is $dt = ds/\left(c + u(s)\cos\alpha\right)$. If the path has total length L, then the total time of travel T_1 is obtained by integrating to give

$$T_1 = \int_0^{T_1} dt = \int_0^L \frac{ds}{c + u(s)\cos\alpha}, \tag{6.1}$$

and repeating for a traverse in the reverse direction against the general direction of flow:

$$T_2 = \int_0^{T_2} dt = \int_0^L \frac{ds}{c - u(s)\cos\alpha}. \tag{6.2}$$

Now we expand the denominators of both integrals by the binomial theorem:

$$T_1 = \frac{1}{c}\int_0^L \left(1 - \frac{u(s)}{c}\cos\alpha\right) ds \quad \text{and} \quad T_2 = \frac{1}{c}\int_0^L \left(1 + \frac{u(s)}{c}\cos\alpha\right) ds, \tag{6.3}$$

where we have ignored terms proportional to the square of the fluid velocity divided by the speed of sound, $u^2(s)/c^2$ (that is, the square of the Mach number of the channel flow!), so this should be an excellent approximation. Evaluating gives

$$T_1 = \frac{L}{c} - \frac{1}{c^2}\int_0^L u(s)\cos\alpha\, ds \quad \text{and} \quad T_2 = \frac{L}{c} + \frac{1}{c^2}\int_0^L u(s)\cos\alpha\, ds. \tag{6.4}$$

Adding the two equations and solving for c and re-substituting we obtain

$$\int_0^L u(s)\cos\alpha\, ds = 2L^2 \frac{T_2 - T_1}{(T_1 + T_2)^2}. \tag{6.5}$$

In computing discharge, it is necessary to compute the flow from the integral of the velocity component transverse to the beam path, which is

$$q = \int_0^L u(s)\sin\alpha\, ds. \tag{6.6}$$

Now we are forced to assume that the angle that the velocity vector makes with the beam is constant over the path (or at least in some rough averaged sense), and so for α constant, taking the trigonometric functions outside the integral signs and combining equations (6.5) and (6.6) we obtain

$$q = 2\,\tan\alpha\, L^2 \frac{T_2 - T_1}{(T_1 + T_2)^2}. \tag{6.7}$$

This shows how the result is obtained by assuming the angle of inclination of the fluid velocity to the beam is constant, but importantly it shows that it is not necessary to assume that velocity u is constant over the beam path.

The expression presented in Standards and textbooks (for example, equation 2.1 of Australian Standard AS 3778.3.7, 1990; or equation 4.63 of Boiten, 2000) is obtained by assuming that $u(s)$ is constant in (6.1) and (6.2), giving

$$T_1 = \frac{L}{c + u\cos\alpha} \quad \text{and} \quad T_2 = \frac{L}{c - u\cos\alpha},$$

from which c can be eliminated and the result for u substituted into equation (6.6) to give

$$q = \frac{\tan\alpha\, L^2}{2}\left(\frac{1}{T_1} - \frac{1}{T_2}\right). \tag{6.8}$$

It can be shown that the relative error of using this equation, obtained by assuming that the velocity is constant, compared with the one derived more rationally, equation (6.7), is of the order of the Mach number of the streamflow, u/c, which is very small. It is fortunate that the end result, presented in Standards and trade brochures and implemented in practice, is sufficiently correct for practical purposes.

6.2 Vertical integration of beam data

The mean velocities on different levels obtained from the beam data are considered to be highly accurate, provided all the technical problems associated with beam focussing *etc.* are overcome, and the streamflow has a constant angle α to the beam. The problem remains to calculate the discharge in the channel by evaluating the vertical integral of q, which, as shown by equation (6.6), is the integral along the beam of the velocity transverse to the beam. The problem is then to evaluate

$$Q = \int_0^h q(z)\, dz, \tag{6.9}$$

where in practice the information available is that $q = 0$ on the bottom of the channel $z = 0$ and the two to four values of q which have been obtained from beam data, as well as the total depth h. It is in the evaluation of this integral that the performance of the trade and scientific literature has been poor. Several trade brochures advocate the routine use of a single beam, or maybe two, suggesting that that is adequate (see, for example, Boiten 2000, p141). In fact, with high-quality data for q at two or three levels, there is no reason not to use accurate integration formulae. However, practice in this area has been quite poor, as trade brochures that the author has seen use the inaccurate Mean-Section Method for integrating vertically over only three or four data points, when its errors would be rather larger than when it is used for many verticals across a channel, as described previously. This seems to be a ripe area for research. The author has applied some of the methods described previously for determining the velocity on a vertical, but as in Figure 3-2 there was some variation between them. They were however, better than traditional practice, which handles the top and bottom rather clumsily.

7. Correcting for rising or falling stage – the looped flood trajectory

Consider the problem of a flood wave propagating along a waterway and the effects on the surface slope at a gauging station. Ahead of a flood, the slope is greater than at the rear, and so according to elementary hydraulic theory, for a given stage the discharge will be greater ahead of a flood event than after it. This leads to a looped flood trajectory – see Fenton & Keller (2001) and Fenton (2001). In those papers two methods for calculating the discharge for rising or falling stage were obtained. The first method used the full long wave equations and gave a differential equation for dQ/dt in terms of Q and stage and the derivatives of stage $d\eta/dt$ and $d^2\eta/dt^2$, which could be calculated from the record of stage with time. It could be solved numerically. The second method is rather simpler, and is based on a low-inertia approximation to the long wave equations, where inertial terms, which are of the order of the square of the Froude number, are ignored, giving an advection-diffusion equation which approximates motion in most waterways quite well. In that equation, the surface slope is expressed in terms of the first two time derivatives of stage. The resulting expression is:

$$Q = Q_{\mathrm{r}}(\eta)\sqrt{\underbrace{\underbrace{1}_{\text{Rating curve}} + \frac{1}{c\bar{S}}\frac{d\eta}{dt}}_{\text{Jones formula}} \underbrace{- \frac{D}{c^3\bar{S}}\frac{d^2\eta}{dt^2}}_{\text{Diffusion term}}} \tag{7.1}$$

where Q is the discharge at the gauging station, $Q_{\mathrm{r}}(\eta)$ is the rated discharge for the station as a function of stage, $\bar{S}$ is the bed slope, c is the kinematic wave speed given by:

$$c = \frac{\sqrt{\bar{S}}}{B}\frac{dK}{d\eta} = \frac{1}{B}\frac{dQ_{\mathrm{r}}}{d\eta},$$

in terms of the gradient of the conveyance (K) curve or gradient of the rating curve; B is the width of the water surface; and the coefficient D is the diffusion coefficient in advection-diffusion flood routing:

$$D = \frac{K}{2B\sqrt{\bar{S}}} = \frac{Q_{\mathrm{r}}}{2B\bar{S}}. \tag{7.2}$$

In equation (7.1) it is clear that the extra diffusion term is a simple correction to the Jones formula, allowing for the subsidence of the wave crest as if the flood wave were following an advection-diffusion law, a good approximation for much flood propagation. Equation (7.1) provides a means of analysing stage records and correcting for the effects of unsteadiness and variable slope. It can be used in either of two directions:

1. If a gauging exercise has been carried out while the stage has been varying (and been recorded), the value of Q obtained can be corrected for the effects of variable slope, giving the steady-state value of discharge for the stage-discharge relation, Q_{r}, or
2. Proceeding in the other direction, in operational practice, the equation can be used for the routine analysis of stage records to correct for any effects of unsteadiness.

Corrections are largest for rivers where the conditions change quickly but which are otherwise slow-moving with a mild slope. In fact, these conditions are often mutually exclusive, such that slow-moving rivers are likely to be slow to rise and fall. Nevertheless, it is quite possible that there are stations where the corrections are necessary.

7.1 Implementation of the theory for practical problems

The theory described above could be implemented at two levels. The first would be a screening of a particular gauging station and its records simply to determine whether it is necessary to correct for unsteady effects. This will mean initially estimating the slope of the river. Then the data from a particular flood can be taken and the stage record processed. In these formulae it would be simplest and quite accurate enough, provided the time interval of the readings is small enough to describe the variation, especially at the crest, to use the three-point finite difference approximations for the derivatives:

$$\left.\frac{d\eta}{dt}\right|_i \approx \frac{\eta_{i+1}-\eta_{i-1}}{2\delta} \quad \text{and} \quad \left.\frac{d^2\eta}{dt^2}\right|_i \approx \frac{\eta_{i+1}-2\eta_i+\eta_{i-1}}{\delta^2}, \tag{7.3}$$

where η_{i-1}, η_i, and η_{i+1} are three successive stage readings, taken with a time interval between readings of δ. In this preliminary screening case it has only been necessary to take representative values of velocity, which might involve using a representative value of area, and mean depth.

All that is necessary is, firstly the traditional information to hand:

1. The stage record, giving measured values of stage at equally-spaced times separated by an interval δ: $\eta_i, i = 1, 2, \ldots$. At a particular reading the first and second derivatives can be calculated numerically using the difference approximations (7.3). If the data is not equally spaced, then different formulae for the time derivatives are necessary, and spline interpolation might be useful.
2. The rating curve in the form of a number of data pairs of stage and rated discharge: $\left(\eta_j, Q_{\mathrm{r},j}\right)$, for $j = 1, 2, \ldots$. In Australian hydrographic practice these might have to be converted from Megalitre/day to cubic metres per second by dividing by a conversion factor of 86.4. It is necessary to be able to interpolate in this data to be able to calculate a discharge for an arbitrary stage value.
3. The rating curve has to be able to be differentiated to give the $dQ/d\eta$, also at an arbitrary stage. For sufficiently large numbers of data points (small intervals) simple finite difference formulae could be

used, however it might be reasonable to develop a global approximation.

4. The mean bed slope $\bar{S}$.
5. The cross-sectional geometry simply in the form of a number of data pairs of stage and surface width: (η_k, B_k), for $k = 1, 2, \ldots$. These would probably be obtained from cross-sectional data in the form of tables of readings of position and elevation, as well as judgement and knowledge of the site.

It is interesting that to implement the method, little extra data is necessary beyond that used to implement the simple rating curve method.

7.2 A theoretical example

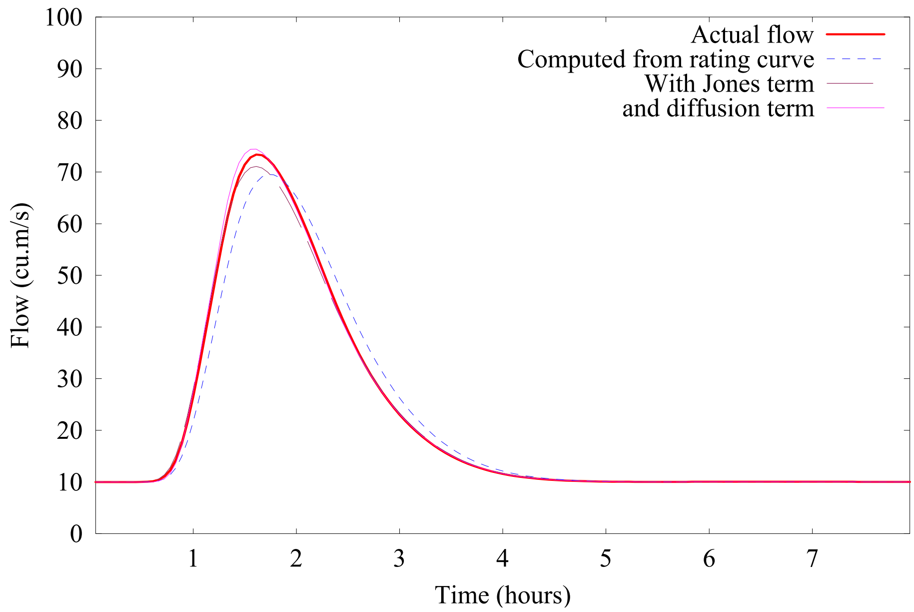

Figure 7-1. Simulated flood with hydrographs computed from stage record using three levels of approximation

A numerical solution was obtained for the particular case of a fast-rising and falling flood in a stream of $10\,\mathrm{km}$ length, of slope 0.001, which had a trapezoidal section $10\,\mathrm{m}$ wide at the bottom with side slopes of 1:2, and a Manning's friction coefficient of 0.04. The downstream control was a weir. Initially the depth of flow was $2\,\mathrm{m}$, while carrying a flow of $10\,\mathrm{m}^3\,\mathrm{s}^{-1}$. The incoming flow upstream was linearly increased ten-fold to $100\,\mathrm{m}^3\,\mathrm{s}^{-1}$ over 60 mins and then reduced to the original flow over the same interval. The initial backwater curve problem was solved and then the long wave equations in the channel were solved over six hours to simulate the flood. At a station halfway along the waterway the computed stages were recorded (the data one would normally have), as well as the computed discharges so that some of the above-mentioned methods could be applied and the accuracy of this work tested.

Results are shown on Figure 7-1. It can be seen that the application of the diffusion level of approximation has succeeded well in obtaining the actual peak discharge. The results are not exact however, as the derivation depends on the diffusion being sufficiently small that the interchange between space and time differentiation will be accurate. In the case of a stream such as the example here, diffusion is relatively large, and our results are not exact, but they are better than the Jones method at predicting the peak flow. Nevertheless, the results from the Jones method are interesting. A widely-held opinion is that it is not accurate. Indeed, we see here that in predicting the peak flow it was not accurate in this problem. However, over almost all of the flood it was accurate, and predicted the time of the flood peak well, which is also an important result. It showed that both before and after the peak the "discharge wave" led the "stage wave", which is of course in phase with the curve showing the flow computed from the stage graph and the rating curve. As there may be applications where it is enough to know the arrival time of the flood peak, this is a useful property of the Jones formula. Near the crest, however, the rate of rise became small and so did the Jones correction. Now, and only now, the inclusion of the extra diffusion term gave a significant correction to the maximum flow computed, and was quite accurate in its prediction that the real flow was some 10% greater than that which would have been calculated just from the rating curve. In this fast-rising example the application of the unsteady corrections seems to have worked well and to be justified. It is no more difficult to apply the diffusion correction than the Jones correction, both being given by derivatives of the stage record.

8. Conclusions

Some more-accurate formulae for the velocity-area method of streamgauging have been obtained and presented for cases of two, three, and four velocity measurements. Traditional formulae require that the velocities be measured at pre-determined depths. The formulae presented here are applicable to arbitrary depths, so that hydrographers need not compute required depths and set the velocity meter there. Instead, the arbitrary actual depths used and corresponding measured velocities can simply be recorded and processed.

It has been shown that the traditional formulae are not as accurate as is sometimes believed. The new formulae presented here should be more accurate, as they are based on a rational approximation of the theoretical logarithmic profile, but where they allow the profile to bend forwards or backwards as is often observed. Two families of methods have been presented, one based on interpolation, where the function describing the velocity profile passes through every data point; and one based on approximation, where the points are described in an approximate sense, minimising the error of the approximation.

When compared with some streamgauging results, the new formulae generally gave good and consistent results. However in one profile, where the measurements showed irregularities and inconsistencies, the deduced profiles seemed inaccurate, even though the calculated mean velocities were not greatly in error.

In view of the sensitivity of the higher-accuracy methods to irregular results, the new two-point formula may be accurate enough for most purposes. However, a procedure was suggested which would obviate use of all the formulae considered here, and give a robust and accurate method for calculating the mean by simply traversing the current meter at constant velocity vertically through the water so that it performs the integration automatically to obtain the mean.

In integrating the results of such data across the channel, the traditional Mean-Section method for calculating the discharge has been shown to be wrong, and it should not be used.

Turning to ultrasonic velocimetry, the basic equation for ultrasonic flow measurement has been derived on a more satisfactory basis, without having to assume a constant velocity of flow. Then, the problem of vertical integration of beam data for the discharge has been considered. It has been noted that the trade literature and conventional practice are quite defective, and that flow results from ultrasonic meters are not as accurate as they might be, partly through use of the inaccurate Mean-Section method and partly through crude treatment of velocities above the highest beam and below the lowest. To obtain more accurate results it has been suggested that the methods described above for mean velocity in the vertical could be used.

Finally, a method is proposed and tested for allowing for the effects of rising and falling stage while gauging is in progress, or afterwards when a rating curve has been obtained.

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