A note on two approximations to the linear dispersion relation for surface gravity water waves

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Abstract

A comparison is made between two approximations to the dispersion relation. The two both have adequate accuracy, given the approximations inherent in the equation they are approximating. The more recent solution by Guo (2002) is very slightly simpler and is probably to be preferred. It is doubtful that more research into this problem is justified.

Figure 1. Comparison of approximations and the linear dispersion relation

A common problem in coastal engineering is to obtain the length $\lambda$ of a wave train if the wave period $\tau$ and depth $d$ are known. To a first level of approximation, ignoring all nonlinear effects and all effects due to current shifting of the wave period, the well-known linear dispersion relation is

$$ \sigma \sqrt{\frac{g}{d}} = \sqrt{k d \tanh kd}, \quad \text{or} \quad \frac{2\pi}{\tau g/d} = \sqrt{\frac{2\pi d}{\lambda} \tanh \frac{2\pi d}{\lambda}}, $$

(1)

where it has been convenient to introduce the wavenumber $k = 2\pi/\lambda$ and the radian frequency $\sigma = 2\pi/\tau$. If period $\tau$ is measured, as is often the case, this is a nonlinear transcendental equation for wavelength $\lambda$, which can be solved by standard numerical means such as trial and error, bisection, or Newton’s method. Fenton & McKee (1990) gave an empirical explicit approximation, with little theoretical justification:

$$ kd = \frac{\sigma^2 d}{g} \left( \coth \left( \sigma \sqrt{d/g} \right)^{3/2} \right)^{2/3}. $$

(2)

This expression is accurate to within 1.5% over all wavelengths, which is probably accurate enough for
practical purposes, in view of the approximations noted above. It is exact in both long wave (shallow water) and short wave (deep water) limits respectively. It can be used as an initial approximation if greater numerical refinement were required.

A more recent approximation formula by Guo (2002):

$$k d = \frac{\sigma^2 d}{g} \left( 1 - e^{-\left(\sigma \sqrt{d/g}\right)^{5/2}} \right)^{-2/5}$$

(3)

has a maximum error of about 0.7%, about half the error of equation (2). It, also, is exact in the short and long wave limits. When the exact expression (1), and the approximations (2) and (3) are plotted, as shown on Figure 1, it can be seen that both approximations are accurate enough and that there is little significant difference between the two.

As noted by Fenton & McKee (1990), the approximations made in obtaining equation (1) have introduced much more error than either of these approximations to that equation. Hence, obtaining highly accurate results is not important, and on that basis the two approximations are of comparable adequate accuracy. However, equation (3) is slightly simpler than equation (2) and hence is probably to be preferred. There seems to be little justification in pursuing further research in this direction.

References
