J. Fluid Mech. (2000), *vol.* 418, *pp.* 101–117. Printed in the United Kingdom

Potential-flow instability theory and alluvial stream bed forms

By S. E. COLEMAN[1] AND J. D. FENTON[2]

[1]Department of Civil and Resource Engineering, The University of Auckland, Private Bag 92019, Auckland, New Zealand
e-mail: s.coleman@auckland.ac.nz

[2]Cooperative Research Centre for Catchment Hydrology, Department of Civil Engineering, Monash University, Clayton, Victoria, Australia 3168
e-mail: fenton@labyrinth.net.au

(Received 15 February 1999 and in revised form 16 March 2000)

The present work constitutes a reassessment of the role of potential-flow analyses in describing alluvial-bed instability. To facilitate the analyses, a new potential-flow description of unsteady alluvial flow is presented, with arbitrary phase lags between local flow conditions and sediment transport permitted implicitly in the flow model. Based on the present model, the explicit phase lag between local sediment transport rate and local flow conditions adopted for previous potential-flow models is shown to be an artificial measure that results in model predictions that are not consistent with observed flow system behaviour. Previous potential-flow models thus do not provide correct descriptions of alluvial flows, and the understanding of bed-wave mechanics inferred based upon these models needs to be reassessed. In contrast to previous potential-flow models, the present one, without the use of an explicit phase lag, predicts instability of flow systems of rippled or dune-covered equilibrium beds. Instability is shown to occur at finite growth rates for a range of wavelengths via a resonance mechanism occurring for surface waves and bed waves travelling at the same celerity. In addition, bed-wave speeds are predicted to decrease with increasing wavelength, and bed waves are predicted to grow and move at faster rates for flows of larger Froude numbers. All predictions of the present potential-flow model are consistent with observations of physical flow systems. Based on the predicted unstable wavelengths for a given alluvial flow, it is concluded that bed waves are not generated from plane bed conditions by any potential-flow instability mechanism. The predictions of instability are nevertheless consistent with instances of accelerated wave growth occurring for flow systems of larger finite developing waves. Potential-flow description of alluvial flows should, however, no longer form the basis of instability analyses describing bed-form (sand-wavelet) generation from flat bed conditions.

1. Previous potential-flow instability models

Reynolds (1976) in reviewing the then current understanding of stream-bed stability commented that '... available mathematical techniques have revealed the general features of the interaction between flow and bed material'. Foremost among the techniques at that time were potential-flow instability models (Kennedy 1963, 1969; Gradowczyk 1970; Jain & Kennedy 1974) and rotational-flow instability models (for example, Engelund 1970). Advanced rotational-flow models were later proposed by Richards (1980) and Sumer & Bakioglu (1984).

Kennedy (1969) presented a quasi-steady potential-flow analysis incorporating an explicit lag distance, β, between local flow conditions and local sediment transport rate. He considered dominant wavelengths and velocities of bed features generated from plane bed conditions and presented diagrams indicating the occurrence of various types of bed forms appropriate to various flow conditions.

A principal result of Kennedy (1969) is that for a given alluvial sediment bed, dunes and antidunes are predicted to form only for $Fr^2 < \tanh kH/kH$ and $Fr^2 > \tanh kH/kH$ respectively, where Fr is the Froude number of the flow, $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength of the bed waves, and H is the undisturbed mean flow depth.

Hayashi (1970) included local bed slope in considerations of Kennedy's phase lag. Gravity and expansion losses were shown to significantly affect values of this phase lag along with the resulting quasi-steady potential-flow model predictions of bed waves.

Jain & Kennedy (1974) incorporated spectral descriptions of the streamlines, the bed profile and the water surface profile as functions of time into a quasi-steady potential-flow model. For given fluid–sediment flow systems, they produced graphs of normalized bed-wave growth rate as a function of normalized wavelength, and compared numerical model predictions with experimental results for the wavelengths of waves generated from plane bed conditions.

The assumption of quasi-steady conditions in these potential-flow models (and also the recent work of Pantin 1990) results in a singularity in these analyses at the condition

$$Fr^2 = \frac{\tanh kH}{kH}. \tag{1.1}$$

Owing to this singularity, Jain & Kennedy (1974) predicted waves of lengths defined by equation (1.1) to be generated at infinite growth rates from plane bed conditions. This singularity has to date prevented determination of true flow-system behaviour at this condition.

A second limitation of previous potential-flow analyses is the difficulty associated with defining appropriate values of the phase lag β for flow systems. For example, for explicit phase lag values of zero, flow system instability and bed-wave generation and growth are not predicted by Kennedy (1969), Hayashi (1970) or Jain & Kennedy (1974).

Given these limitations of potential-flow analyses, Gradowczyk (1968) commented about the Kennedy analyses '... is it possible to obtain the same wave configurations by means of a potential flow theory which does not assume quasi-steadiness and without the lag β? The answer to this question may help to clear up the meaning of the (phase lag) parameter'. By considering an unsteady, rather than quasi-steady, potential-flow model, Gradowczyk (1970) did obtain a reasonable explanation of some features of bed-form regimes without inclusion into the analyses of Kennedy's explicit phase lag. Gradowczyk nevertheless predicted only neutral bed forms of zero growth rate. Gradowczyk also indicated that the singularity evident for previous potential-flow models can be removed, although this is not carried out for the analyses he presented. Despite the subsequent investigations of numerous authors, the questions posed by Gradowczyk (1968) remain unanswered over three decades later.

In reviewing the understanding of bed waves at the time, Kennedy (1980) commented that Kennedy (1969) '... treats in almost exhaustive detail the potential flow theory of bed forms ...'. Kennedy (1980) nevertheless did not answer several impor-

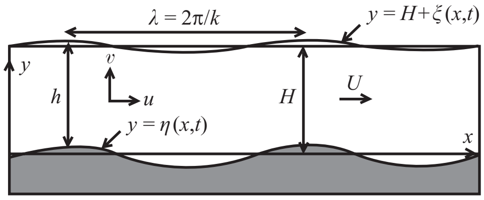

FIGURE 1. Flow system to be modelled.

tant criticisms, such as those indicated above, of the potential-flow theory to date and its implications. Indeed Kennedy (1980) is found to concur with most of the criticisms of potential-flow theory levelled by Reynolds (1976).

In addition to theoretical conjecture as to the validity of conventional models of alluvial-bed instability, recent data also bring into question the physical validity of the predictions of these instability models. In particular, the measured data describing the lengths of waves first generated from plane bed conditions have been assessed and collated in Coleman (1996) and Coleman & Melville (1996). Conventional instability models, which are typically designed to describe these initial wavelengths, are nevertheless generally found to incorrectly predict the magnitudes of and trends in these measured lengths (Coleman & Melville 1996).

The present work constitutes a comprehensive assessment of the potential-flow model of alluvial-bed instability. With the fluid–sediment flow system considered to be fully unsteady, and with any lags between local sediment transport rate and local flow conditions being implicit in analyses, the potential-flow model formulated herein provides answers to the lingering questions of Gradowczyk (1968). By subsequently introducing the explicit phase lag β of earlier models, the true nature and implications of this lag are revealed. Predictions of the present potential-flow model are used to critically examine over three decades of conjecture regarding the potential-flow description of alluvial-flow instability. The present results also have implications for rotational-flow models, because the flow behaviour described by potential-flow theory underpins the behaviour described by rotational-flow theory.

2. Unsteady linearized potential flow with unconstrained wave phases

The two-dimensional, irrotational, unsteady, free-surface flow to be modelled is shown in figure 1. The fluid is assumed to be incompressible and inviscid. The erodible bed profile provides the initial system disturbance in the form of a sinusoid of small amplitude. More complex disturbance profiles can be being obtained by linear superposition of any number of such functions.

Irrotational flow of an incompressible fluid is described by

$$\nabla^2 \phi = 0, \tag{2.1}$$

where ϕ is a scalar function of position such that the velocity vector $\boldsymbol{u} = (u, v) = (\partial\phi/\partial x, \partial\phi/\partial y)$, where u and v are horizontal and vertical components respectively. With no relative motion of the fluid across any interface, the two boundary conditions required for solution of (2.1) can be expressed as

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} - v = 0 \quad \text{on the water surface } y = H + \xi, \tag{2.2}$$

and

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - v = 0 \quad \text{on the bed surface } y = \eta, \tag{2.3}$$

where $\xi(x,t)$ and $\eta(x,t)$ describe the water-surface and sediment bed profiles respectively. Having introduced the two extra variables ξ and η, two further equations are necessary to relate these variables to the variables u and v. The dynamic boundary condition for the water surface is obtained from the pressure or unsteady Bernoulli equation for irrotational, unsteady flow of an inviscid, incompressible fluid:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + g\xi = f(t) - gH \quad \text{on } y = H + \xi, \tag{2.4}$$

where H is the undisturbed mean flow depth and $f(t)$ represents some function of time. The erosion equation for sediment movement provides the final equation relating the flow to the fluid–sediment boundary:

$$\frac{\partial \eta}{\partial t} + \frac{\partial q_s}{\partial x} = 0 \quad \text{on } y = \eta. \tag{2.5}$$

Recognizing forms of general sediment transport equations such as the Engelund–Hansen expression (given later as equation (2.17)), and also that local sediment transport responds to local rather than average flow conditions, the sediment transport rate per unit width of flow can be expressed generically as

$$q_s = f(u_\eta, h, S, \rho, \rho_s, d, g), \tag{2.6}$$

where u_η is the component of point fluid velocity at the bed level that is parallel to the overall bed slope S, h is flow depth, ρ is fluid density, ρ_s is sediment density, and d is a representative sediment size. Equation (2.6) can be taken to describe the unsteady response of local sediment transport q_s to unsteady fluctuations in local flow conditions u_η and h. Equation (2.5) can be rewritten as

$$\frac{\partial \eta}{\partial t} + q_1\frac{\partial u_\eta}{\partial x} + q_2\left(\frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial x}\right) = 0 \quad \text{on } y = \eta, \tag{2.7}$$

where $q_1 = \partial q_s/\partial u_\eta$, $q_2 = \partial q_s/\partial h$ and $h = H + \xi - \eta$.

For the perturbed flow system, the initial disturbance to the system of otherwise uniform fluid flow ($\boldsymbol{u} = (U, 0)$) is assumed to be in the form of a sinusoid of small amplitude such that η, ξ and ϕ can then be taken to be of the respective forms

$$\eta = \varepsilon C\, \mathrm{e}^{\mathrm{i}k(x-ct)}, \tag{2.8}$$

$$\xi = \varepsilon D\, \mathrm{e}^{\mathrm{i}k(x-ct)}, \tag{2.9}$$

$$\phi = Ux + \varepsilon(A\sinh ky + B\cosh ky)\, \mathrm{e}^{\mathrm{i}k(x-ct)}, \tag{2.10}$$

where ε is a small perturbation parameter, U is undisturbed mean fluid velocity, c is wave celerity, angular wavenumber is $k = 2\pi/\lambda$, and λ is wavelength.

The adopted form of ϕ can be shown to satisfy equation (2.1). Substituting for η, ξ and ϕ from equations (2.8) to (2.10) into equations (2.2), (2.3), (2.4) and (2.7) and linearizing the resulting equations gives

$$f(t) = \tfrac{1}{2}U^2 + gH, \tag{2.11}$$

from the equating of coefficients of ε^0. More importantly, the following system of equations results from equating coefficients of ε^1:

$$\begin{bmatrix} \cosh kH & \sinh kH & 0 & -\mathrm{i}(U-c) \\ 1 & 0 & -\mathrm{i}(U-c) & 0 \\ \mathrm{i}(U-c)\sinh kH & \mathrm{i}(U-c)\cosh kH & 0 & g/k \\ 0 & \mathrm{i}kq_1 & -c-q_2 & q_2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.12}$$

Equation (2.12) is an homogeneous system of four equations in four unknowns for which the trivial solution $A = B = C = D = 0$ exists (indicating an unperturbed flow system). For a non-trivial solution to exist, the determinant of the leading matrix must be zero. This condition gives the general dispersion relation

$$\delta_*^3 + a_2\delta_*^2 + a_1\delta_* + a_0 = 0, \tag{2.13}$$

where

$$a_0 = \frac{(-U_* - q_{2*})\tanh kH}{1 + q_{1*}\tanh kH}, \tag{2.14}$$

$$a_1 = \frac{-q_{1*} - \tanh kH}{1 + q_{1*}\tanh kH}, \tag{2.15}$$

$$a_2 = \frac{U_* + q_{2*}(1 - \operatorname{sech} kH)}{1 + q_{1*}\tanh kH}, \tag{2.16}$$

for the flow system of figure 1, where equation (2.13) has been non-dimensionalized by multiplying through by $(k/g)^{3/2}/\cosh kH$, and where $U_* = U(k/g)^{1/2}$, $\delta_* = (c-U)(k/g)^{1/2}$, $q_{1*} = q_1 k$ and $q_{2*} = q_2(k/g)^{1/2}$.

With linearization of the governing equations for the flow system, q_1 and q_2 become $\partial q_s/\partial U$ and $\partial q_s/\partial H$ respectively and a sediment transport equation of the form $q_s = f(U, H, S, \rho, \rho_s, d, g)$ is required for solution of the dispersion relation of (2.13). The equation adopted for the present analyses is the Engelund–Hansen formula for total sediment load (Vanoni 1975). This relation can be written in the form

$$q_s = \frac{0.05U^2H^{3/2}S^{3/2}}{d(s-1)^2g^{1/2}}, \tag{2.17}$$

where $s = \rho_s/\rho$, and shear stress is given by $\tau = \rho gHS$ for steady, uniform flow over an essentially flat bed. This transport relation is appropriate to the data of Jain & Kennedy (1971, 1974) and Guy, Simons & Richardson (1966) analysed herein. Other transport relations have also been used with only minor numerical variations in the predictions of the analyses.

For a given flow system of U, H, S, ρ, ρ_s and d, $q_1 = \partial q_s/\partial U$ and $q_2 = \partial q_s/\partial H$ can be evaluated, and wave celerities for different wavelengths can be calculated from equation (2.13) based on standard cubic solving techniques (for example, based on Abramowitz & Stegun 1968, case 3.8).

The dispersion relation of (2.13) is a cubic in terms of wave celerity. For any given wavelength–flow system combination, two of the celerities evaluated from this equation will apply to the movement of water-surface waves. These waves are distinguished herein as the faster-moving (case 1) and slower-moving (case 2) water-surface waves. The remaining celerity will be related to the motion of waves in the bed surface.

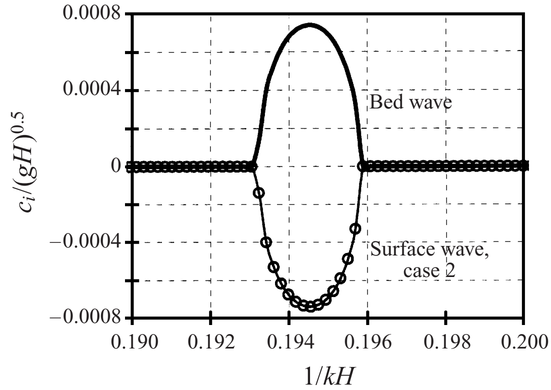

FIGURE 2. Normalized growth parameters predicted by the present model for Run S8 of Jain & Kennedy (1974).

The forms of (2.8) to (2.10) indicate that for a given flow system, any wavelength for which (2.13) predicts complex solutions of $c_i > 0$ (where $c = c_r + \mathrm{i}c_i$, and c_r and c_i are the real and imaginary components of c) is a wavelength for which the given flow system is predicted to be unstable. The wavelength of the waves that can be expected to form in such an unstable flow system is conventionally assumed to be that for which the value of the growth parameter, c_i, is the greatest.

3. Instability predictions of the present model

For the purpose of comparison of the present potential-flow instability model with previous models, flow system instability predictions were analysed for Run S8 of Jain & Kennedy (1971, 1974). The values of U, H, S, Fr and shear velocity u_* for this run are 0.3847 m s^{-1}, 0.0771 m, 0.00267, 0.442, and 0.0418 m s^{-1} respectively. The sediment used was a quartz ($s = 2.65$) sand of geometric mean size $d_g = 0.25$ mm.

Calculated values of the normalized growth parameter, $c_i/(gH)^{1/2}$, are shown in figure 2, where $c_i = 0$ for all other values of kH, and $c_i = 0$ for the faster-moving (case 1) water-surface waves. Run S8 is predicted by the present model to be unstable ($c_i > 0$) for wavelengths of 0.0936 m to 0.0948 m ($kH = 5.11$ to 5.175). The wavelength of the waves expected to form in this flow system is 0.0942 m ($kH = 5.142$).

Predicted normalized wave speeds, $c_r/(gH)^{1/2}$, as functions of wavelength for Run S8 are presented in figure 3. Bed-wave speed is predicted to generally decrease with increasing wavelength, shorter bed waves travelling faster than longer ones.

Wave speeds in the region of instability are highlighted in figure 4. It can be seen that when the flow system is unstable (cf. figure 2), the speeds of the bed waves and the slower-moving (case 2) water-surface waves are equal.

With the dispersion relation of (2.13) being a cubic in terms of wave celerity, complex solutions occur in conjugate pairs, and flow system instability is predicted to occur ($c_i > 0$) for wavelengths for which bed waves are predicted to move at the same speed as water-surface waves travelling upstream relative to U (that is, $c_{r_{\text{bed}}} = c_{r_{\text{water surface}}}$). The instability mechanism is then a bed–water-surface resonance phenomenon, the phenomenon arising with bed and water surface waves remaining stationary relative to each other. This is the sole instability mechanism predicted by potential-flow instability theory.

In order to further analyse the bed–water-surface resonance instability mechanism

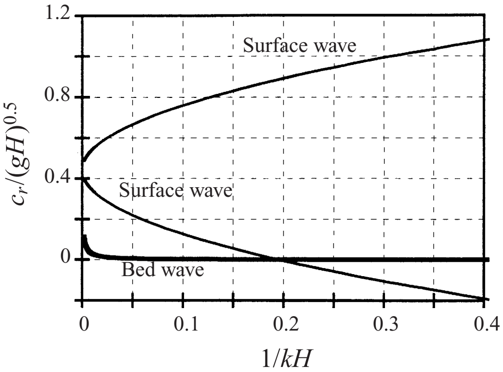

FIGURE 3. Normalized wave speeds predicted by the present model for Run S8 of Jain & Kennedy (1974).

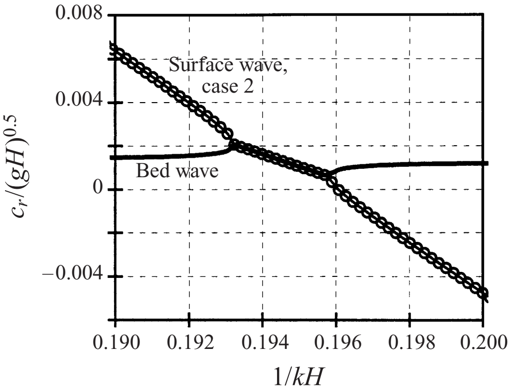

FIGURE 4. Normalized wave speeds predicted by the present model for Run S8 of Jain & Kennedy (1974).

predicted by potential-flow theory, flow system instability predictions were analysed for the experiments of Guy *et al.* (1966) in which sediments of $d = 0.19$ mm and $d = 0.93$ mm respectively were used in a 2.44 m wide flume. Each point of the respective data series of $d = 0.19$ mm and $d = 0.93$ mm in figure 5 reflects the resulting wavelength predicted to form from flat bed conditions for an individual experiment of Guy *et al.* (1966), 31 and 35 experiments having been analysed by them for the 0.19 mm and 0.93 mm sands respectively. The predicted wavelength in each case is that giving the maximum positive growth parameter for the associated flow system. Each predicted wave magnitude approximately lies on the curve described by equation (1.1). The wavelengths predicted to form for a given fluid–sediment flow system can be seen to be independent of the size of the bed sediment. The interaction of bed and surface waves (figure 4), and thereby flow system instability, is predicted not to occur for flows of $Fr > 0.8$ (figure 5, and Coleman 1991).

Although not indicated by the data of figure 5, the range of wavelengths predicted to be unstable for a flow system generally increases with increasing Fr (Coleman 1991). The size of the sediment comprising the erodible bed affects to only a minor degree this range of wavelength values. As Froude numbers increase from threshold conditions, growth parameters, c_i, are predicted (Coleman 1991) to increase (initially

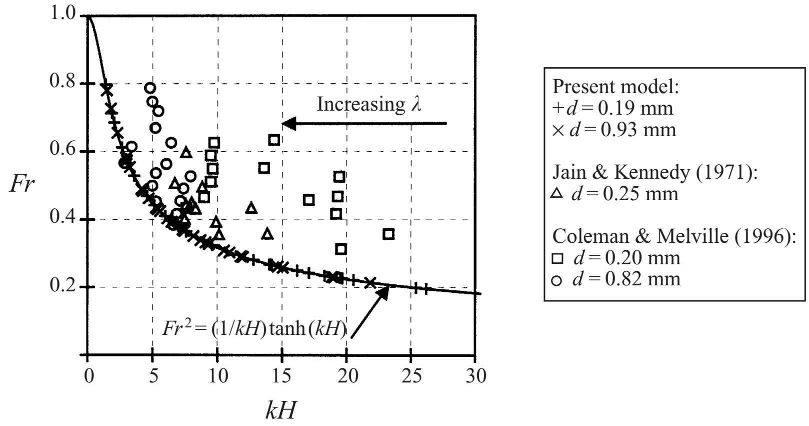

FIGURE 5. Lengths of sand wavelets generated from flat bed conditions.

rapidly), approach a maximum value, and then decrease slightly in magnitude before becoming unconditionally zero (indicating no flow system instability). Similar trends are predicted for wave speeds, c_r (Coleman 1991). These predictions reflect observations of bed waves growing and moving at faster rates for flows of larger Froude numbers (Coleman & Melville 1994, 1996).

Linearized potential-flow theory is designed to describe the growth of small-amplitude disturbances to a flow system. Such a flow theory should describe the wavelengths for the waves first generated from plane bed conditions (termed sand wavelets by Coleman & Melville 1996) for a given flow system. Such measured wavelengths for flow systems (Coleman & Melville 1996) are given in figure 5 for comparison with predictions of the present theory. The smaller of the two respective wavelengths given by Jain & Kennedy (1971) for each run is utilized here because their presented spectral results indicate the spectral peak of larger kH (smaller λ) to be markedly more significant than the spectral peak of smaller kH for a run.

Figure 5 indicates that potential-flow theory predicts lengths for sand wavelets generated from flat bed conditions that are markedly greater than observed lengths. Furthermore, sand-wavelet lengths predicted by the present potential-flow instability model can be seen to be a function of flow conditions but independent of sediment size (figure 5). In contrast, the measured data show these lengths to be relatively insensitive to applied flow conditions and primarily a function of the size of the sediment comprising the bed (Coleman 1996; Coleman & Melville 1996). Sand wavelets are therefore not generated by any potential-flow instability mechanism.

The potential-theory bed–water-surface resonance mechanism elucidated herein is nevertheless a valid instability mechanism and can be seen to promote accelerated wave growth for a flow system of larger developing bed waves (not the initial sand wavelets). To this end, the experimental results of figure 6 indicate rapid development of bed waves in the vicinity of $\lambda = 0.2169$ m for the flow system, instability for this wavelength–flow system combination being predicted by the present theory (as indicated by equation (1.1)). Coleman & Fenton (1996) presented similar rapid wave development in the vicinity of $\lambda = 0.4305$ m for a flow system of $H = 0.126$ m and $Fr = 0.719$. Many other examples of this period of accelerated growth at the predicted wavelength can be cited, but only two are selected for the example purposes of the present paper. It is postulated that the resonance mechanism of instability provides

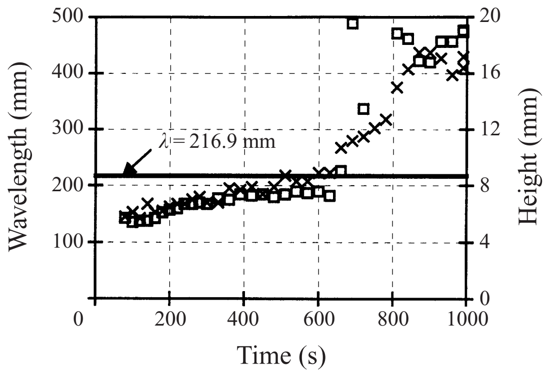

FIGURE 6. Development of bed-form height (×) and wavelength (□) for Run C8 of Coleman & Melville (1996); $d = 0.82$ mm, $H = 0.119$ m, and $Fr = 0.538$.

the means whereby the sediment waves commonly referred to as 'dunes' develop from smaller sand waves growing on a sediment bed. In contrast, this instability mechanism is conjectured not to occur for flow systems of rippled equilibrium beds. Assessment of the association of the occurrence of this instability mechanism with the formation of dunes is continuing.

For the commonly accepted arbitrary boundary for 'deep-water' waves of $H/\lambda > 1/2$, that is kH $(= 2\pi H/\lambda) > \pi$, the ratio of the velocity on the bed to that at the water surface is given by $\exp(-kH) = \exp(-\pi) \approx 0.043$. Despite this value being small, it is recognized that for the present flow system instability considerations, even minor perturbations to the system can be of significance. The authors therefore believe that there is no justification for a hard classification of the present waves generated by system instabilities into conventional 'deep-water' or 'shallow-water' waves, with an accompanying dismissal of any conventional 'deep-water' results for the present resonance mechanism of instability. The authors' belief is reflected in there being no obvious discontinuity in the results in figure 5, the system not seeming to recognize any distinction between conventional 'deep-water' and 'shallow-water' conditions.

4. Present model predictions with an explicit phase lag

In order to investigate the appropriate role of the explicit phase lag proposed by Kennedy (1969) in potential-flow instability analyses, the present potential-flow instability model was reformulated to include this phase lag.

The sediment transport rate per unit width of flow can be related to flow conditions at a distance β upstream using

$$q_s(x,t) = f(u_\eta(x-\beta,t), h(x-\beta,t), S, \rho, \rho_s, d, g). \tag{4.1}$$

Using (4.1), equation (2.7) can again be derived from (2.5) if the revised definitions of $q_1 = [\partial q_s/\partial u_\eta(x-\beta,t)]\,\mathrm{e}^{-\mathrm{i}k\beta}$ and $q_2 = [\partial q_s/\partial h(x-\beta,t)]\,\mathrm{e}^{-\mathrm{i}k\beta}$ are adopted. The potential-flow model formulation is then as presented earlier, with the exception of the revised coefficients q_1 and q_2, which are now complex in nature. With linearization of the governing equations for the flow system, $\partial q_s/\partial u_\eta(x-\beta,t)$ and $\partial q_s/\partial h(x-\beta,t)$ again become $\partial q_s/\partial U$ and $\partial q_s/\partial H$ respectively. These terms can then be evaluated for a given fluid–sediment flow system based on the Engelund–Hansen transport formula of equation (2.17).

Run S8 of Jain & Kennedy (1971, 1974) was analysed using the reformulated

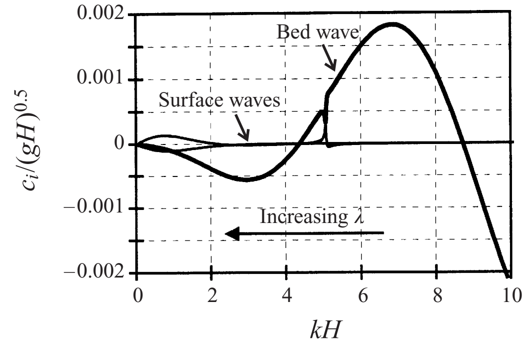

FIGURE 7. Normalized growth parameters predicted based on $\beta/H = 0.72$ for Run S8 of Jain & Kennedy (1974).

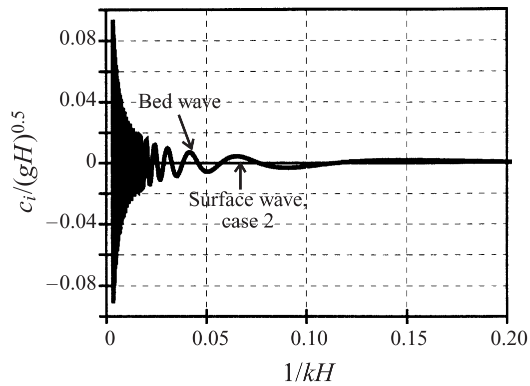

FIGURE 8. Normalized growth parameters predicted based on $\beta/H = 0.72$ for Run S8 of Jain & Kennedy (1974).

model, where the flow and sediment parameters for this run are given earlier in this paper. In line with the theoretical analyses of Jain & Kennedy (1974), $\beta/H = 0.72$ was adopted for the present analyses.

Calculated values of the normalized growth parameter, $c_i/(gH)^{1/2}$, are shown in figure 7 for $kH < 10$. The bed-wave results of this figure, with the abscissa plotted in terms of kH, are directly comparable with those of figure 2 of Jain & Kennedy (1974). Particular details of the two graphs differ owing to Jain & Kennedy additionally incorporating an allowance for the effect of local bed slope on sediment transport rate into their analyses.

Figure 7 confirms that the singularity evident for previous potential-flow instability models can be removed by appropriate formulation of these models. Infinite growth rate is predicted by Jain & Kennedy for $kH = 5.12$, with a maximum in growth rate predicted for $kH = 7.54$. For the present results of figure 7, the corresponding maximum in growth rate occurs at $kH = 6.84$, with no local maximum occurring at the condition of equation (1.1). In contrast to the results of Jain & Kennedy (1974), no waves are then predicted to form at this latter value of wavelength ($kH = 5.12$).

Jain & Kennedy (1971) indicate that the graph of bed-wave growth rate '... has a series of maxima at successive higher values of kH ...', Kennedy concluding that '... the maximum corresponding to the largest value of wavelength will dominate'. This

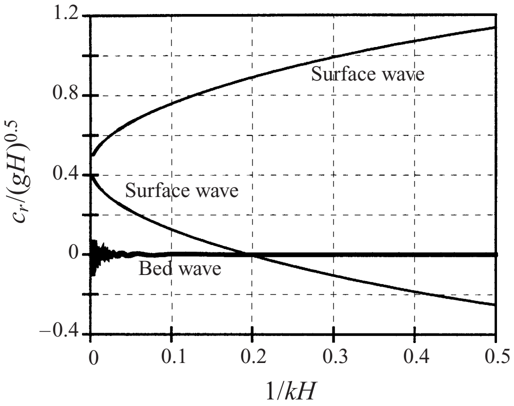

FIGURE 9. Normalized wave speeds predicted based on $\beta/H = 0.72$ for Run S8 of Jain & Kennedy (1974).

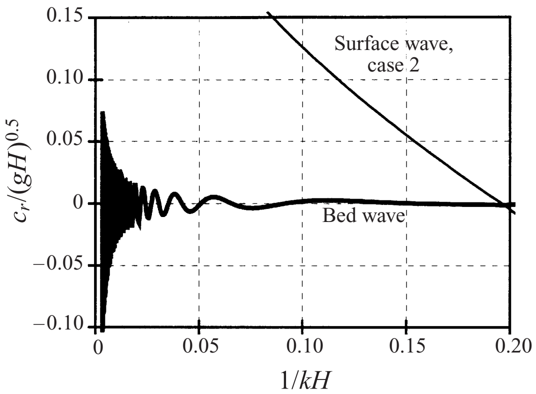

FIGURE 10. Normalized wave speeds predicted based on $\beta/H = 0.72$ for Run S8 of Jain & Kennedy (1974).

maximum is in addition to that given by the singularity evident for the quasi-steady potential flow model.

Figure 8 illustrates the series of maxima in bed-wave growth rate predicted by the present model, where $c_i = 0$ for the faster-moving (case 1) water-surface waves over the range of kH values shown. Based on conventional fluid stability considerations, whereby the wavelength of the waves expected to form from plane bed conditions is that of the largest associated (positive) predicted growth rate, the results of figure 8 indicate that waves of infinitesimal length ($kH \to \infty$ and $\lambda \to 0$) are actually predicted by the present model to form for Run S8 with $\beta/H = 0.72$. This prediction is not reflected in the experimental results presented by Jain & Kennedy (1971, 1974).

Normalized wave speeds, $c_r/(gH)^{1/2}$, as functions of wavelength for Run S8 are presented in figure 9. Wave speed results at larger values of kH are highlighted in figure 10. In contrast to the results for no explicit phase lag ($\beta = 0$), bed-wave speeds are predicted to oscillate between positive and negative values with decreasing wavelengths. That waves of wavelengths indicated by figure 8 to be unstable (for example, $c_i > 0$ for $kH = 14$) should be indicated by figure 10 to move in an upstream direction ($c_r < 0$) for Run S8 (of $Fr = 0.442$) is clearly in conflict with experimental observations.

The incorporation into conventional potential-flow analyses of a phase lag between sediment transport rate and local flow conditions in the form of the explicit lag β can be seen to result in analytical predictions contrary to observed flow system behaviour in terms of predicted lengths and also wave speeds for the unstable waves.

5. Discussion

5.1. *Significance of the present potential-flow model*

The present model advances previous potential-flow models in two principal respects. Previously, the change in bed-wave amplitude with time ($\partial\eta/\partial t$ in equation (2.3)) has been neglected based on the assumption that this term is small relative to the other two terms in the equation. This assumption is perhaps particularly questionable for sand-wave growth from plane bed conditions (for which $\partial\eta/\partial t$ can be large and $\partial\eta/\partial x$ tends to zero), which are precisely the conditions to which the assumption is applied for linearized potential-flow analyses of sand-wave development. The assumption of neglecting the term $\partial\eta/\partial t$ in equation (2.3) is also questionable in that the focus of the analyses is bed-wave growth. The present analyses advance the previous quasi-steady models by being formulated for a fully unsteady flow system, with no time-derivative terms being neglected. Sediment transport is accordingly modelled as responding to the unsteady local flow velocities and depths (equation (2.6)). Subsequent linearization of the equations then facilitates the adoption of a steady-state expression for sediment transport in the analyses.

The second principal advance comes in the formulation of the analyses. It is conventional understanding that phase shifts between the perturbations to flow, bed and water surface are necessary for prediction of flow system instability (e.g. Kennedy 1980). As indicated above, an artificial explicit phase lag between local sediment transport and local flow conditions has previously been introduced into potential-flow analyses in order to facilitate predictions of flow system instability. In contrast to previous models, the present analyses implicitly facilitate free variation of phase between the perturbations to the flow system. To this end, the flow system perturbations of equations (2.8), (2.9) and (2.10) are sinusoids incorporating the coefficients A, B, C and D which are complex, e.g. $C = c\ \exp(\mathrm{i}\beta_c)$, where c is a real coefficient and β_c is a phase shift of the bed perturbation relative to the perturbations in the flow and the water surface. This implicitly permits phase shifts between the flow, bed and water surface perturbations (of respective phase lags of β_a, β_b, β_c, and β_d). Both the determination of the dispersion relation of equation (2.13) and also the subsequent analyses of flow system instability are independent of the coefficients A, B, C, and D, and are thereby implicitly valid for all phase lags between the flow system perturbations.

For sediment being moved by the flow shear stress acting on the sediment grains, it is the lag between the bed topography and the flow shear stress that is conventionally viewed as being required for bed-form growth. By freely facilitating all lags between the bed surface η, the flow velocity ($= f(\phi)$), and the water surface ξ (or the flow depth $h = H + \xi - \eta$, or the shear stress τ or shear velocity u_*, where $\tau = \rho u_*^2 = \rho g h S$, or alternatively $\tau = f(\rho, U, h, d_{50})$ based on the turbulent-flow velocity profile), the present analyses then also appropriately facilitate lags between the flow shear stress and the bed.

An additional advance of the present model is that for the first time celerities are calculated for both the bed wave and also the two surface waves for each flow system–wavelength combination analysed. In contrast to previous models, the present model can then be used to investigate in detail any interrelations between the flow system bed and surface waves.

5.2. *Potential-flow descriptions for* $\beta \neq 0$

The similarities between the results of figure 7 and the results of the equivalent figure 2 of Jain & Kennedy (1974) confirm the accuracy of our analyses for the inclusion of an explicit phase lag β between flow conditions and sediment transport. The differences in results that are apparent are nevertheless very significant.

Previous potential-flow theories are recognized to not describe plane-bed instability in the absence of an explicit phase lag between flow conditions and sediment transport ($\beta = 0$). The present analyses reveal that for the alternative situation of $\beta \neq 0$, flow system instability is predicted by potential-flow models for a variety of wavelengths, with waves of infinitesimal lengths ($\lambda \to 0$) actually predicted to form for a given flow system. In addition, the bed waves generated from plane bed conditions are predicted to move either upstream or downstream, depending upon the length of the wave. Both of these central results are clearly contrary to observed flow system behaviour, with bed waves being observed to move downstream owing to the propagation-of-mass nature of their movement (Coleman & Melville 1994), and with waves of distinct finite lengths being observed to appear from plane bed conditions (Coleman & Melville 1996). These erroneous predictions of physical behaviour by potential flow theories for $\beta \neq 0$ have not previously been recognized.

In addition, from figures 8 and 9, instability ($c_i > 0$) is predicted for $\beta \neq 0$ in the absence of any resonance interaction between the bed and water-surface waves, with unstable (growing) bed waves predicted to move upstream and downstream at celerities significantly less than corresponding free-surface-wave celerities. For $\beta \neq 0$, instability is then arbitrarily predicted for bed waves without an apparent physical basis.

Furthermore, Jain & Kennedy (1971, 1974) and Kennedy (1980) propose a variance cascade mechanism of progressive bed development whereby smaller waves travel at faster speeds, and wave growth occurs through the smaller waves catching and coalescing with larger waves. This is in essence a valid mechanism of bed development as indicated by Coleman & Melville (1994). Nevertheless, with bed-wave speed not predicted to consistently decrease with increasing bed-wave length (figure 10), and with waves of different lengths predicted to move in upstream and downstream directions, previous potential flow instability models for $\beta \neq 0$ cannot correctly represent this mechanism of bed development. The basis of the mechanism proposed was purely conjecture, although of remarkable insight.

Finally, in contrast to the results of Jain & Kennedy (1971, 1974), the removal of the singularity at condition (1.1) for the present unsteady model results in instability not necessarily being predicted at the associated wavelength for the model with $\beta \neq 0$ (figure 7). The twin modes of instability basis for bed development processes as conjectured by Jain & Kennedy (1971, 1974) is then not predicted for a correctly formulated potential-flow description of bed development.

To date, it has been accepted that the explicit phase lag of $\beta \neq 0$ is valid and is the sole means by which potential flow analyses can be made to describe flow system instability. Based on the present findings, this explicit phase is actually an artificial measure that results in model predictions that are not consistent with observed flow system behaviour. Conventional use of the explicit phase lag of $\beta \neq 0$ in the potential-flow approach is accordingly concluded herein to be erroneous. In the absence of the explicit lag, conventional potential-flow analyses predict no flow system instability.

5.3. *Potential-flow descriptions for* $\beta = 0$

In contrast with previous models, the present potential-flow model of no explicit phase lag between flow conditions and sediment transport ($\beta = 0$) predicts flow system instability for $Fr < 0.8$. Instability is predicted to occur at finite growth rates for a range of wavelengths via a resonance mechanism occurring for surface waves and bed waves travelling at the same celerity. In addition, bed-wave speeds are predicted to decrease with increasing wavelength, and bed waves are predicted to grow and move at faster rates for flows of larger Froude numbers. These predictions are entirely consistent with observations of flow systems (Coleman & Melville 1994, 1996). With bed-wave speed predicted to consistently decrease with increasing bed-wave length, and with waves predicted to move in the downstream direction, the present potential-flow instability model of $\beta = 0$ is furthermore consistent with the recognized variance cascade mechanism of progressive bed development (Coleman & Melville 1994).

The identified resonance mechanism of flow system instability has not previously been shown to exist. Up to now (e.g. Jain & Kennedy 1971, 1974, and others thereafter), it has only been hypothesized that instability arises when bed waves and surface waves travel at a common celerity. Comparisons of measured or calculated bed-wave and surface-wave celerities have not previously been presented to confirm the occurrence of the resonance instability mechanism. For the first time, the present work calculates the three (two free-surface and one bed) wave celerities for a flow system and shows that at the point of instability, the celerity of one of the surface waves equals that of the bed wave. The present model thereby for the first time proves the existence of the bed-wave–water-surface-wave resonance mechanism of instability.

Further to the previous postulated nature of resonance for a flow system, Jain & Kennedy (1974) quote instability arising for a surface-wave train moving relative to the fluid with velocity just equal in magnitude but opposite in direction to the mean flow velocity. They then conjectured instability to occur when the surface waves appeared to be stationary. As proved by the present analyses (figure 4), however, with small bed-wave speeds of $c_{r_{\mathrm{bed}}} \ll U$, instability actually arises when the surface waves travel at celerities equal to the small bed-wave speeds, i.e. $c_{r_{\mathrm{water\ surface}}} = c_{r_{\mathrm{bed}}} \ll U$. The present work thus clarifies previous conjecture regarding the occurrence of resonance for flow systems.

The present analyses also advance previous hypotheses of resonance generation of bed waves by predicting that the resonance mechanism will not occur for large Froude numbers ($Fr > 0.8$). To this end, the present model is the first potential-flow model to describe in detail the complex interactions of bed-wave and surface-wave speeds as shown in figure 4. For larger Froude numbers, the present model shows bed and surface waves interacting to prevent their speeds from matching and resonance occurring (Coleman 1991). There is no equivalent prediction of stability at larger Froude numbers for previous potential-flow models, these models either being unduly influenced by incorporation of the erroneous explicit phase lag of $\beta \neq 0$, or predicting instability unconditionally for all flows at the singularity condition (1.1).

As indicated by figure 5 (with the exception of flow systems of $Fr > 0.8$, for which no flow system instability is predicted by the present model), the wavelength of maximum growth rate predicted by the present model with $\beta = 0$ can be described by equation (1.1). Previously, flow system behaviour at this condition could not be determined, as equation (1.1) defined a singularity in instability theory (Kennedy 1969; Gradowczyk 1970; Jain & Kennedy 1974). This relation (1.1) has nevertheless previously been interpreted as delineating regions of occurrence for ripples and dunes collectively and antidunes (Kennedy 1969; Gradowczyk 1970), or defining the lengths of waves generated at infinite growth rates from plane bed conditions (Jain & Kennedy 1974). In contrast, the present fully unsteady model has resulted in the removal of the singularity evident for previous models, with equation (1.1) defining the wavelength of bed waves expected to develop at finite growth rates from plane bed conditions for a flow system of $Fr < 0.8$.

Based on predicted unstable wavelengths (approximated by (1.1)) markedly exceeding observed sand-wavelet lengths (figure 5), and based also on respective trends in predicted and observed wavelengths with increasing flow strength, it is concluded that bed waves are not generated from plane bed conditions by any potential-flow instability mechanism. Conventional thinking is that potential-flow analyses, while being superseded by more complex models, still provide valid descriptions of plane-bed instability, with the explicit phase lag adjusted to produce the necessary results. That potential-flow analyses are inadequate for the description of flat-sediment-bed instability has not previously been shown.

The true mechanism of instability for plane alluvial beds thus lies in flow features not apparent in potential-flow descriptions of flow, and not just in phase-lag descriptions advanced from those of previous potential flows. This is an important understanding, both for use of potential-flow analyses and also for understanding of what actually is occurring when a plane bed becomes unstable. Potential-flow descriptions of flows should thus no longer form the basis of instability analyses describing bed-wave generation from plane bed conditions. The potential-theory bed–water-surface resonance mechanism is nevertheless a valid instability mechanism that is found to promote accelerated wave growth for a flow system of larger finite developing waves. As indicated above, correctly formulated potential-flow theory is also consistent with a number of further aspects of observed physical flow system behaviour.

6. Summary

The present work constitutes a review of the role of potential-flow analyses in describing alluvial-bed instability, the analyses and findings answering lingering questions posed by Gradowczyk (1968) and completing a series of research by numerous authors spanning over three decades.

The basis of the present work is an unsteady potential-flow description of alluvial-bed instability, the model predicting instability of flow systems of rippled or dune-covered equilibrium beds without inclusion of the conventional artificial explicit phase lag adopted in previous potential-flow models. The form of the present model and the associated finding of instability in the absence of an explicit phase lag have eluded all previous potential-flow analyses.

Predictions of the revised flow model prove the inadequacies of conventional potential-flow models incorporating explicit phase lags between sediment transport and local flow conditions.

In contrast, the present potential-flow model with no explicit phase lag predicts instability to occur at finite growth rates for a range of wavelengths via a resonance mechanism occurring for surface waves and bed waves travelling at the same celerity. In addition, bed-wave speeds are predicted to decrease with increasing wavelength, and bed waves are predicted to grow and move at faster rates for flows of larger Froude numbers. All predictions of the present potential-flow model are consistent with observations of physical flow systems. Based on the predicted unstable wavelengths for a given alluvial flow, it is concluded that bed waves are not generated from plane bed conditions by any potential-flow instability mechanism. Potential-flow description of alluvial flows should thus no longer form the basis of instability analyses describing bed-form generation from flat bed conditions. Nevertheless, potential-flow theory is actually shown to correctly predict flow system instability in the form of accelerated wave growth for flow systems of larger finite developing waves.

The present work corrects three decades of misunderstanding of basic flow behaviour, clarifies flow understanding that underpins other flow descriptions, and corrects several aspects of presumed established knowledge that have actually never before been proven. The understanding of the mechanics of alluvial stream bed forms as inferred from previous potential-flow instability models, which require $\beta \neq 0$ for prediction of flow system instability, must be reconsidered with the implications of the present results in mind.

The relevance of the present work to the description of alluvial bed waves is proven by the centrality of the potential-flow approach (over three decades) to describing the given phenomenon, the importance of the present findings to evaluation of the validity of the potential-flow approach to describing alluvial-bed instability, and the implications of the present findings for more complex instability analyses. In regard to the last perspective, it is recognized that analytical descriptions of flows have advanced since the potential-flow descriptions of the 1960s, but these advances have come with an inadequate understanding of the underlying potential-flow description. In order to properly understand rotational-flow predictions, one first needs to recognize what potential-flow analyses really predict, and the true limitations of these underlying analyses. Similarly, in order to determine which aspect of the alluvial flow system produces plane-bed instability, the true contribution to instability of the potential-flow aspect of the flow, as highlighted in the present work, needs to be recognized.

The present advances in theory and findings are entirely consistent with potential-flow description of alluvial flows. With current understanding of alluvial-bed instability built on flawed analyses and flawed understandings of flows, the present work provides a necessary step towards obtaining a full understanding and description of this phenomenon. In this regard, basic analyses and understanding are corrected herein such that crucial terms in the analyses can be identified, correct understanding of flows can be obtained, and paths to be pursued to achieve the goal of correct description of alluvial-bed instability can be refined.

The financial support of these studies by the Stiftung Volkswagenwerk, Hanover, is gratefully acknowledged. The writers also wish to express their thanks to Associate Professor B. W. Melville and Emeritus Professor A. J. Raudkivi for their comments and suggestions. The direction and comments of the anonymous reviewers is also gratefully acknowledged.

REFERENCES

Abramowitz, M. & Stegun, I. A. 1968 *Handbook of Mathematical Functions*. Dover.

Coleman, S. E. 1991 The mechanics of alluvial stream bed forms. PhD thesis, The University of Auckland, Auckland, New Zealand.

Coleman, S. E. 1996 Wave generation and development on a sandy river bed. Discussion of 'The stability of a sandy river bed', by J. Fredsøe. In *Issues and Directions in Hydraulics* (ed. T. Nakato & R. Ettema), pp. 145–155. Balkema.

Coleman, S. E. & Fenton, J. D. 1996 Potential flow instability theory and bed forms. *Proc. North American Water and Environment Congress '96, Anaheim, California, USA*. ASCE 6 pages.

Coleman, S. E. & Melville, B. W. 1994 Bed-form development. *J. Hydr. Engng ASCE* **120**, 544–560.

Coleman, S. E. & Melville, B. W. 1996 Initiation of bed forms on a flat sand bed. *J. Hydr. Engng ASCE* **122**, 301–310.

Engelund, F. A. 1970 Instability of erodible beds. *J. Fluid Mech.* **42**, 225–244.

Gradowczyk, M. H. 1968 Wave propagation and boundary instability in erodible-bed channels. *J. Fluid Mech.* **33**, 93–112.

Gradowczyk, M. H. 1970 Interfacial instability between fluids and granular beds. *IUTAM Symp. on Instability of Continuous Systems, Herrenalb, 1969* (ed. H. Leipholz), pp. 143–150.

Guy, H. P., Simons, D. B. & Richardson, E. V. 1966 Summary of alluvial channel data from flume experiments, 1956–61. *US Geol. Surv. Prof. Paper* 462-I. Washington, DC.

Hayashi, T. 1970 Formation of dunes and antidunes in open channels. *J. Hyd. Div. ASCE* **96**, 357–366.

Jain, S. C. & Kennedy, J. F. 1971 The growth of sand waves. *Proc. Intl Symp. on Stochastic Hydr.*, pp. 449–471. Pittsburgh University Press.

Jain, S. C. & Kennedy, J. F. 1974 The spectral evolution of sedimentary bed forms. *J. Fluid Mech.* **63**, 301–314.

Kennedy, J. F. 1963 The mechanics of dunes and antidunes in erodible-bed channels. *J. Fluid Mech.* **16**, 521–544.

Kennedy, J. F. 1969 The formation of sediment ripples, dunes and antidunes. *Ann. Rev. Fluid Mech.* **1**, 147–168.

Kennedy, J. F. 1980 Bed forms in alluvial streams: some views on current understanding and identification of unresolved problems. In *Application of Stochastic Processes in Sediment Transport* (ed. H. W. Shen & H. Kikkawa), pp. 6a/1–6a/13. Water Resources Publications, Littleton, CO.

Pantin, H. M. 1990 Potential flow over a wavy bed: a persistent error. *Proc. Intl Symp. on Sediment Transport Modeling* (ed. S. Y. Wang), pp. 76–81. ASCE.

Reynolds, A. J. 1976 A decade's investigation of the stability of erodible stream beds. *Nordic Hydrology* **7**, 161–180.

Richards, K. J. 1980 The formation of ripples and dunes on an erodible bed. *J. Fluid Mech.* **99**, 597–618.

Sumer, B. M. & Bakioglu, M. 1984 On the formation of ripples on an erodible bed. *J. Fluid Mech.* **144**, 177–190.

Vanoni, V. A. 1975 *Sedimentation Engineering. Manuals and Reports on Engineering Practice*, Vol. 54. ASCE.