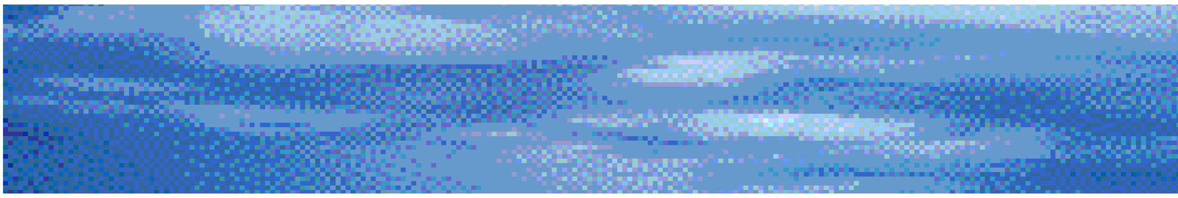
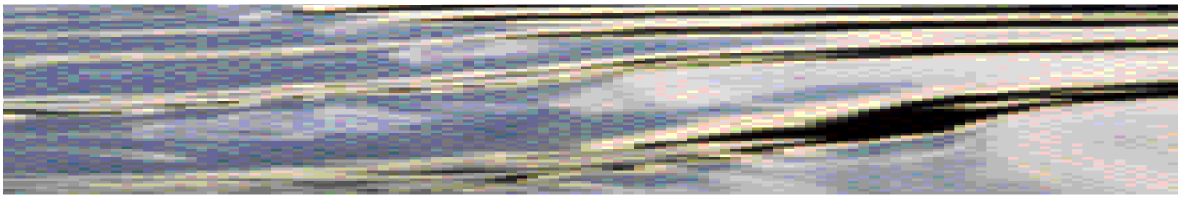
COOPERATIVE RESEARCH CENTRE FOR **CATCHMENT HYDROLOGY**

THE CALCULATION OF STREAMFLOW FROM MEASUREMENTS OF STAGE

TECHNICAL REPORT
Report 01/6
September 2001

John D. Fenton and Robert J. Keller





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Preface

A major dilemma in flood estimation is that the data on the largest observed floods, which should carry the greatest amount of information relevant to the estimation of extreme floods, are also the ones with the lowest level of accuracy and reliability. The inadequacy of rating curves, used to convert measured water levels to estimated flow rates, is a major source of the error.

Rating curve errors impose severe limitations on further improvements in flood estimation methods. They introduce bias into both at-site and regional flood frequency estimates and make it difficult to identify the true degree of non-linearity in catchment response to storm rainfall. Accordingly, improvements in the determination of the high flow end of rating curves is regarded as vital to further improvements in the reliability of flood estimates and was a major objective of Project FL3.

In this report, a number of issues with regard to rating curve determination are addressed. By building on basic hydraulic principles, this report demonstrates that existing practices are inadequate and the authors document a number of techniques to improve current practice.

Robert Keller
Project Leader
Initial CRC Flood Hydrology Program (1997-99)

Abstract

The calculation of river flow from measurements of surface elevation (stage) is a fundamental problem in river engineering that has been surprisingly little investigated. Existing International and Australian Standards reflect this lack of knowledge. The usual way in which stage and flow at a station are shown as being related is via a rating curve. Yet, the concept of a unique rating curve at a station is theoretically flawed. The hydraulics of river flow show that slope as well as stage is a determinant of flow, and in principle it would be best also to measure the slope at a gauging station, and use this also to calculate discharge. In practice the slope often does not vary much over a range of flows and rates of changes of flow at a point; the reasonable and convenient assumption is implicitly made that flow and stage are connected by a unique relationship that can be represented by a rating curve. Unfortunately the accuracy of this approach is rarely examined in practice.

This report considers a number of aspects of the problem of representing stage and discharge measurements and using them to calculate flow from subsequent stage measurements. A critique of existing Standards is presented. The hydraulics of flow in channels and at gauging stations is considered. Procedures are described which could be implemented if slope also were measured. As this is rarely done, a method is developed here which gives a correction to the flow calculated from a rating curve allowing for the variation of elevation (and slope) with time. This provides a correction to the well-known Jones method. Formulae are presented to estimate when these unsteady effects are significant. For most gauging stations they are small, and a unique rating curve is a good approximation. Nevertheless the unsteady corrections are simply implemented, and a number of practical details for that are presented here.

A theoretical model is developed for a river with a gauging station affected by a local control. This is used to give an expression for the rating curve for low flows. Another expression is given for higher flows when the local control washes out, and the channel itself provides the control. The theory for both these ranges shows that in many cases the stage will vary approximately like the square root of discharge. This can be used to calculate an approximate rating curve in the absence of other information, or preferably, together with one or two ratings, to calibrate such a model. It is suggested that in presenting and approximating rating curves, plotting the square root of discharge against the stage has some advantages. Many data points from gaugings should plot roughly as a straight line, which can help the determination of the cease-to-flow point, as well as the possible extrapolation of the curve at high flows. It is shown how global approximation of the rating data can be implemented via a robust numerical method.

Summary

The usual measurement of the state of a river at a gauging station is that of the stage, the surface elevation. While this is important in determining the danger of flooding, the volume flow rate past the gauging station is also important, notably to hydrologic investigations and practical operations with a river. Stage and discharge at a station are usually related by a rating curve determined from measurements. It is surprising that there has been little research done on the theoretical and practical aspects of this relationship. Reference books and publications of standards authorities show how limited is the state of the art. There has been little effort spent on applying hydraulics to the problem. Instead, relatively crude models and assumptions are still conventional practice, and usually the accuracy of these has not been tested. The main aims of the research of which this report is the main product have been

- To improve current methods of converting measured water levels to flow rates, especially for high flows; and
- Thereby to improve the reliability of flood estimates.

The report is divided into two main parts. The first part is a more descriptive presentation that is intended to be able to be read without it being necessary to refer to the second, which consists of appendices providing technical details, as well as a presentation of the hydraulics of river flow. Initially in the first part the physical problem of river flow past a gauging station is described. It is shown how the conventional method of relating flow uniquely to stage is only part of the necessary physical description, and that the concept of a unique rating curve is theoretically flawed. If the slope of the river surface at the gauging station changes, whether due to conditions downstream changing, or to the passage of a flood, then calculating flows from the rating curve may be in error. At this stage reference is made to an appendix containing a review of the Australian and International Standards for calculating flow from stage, and some strong criticisms are made of them for providing little serious help in solving the technical problems encountered.

An introduction to river hydraulics is then provided. It is shown how it would be better if the slope were measured routinely, and instead of stage-discharge relationships, stage-conveyance relationships were used, automatically including effects of backwater from downstream and unsteady flood effects.

Next, the problem of correcting for a varying surface slope without measuring it directly is considered. A review of previous approaches is given, and a method is developed which expresses the surface slope in terms of the time derivatives of the stage. This gives a correction to the flow that is effectively an extension to the well-known Jones method. Formulae are given to estimate when unsteady effects are worth correcting for, and some examples are presented. It is concluded that for most rivers the unsteady effects are small, but it is emphasised that relatively little extra effort is necessary to implement the corrections.

The problem considered next is that of the hydraulic derivation of rating curves when there is little rating information available. A mathematical model is developed for a reach of river with a gauging station and local control. This is then used to predict the rating curve for low flows, and when the control washes out, for high flows. In practice, the nature of the local control will often be too complicated to calculate the low flow end of the rating curve. However the use of theory for both low and mid- to high flows leads to the deduction that in many cases the stage varies like the square root of discharge, both at low flows and for mid to high flows. It is concluded that in the absence of any other information this can be used, together with one or two ratings, to generate a rating curve.

In the next section, the representation and approximation of rating curves is considered. A critique is made of log-log plots, and it is suggested that plotting the square root of discharge against the stage has some advantages. In particular, both for low and high flows, many data points from gaugings should plot in almost straight lines, which can help the determination of the cease-to-flow point, as well as the possible extrapolation of the curve at high flows. A number of examples are considered and it is concluded that this suggestion has promise. Finally it is shown how global approximation of the rating data can be implemented via a robust numerical method to generate rating curves.

Acknowledgements

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1. Introduction

1.1 Background

At the AWRC Workshop on Surface Water Data in Canberra in 1983, J.A.H. Brown, the surface water resource consultant for the Commonwealth Government's 'Water 2000' study, called for the need to examine and "rationalise the whole question of discharge rating curves". It "has received relatively little attention in technical literature" and "is covered in a general fashion in the stream gauging manuals", quoted in Chester (1986).

Since 1983 there has been little progress along those suggested directions, despite the importance of determining the actual state of a river, both under extreme events and in daily hydrologic practice. To address some of problems identified by the water industry and by researchers, in September 1997 the Cooperative Research Centre for Catchment Hydrology began a research program "Hydraulic Derivation of Stream Rating Curves", which had as its main objectives:

- To improve current methods of converting measured water levels to flow rates, especially for high flows; and
- Thereby to improve the reliability of flood estimates.

This report is the main outcome of the project.

1.2 Nature of the problem - river hydraulics and rating curves

Almost universally the routine measurement of the state of a river is that of the *stage*, the surface elevation at a gauging station, usually relative to an arbitrary local datum. While surface elevation is an important quantity in determining the danger of flooding, another important quantity is the actual flow rate past the gauging station. Accurate knowledge of this instantaneous discharge - and its time integral, the total volume of flow - is crucial to many hydrologic investigations and to practical operations of a river and its chief environmental and commercial resource, its water. Examples include decisions on the allocation of water resources, the design of reservoirs and their associated spillways, the calibration of models, and the interaction with other computational components of a network.

The traditional way in which volume flow is inferred is for a *rating curve* to be derived for a particular gauging station, which is a relationship between the stage measured and the flow passing that point. The measurement of flow is done at convenient times by traditional hydrologic means, with a current meter

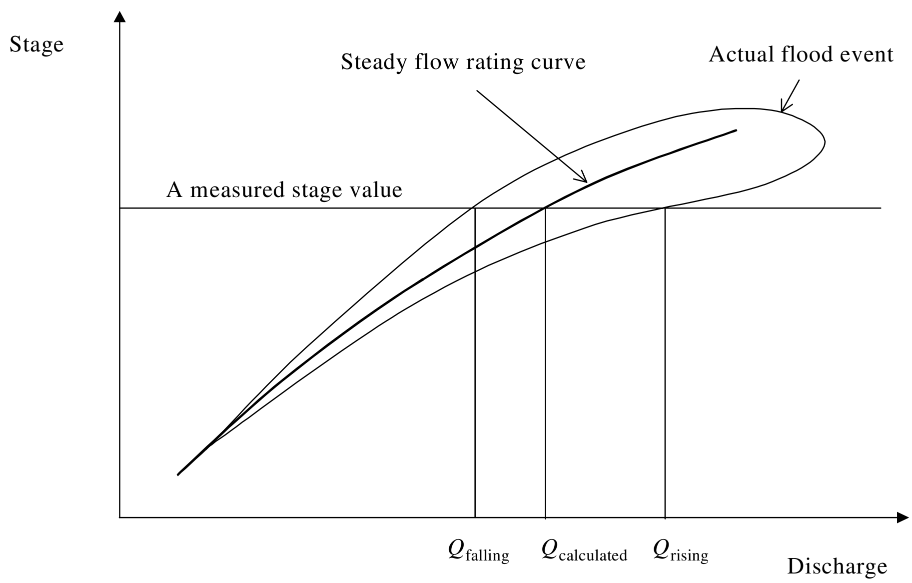

Figure 1.1 Stage-discharge diagram showing the steady-flow rating curve and a possible trajectory of a particular flood event showing the effect of a larger surface slope on the rising limb and smaller slope on the falling limb.

measuring the flow velocity at enough points over the river cross section so that the volume of flow can be obtained for that particular stage, measured at the same time. By taking such measurements for a number of different stages and corresponding discharges over a period of time, a number of points can be plotted on a *stage-discharge diagram*, and a curve drawn through those points, giving what is hoped to be a unique relationship between stage and flow, the *rating curve*, as shown in Figure 1.1. This is then used in the future so that when stage is routinely measured, it is assumed that the corresponding discharge can be obtained from that curve, such as the discharge for a particular value of stage, $Q_{\mathrm{calculated}}$, shown in the figure. The possibly looped trajectory of a flood event shown on the figure will be explained shortly below.

There are several problems associated with the use of a rating curve:

- The assumption of a unique relationship between stage and discharge is, in general, not justified.
- Discharge is rarely measured during a flood, and the quality of data at the high flow end of the curve might be quite poor.
- It is usually some sort of line of best fit through a sample made up of a number of points - sometimes extrapolated for higher stages.
- It has to describe a range of variation from no flow through small but typical flows to very large extreme flood events.

There are a number of factors which might cause the rating curve not to give the actual discharge, some of which will vary with time. Boyer (1964) described a list of factors affecting the rating curve, or what he called a *shifting control*. These include:

- The channel changing as a result of modification due to dredging, bridge construction, or vegetation growth.
- Sediment transport - where the bed is in motion, which can have an effect over a single flood event, because the effective bed roughness can change during the event. As a flood increases, any bed forms present will tend to become larger and increase the effective roughness, so that friction is greater after the flood peak than before, so that the corresponding discharge for a given stage height will be less after the peak. This will also contribute to a flood event showing a looped curve on a stage-discharge diagram as shown on Figure 1.1. Simons and Richardson (1962) have extensively examined this phenomenon.
- Backwater effects - changes in the conditions downstream such as the construction of a dam or flooding in the next waterway downstream.
- Unsteadiness - in general the discharge will change rapidly during a flood, and the slope of the water surface will be different from that for a constant stage, depending on whether the discharge is increasing or decreasing. The effect of this is for the trajectory of a flood event to appear as a loop on a stage-discharge diagram such as Figure 1.1.
- Variable channel storage - where the stream overflows onto flood plains during high discharges, giving rise to different slopes and to unsteadiness effects.
- Vegetation - changing the roughness and hence changing the stage-discharge relation.
- Ice - which we can ignore in the Australian context.

Some of these can be allowed for by procedures that we develop in this report.

A typical set-up of a gauging station where the water level is regularly measured is given in Figure 1.2 that shows a longitudinal section of a stream. Downstream of the gauging station is usually some sort of fixed control which may be some local topography such as a rock ledge which means that for relatively small flows there is a relationship between the head over the control and the discharge which passes. This will control the flow for small flows. For larger flows the effect of the fixed control is to become unimportant, to “drown out”, and for some other part of the stream to control the flow. If the downstream channel length is long enough before encountering another local control or waterway, the section of channel downstream will itself become the control, where the control is due to friction in the channel, giving a relationship between the flow and the slope in the channel, the stage, channel geometry, and roughness. Finally, control might be due to a larger river downstream shown as a distant control in the figure, or even the sea. There may be more intermediate controls too, but in practice, the precise natures of the controls are usually unknown.

Something that the concept of a rating curve ignores is the effect of unsteadiness, or variation with time. In a flood event the slope of the water surface will be different from that for a constant stage, depending on whether the discharge is increasing or decreasing. Figure 1.2 shows the increased surface slope as a flood approaches the gauging station. As the flood increases, the surface slope in the river is greater than the slope for steady flow at the same stage, and hence, according to conventional simple hydraulic theory explained below, more water is flowing down the river than the rating curve would suggest. The effect of this is shown on Figure 1.1, with the discharge marked Q_{rising} obtained from the horizontal line drawn for a particular value of stage. When the water level is falling the slope and hence the discharge inferred is less.

The effects of this might be important - the peak discharge could be significantly underestimated during highly dynamic floods, and also since the maximum discharge and maximum stage do not coincide, the arrival time of the peak discharge could be in error and may influence flood warning predictions. Similarly water-quality constituent loads could be underestimated if the dynamic characteristics of the flood are ignored, while the use of a discharge hydrograph derived inaccurately by using a single-valued rating relationship may distort estimates for resistance coefficients during calibration of an unsteady flow model.

1.3 A critique of the International and Australian Standards for the determination of the stage-discharge relation

It might be expected by practitioners that good instructions and guidance would be provided by the International and Australian Standards, which are the same, the latter being a reproduction of the former. However, the Standards seem confusing and poorly prepared. In many places they state the obvious, but where sophisticated or modern methods are required they have almost no serious guidance for practical implementation. In Appendix A of this report a detailed review of the Standards is presented. That review stands alone, and is not described here. Instead, we progress to an introduction to the physical processes at work in a river and at gauging stations.

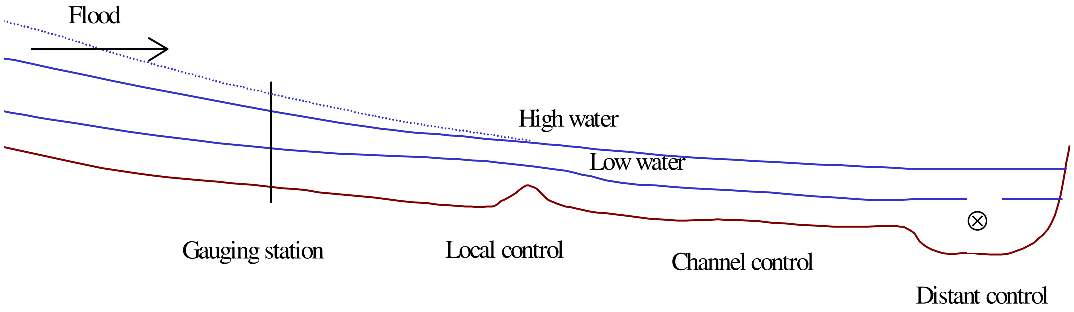

Figure 1.2 Section of river showing different controls at different water levels and a flood moving downstream

2. An Introduction to River Hydraulics - Stage, Slope, and Discharge

Although the factors affecting the stage and discharge at a gauging station shown in Figure 1.2 seem complicated, the underlying processes are capable of quite simple description.

2.1 Low flow

If the flow is small then the gauging station is likely to be under the influence of a local control downstream. There might be something like a rock ledge or an installed weir providing a unique relationship between stage and flow at that point. Between that point and the gauging station upstream, unless they are very close together, the governing equations for open channel flow will apply.

In Appendix D the hydraulics of a downstream control and a gauging station are considered, and the problem is solved assuming that the channel between the two is prismatic, and obtaining an analytic solution to the gradually varied flow equations. In Appendix D.4 it is shown that in the *low flow* limit at the gauging station,

$$h_G = h_{\text{csf}} + H_c(Q) + \text{Terms of order } Q^2, \qquad \text{(D.10)}$$

where h_G is the depth at the gauging station, h_{csf} is the cease-to-flow depth there, $H_c(Q)$ is the head at the control, shown as a function of discharge Q. Generally, for sharp and broad-crested weirs this will vary like $Q^{2/3}$ or $Q^{1/2}$, whereas the neglected terms of order Q^2 give the head loss between the gauging station and the control. For sufficiently low flow these will be negligible, showing that the stage-discharge relationship is given by that of the control, and the reach between gauging station and control is essentially a reservoir with flow through it, where dynamical effects on the surface are negligible, and the surface is horizontal with elevation given by the control. This makes matters rather difficult in deriving a rating curve theoretically, as the complicated geometry of natural controls makes a theoretical determination usually not possible.

What this and the actual detailed solution in Appendix D.3 show is that there is a relationship between stage and discharge at the gauging station when it is under the effects of the local control. This is no longer the case for larger flows where the control is washed out, and control is effectively by stream control, which we now consider.

2.2 Intermediate and high flows

In a typical stream, where all wave motion is of relatively long time and space scales, the governing equations are the long wave equations, which are a pair of partial differential equations for the stage and the discharge at all points of the channel in terms of time and distance along the channel. These are presented in Appendix B.1. One is a mass conservation equation, the other a momentum equation. Under the conditions typical of most flows and floods in natural waterways, however, the flow is sufficiently slow that the equations can be simplified considerably. In Appendix B.2 it is shown that most terms in the momentum equation are of a relative magnitude given by the square of the Froude number, which is U^2 / gd , where U is the fluid velocity, g is the gravitational acceleration, and d is the mean depth of the waterway. In most rivers, even in flood, the square of the Froude number is small. For example, a flow of 1 m/s with a depth of 2m has $F^2 \approx 0.05$. Under these circumstances, as shown in the Appendix, a surprisingly good approximation to the momentum equation of motion for flow in a waterway is the simple equation from (B.26) in the Appendix:

$$\frac{\partial \eta}{\partial x} + S_f = 0, \qquad (2.1)$$

where η is the surface elevation, x is distance along the waterway and S_f is the friction slope. The usual practice is to use an empirical friction law for the friction slope in terms of a conveyance function K, so that we write

$$S_f = \frac{Q^2}{K^2}, \qquad (2.2)$$

in which Q is the instantaneous discharge, and where the dependence of K on stage at a section may be determined empirically, or by a standard friction law, such as Manning's or Chézy's law:

$$\text{Manning's law: } K = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}} \qquad \text{or Chézy: } K = C\frac{A^{3/2}}{P^{1/2}}, \qquad (2.3)$$

where n and C are Manning's and Chézy's coefficients respectively, while A is cross-sectional area and P is wetted perimeter, which are both functions of depth and x, as the cross-section usually changes along the stream. In most hydrographic situations K would be better determined by direct measurements rather than by these formulae as they are approximate only and the roughness coefficients are usually poorly known.

Even though Manning's and Chézy's laws were originally intended for flow which is both steady (unchanging in time) and uniform (unchanging along the waterway), they have been widely accepted as the governing friction equations in more generally unsteady and non-uniform flows. Hence, substituting (2.2) into (2.1) gives us an expression for the discharge, where we now show the functional dependence of each variable:

$$Q(t) = K(\eta(t))\sqrt{S_\eta(t)}\,, \qquad (2.4)$$

where we have introduced the symbol $S_\eta = -\partial\eta/\partial x$ for the slope of the free surface, positive in the downstream direction, in the same way that we use the symbol S_f for the friction slope. This gives us an expression for the discharge at a point and how it might vary with time. Provided we know (1) the stage and the dependence of conveyance K on stage at a point from either measurement or Manning's or Chézy's laws, and (2) the slope of the surface, equation (2.4) gives a formula for calculating the discharge Q which is as accurate as is reasonable to be expected in river hydraulics. It shows how the discharge actually depends on both the stage and the surface slope.

Traditional hydrography assumes that discharge depends on stage alone. In many situations this might be a good approximation to reality, if the slope does not vary much. If the slope does vary under different backwater conditions or during a flood, then a better hydrographical procedure would be to gauge the flow when it is steady, *and to measure the surface slope* S_η, thereby enabling a particular value of K to be calculated for that stage. If this were done over time for a number of different stages, then a *stage-conveyance* relationship could be developed which should then hold whether or not the slope is varying. Subsequently, in day-to-day operations, if the stage and the surface slope were measured, then the discharge calculated from equation (2.4) should be quite accurate, within the relatively mild assumptions made so far.

If hydrography had followed the path described above, of routinely measuring surface slope and using a stage-conveyance relationship, the science would have been more satisfactory. Effects due to the changing of downstream controls with time, downstream tailwater conditions, and unsteadiness in floods would have been automatically incorporated, both at the time of determining the relationship and subsequently in daily operational practice. However, for the most part slope has not been measured, and hydrographical practice has been to use rating curves instead. The assumption behind the concept of a *discharge-conveyance* relationship or *rating curve* is that the slope at a station is constant over all flows and events, so that the discharge is a unique function of stage $Q_r(\eta)$ where we use the subscript r to indicate the rated discharge. Instead of the empirical-rational expression (2.4), traditional practice is to calculate discharge from the equation

$$Q(t) = Q_r(\eta(t))\,, \qquad (2.5)$$

thereby ignoring any effects that downstream backwater and unsteadiness might have, as well as the possible changing of a downstream control with time.

By comparison, equation (2.4), based on a convenient empirical approximation to the real hydraulics of the river, contains the essential nature of what is going on in the stream. It shows that, although the conveyance might be a unique function of stage that it is possible to determine by measurement, because the surface slope will in general vary throughout different flood events and downstream conditions, discharge in general does not depend on stage alone.

The above argument suggests that ideally the concept of a stage-discharge relationship be done away with, and replaced by a stage-conveyance relationship. Of course in many, even most, situations it might well be that the surface slope at a gauging station does vary but little throughout all conditions, in which case the concept of a stage-discharge relationship would be accurate. Below we quantify this and obtain criteria. In most situations it is indeed the case that there is little deviation of results from a unique stage-discharge relationship.

3. The Use of Surface Slope in Determining Flow

We have seen above, that according to the current state of knowledge of river hydraulics, equation (2.4) shows how discharge depends on stage and free-surface slope in the relationship $Q = K(\eta)\sqrt{S_\eta}$, where here we have not shown the time dependence of each quantity, merely noting that if we know the conveyance-stage relationship, then a measurement of stage and of slope gives the discharge. This knowledge is implicit in the method we are about to describe. Subsequently we place it on a more secure footing and recommend it as the ideal method to use, while recognising that it is easier just to use measurements of stage.

3.1 The stage-fall-discharge method of streamflow measurement

This is described at some length in Chapter 8 of Herschy (1995), where there is considerable discussion of the presence of "backwater", and a reader might be forgiven for finding the term confusing. For example, it contains the following paragraph, which is actually an insightful description, but it becomes rather clearer if "backwater" is replaced by "surface slope" in some occurrences:

> *Backwater is one of these factors whereby the velocity is retarded so that a higher stage is necessary to maintain a given discharge than would be necessary if the backwater were not present. Backwater is caused by constrictions such as narrow reaches of a stream channel or artificial structures downstream such as dams or bridges or downstream tributaries. All of these factors can increase or decrease the energy gradient for a given discharge and cause variable backwater conditions. If, however, the backwater caused by a fixed obstruction is always constant at any given stage, the discharge rating is a function of stage only. Constant backwater, as caused by section controls for example, will not adversely affect the simple stage-discharge relation. The presence of variable backwater, on the other hand, does not permit the use of simple stage -discharge relations for the accurate determination of discharge.*

If one adopts the relationship expressed by equation (2.4) then it seems that what is being described is actually the effects of variable surface slope. The introduction of surface slope might clarify some confusing terminology in the literature. Herschy goes on to describe

> *... the so-called stage-fall-discharge method using a reference gauge (base gauge), at which stage is measured continuously and current meter measurements are made occasionally, and an auxiliary reference gauge some distance downstream where stage is also measured continuously. When the two reference gauges are set to the same datum, the difference between the two stage records is the water surface fall and this provides a measure of water surface slope. The shorter the slope reach, the closer the relation between fall and water surface slope. On the other hand, the longer the slope reach, the smaller the percentage uncertainty in the recorded fall.*
>
> *Precise time synchronisation between base gauge and auxiliary gauge is very important if stage changes rapidly, or when fall is small, Reliable records can usually be computed when fall exceeds about 0.1m.*
>
> *... Under backwater conditions, therefore, the fall as measured between the base gauge and the auxiliary gauge is used as a third parameter and the rating becomes a stage-fall-discharge relation.*

We would go further than that and observe that, if the water slope does vary at a station, then the slope (or fall) should *always* be used as an additional parameter.

Both Australian Standard AS 3778.2.3 (1990) and Herschy (1995) describe two Fall methods. One is the *Constant Fall Method*, and the other (variously described as *Normal Fall* or *Limiting Fall*) seems to be a procedure for applying the constant fall method "when discharge is affected by backwater" and using simple rating methods when it is not so affected. To the writers the distinction seems arbitrary, and if one has the measuring and recording equipment in place and operational, then the slope (or fall) should always be used. Our sympathy for the Normal Fall or Limiting Fall methods has not been helped, either by the misleading nomenclature or the inadequate description in Australian Standard AS 3778.2.3, which is the same as the International Standard, or the complicated and arbitrary presentation in Herschy (1995).

The Constant Fall Method, also poorly named, is based on the equation, presented in Australian Standard 3778.2.3 without justification, and which uses misleading and confusing terminology that we try to avoid here. Consider the equation

$$\frac{Q}{Q_n} = f\left(\frac{\Delta\eta}{\Delta\eta_n}\right), \tag{3.1}$$

where Q is the discharge corresponding to the fall in surface level $\Delta\eta$ between the two gauges, Q_n is the discharge which would occur for the same stage if the fall were $\Delta\eta_n$, some standard or reference fall, and $f\,()$ shows a functional relationship. It is supposed that enough information is present to determine the functional relationship, and the references vaguely describe a procedure for determining it. Then it is noted that the procedure may be "aided by the fact that as a first approximation" the functional relationship is such that

$$\frac{Q}{Q_n} = \sqrt{\frac{\Delta\eta}{\Delta\eta_n}}\ . \tag{3.2}$$

The description then goes on to suggest that this may be used to refine the determination of the function $f\,(\Delta\eta/\Delta\eta_n)$. The Standard seems to be suggesting here a simultaneous determination of two functions, both for Q_n as a function of stage, and the relationship $f\,(\Delta\eta/\Delta\eta_n)$, which would seem to be a very ambitious goal, particularly based on the limited data that are usually available.

Equation (3.2) is actually based on Manning's and Chézy's friction laws that are both well established and widely used in other applications. They could surely be adopted, with no need to go to the more general formulation (3.1). It is not the role of the hydrographer to determine the nature of the friction law based on a limited and probably not very accurately known data set. A more satisfactory procedure would be to use equation (2.4), which shows how discharge depends on stage and free-surface *slope* (which determines the fall of course) in the relationship $Q = K(\eta)\sqrt{S_\eta}$, where there is a considerable corpus of knowledge as to how the conveyance depends on cross-section and roughness.

To test the incorporation of slope (fall), with a view to developing below a general method based on two point measurements of stage, we considered several examples provided in the references.

In Figure 3.1 are the results from Figure 5 of Australian Standard AS 3778.2.3 (1990), presented here in Table F.1 in the Appendices, which gives a value for fall for each individual point, noting that the standard fall was 0.30m. We scaled the values from the source using a ruler, then took the individual falls and scaled discharges Q and used equation (3.2) to compute the reference discharge Q_n . Both sets of points are plotted on the figure. It can be seen that incorporating the fall has led to a noticeable clustering of the points onto a curve. We fitted polynomials to each set of points and found that the relative scatter of the slope-corrected points from the curve was about half that of the raw data.

We repeated this exercise for the data given in Figure 8 of Australian Standard AS 3778.2.3 (1990). The values were scaled off, and are presented here in Table F.2 and Figure 3.2. Values of fall were given in the original, but not the reference fall, so that the 'slope corrected' discharge is actually the quantity $Q/\sqrt{\Delta\eta}$, which as the values of the fall were roughly 1m, meant that the two curves are plotted close together. In this case there seems to be little effect of correcting using the fall, and a polynomial fit and computation of the variation showed the two sets of results to have very similar scatter.

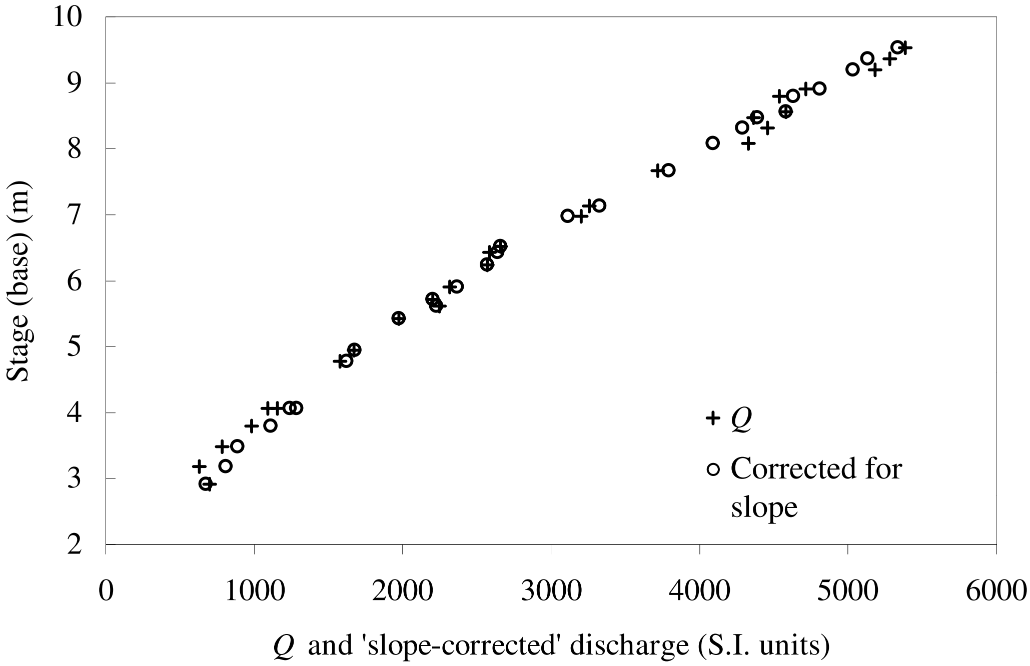

Figure 3.1 Example from Figure 5 of Australian Standard AS 3778.2.3 (1990) showing measured discharges and values corrected for effects of slope.

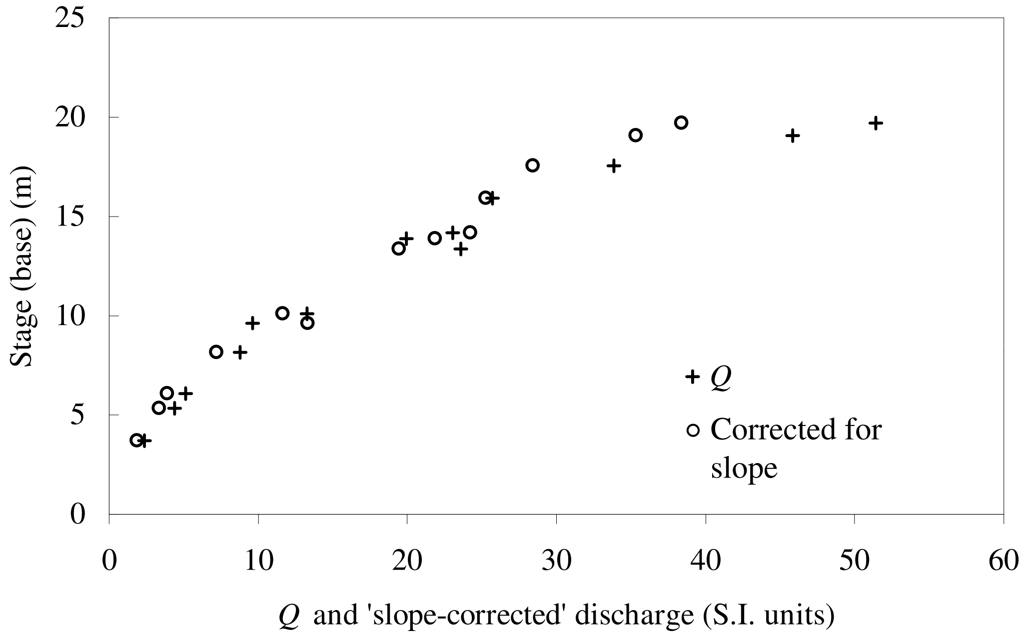

Figure 3.2 Data from Figure 8 of Australian Standard 3778.2.3 (1990)

The results from Table 8.1 of Herschy (1995), reproduced here in Table F.3, show some unusual behaviour, in that there is a markedly linear relationship between stage and discharge, unlike any other results we have seen. The results were described as being for a station with severe backwater from a hydroelectric dam, and it is quite possible that there are unsteady effects present. In spite of this, the results of Figure 3.3 show some condensing onto a single curve.

Of rather more interest are the points from Table 8.5 of Herschy (1995), reproduced in Table F.4, and shown in Figure 3.4. In this case there is a substantial coalescing of the points when discharge was divided by the square root of the fall.

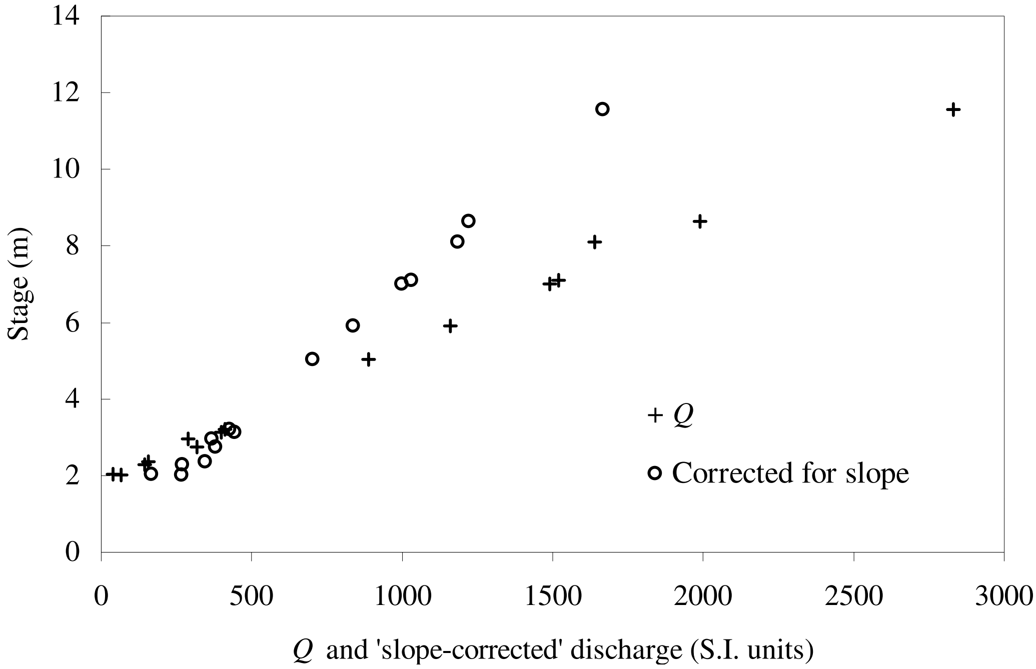

Figure 3.3 Results from Table 8.1 of Herschy (1995).

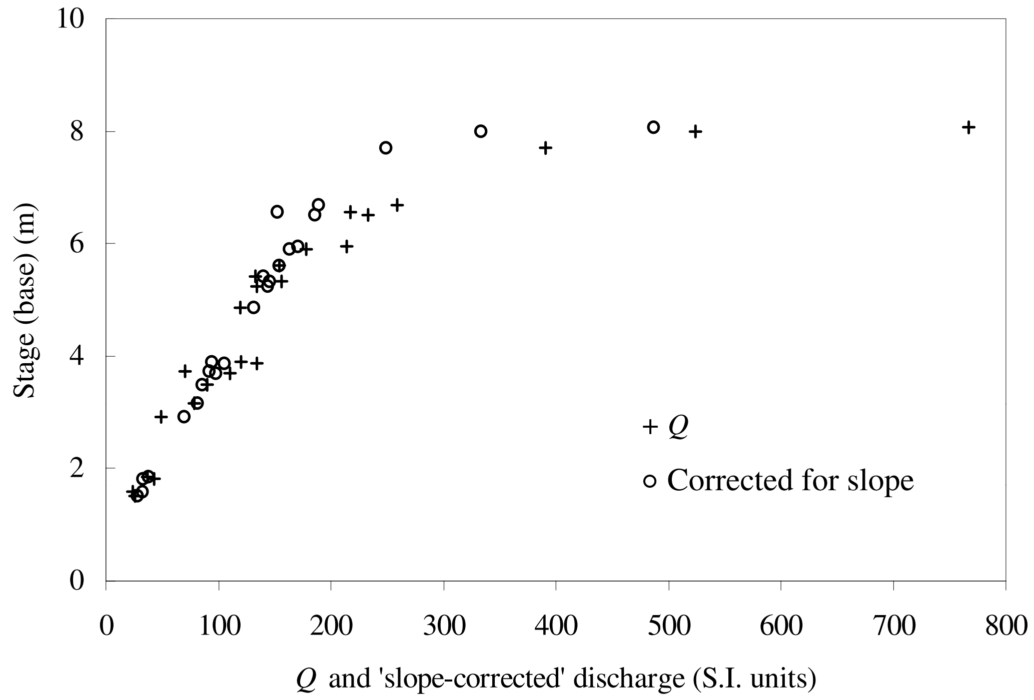

Figure 3.4 Results from Table 8.5 of Herschy (1995)

3.2 The use of slope in determining flow

It is strange that the references cited above seem to have obtained the main results (equations (3.1) and (3.2)) with no apparent justification. We have seen that there is a considerable amount of hydraulic justification for using equation (3.2), but as given by equation (2.4). It could not be claimed that this is a *theoretical* justification, as it is based on empirical friction laws but, based on the cases studied above, the incorporation of slope appears to give a superior and more fundamental description of the processes at work, and handles both long-term effects due to downstream conditions changing and short-term effects due to the flow changing.

This suggests that a better way of determining streamflows in general, but primarily where backwater and unsteady effects are likely to be important, is for the following procedure to be followed:

1. At a gauging station, two measuring devices for stage be installed, so as to be able to measure the slope of the water surface at the station. One of these could be at the section where detailed flow-gaugings are taken, and the other could be some distance upstream or downstream such that the stage difference between the two points is enough that the slope can be computed accurately enough. As a rough guide, this might be, say 10cm, so that if the water slope were typically 0.001, they should be at least 100m apart.
2. Over time, for a number of different flow conditions the discharge Q would be measured using conventional methods such as by current meter. For each gauging, both surface elevations would be recorded, one becoming the stage η to be used in the subsequent relationship, the other so that the surface slope S_η can be calculated. Using equation (2.4), $Q(t) = K(\eta(t))\sqrt{S_\eta(t)}$, this would give the appropriate value of conveyance K for that stage, automatically corrected for effects of unsteadiness and downstream conditions.
3. From all such data pairs (η_i, K_i) for i = 1,2,..., the conveyance curve (the functional dependence of K on η) would be found, possibly by piecewise linear or by global approximation methods, in a similar way to the description of rating curves described below. Conveyance has units of discharge, and as the surface slope is unlikely to vary all that much, all the remarks we make below in Section 6 apply, where we note that there are certain advantages in representing rating curves on a plot using the square root of the discharge, and it my well be that the stage-conveyance curve would be displayed and approximated best using $(\sqrt{K}, \eta)$ axes.
4. Subsequent routine measurements would obtain both stages, including the stage to be used in the stage-conveyance relationship, and hence the water surface slope, which would then be substituted into equation (2.4) to give the discharge, corrected for effects of downstream changes and unsteadiness.

3.3 Slope-area method

The Slope-area method is used where there have not been measurements of discharge to provide a rating curve. It uses cross-sectional information and the Manning and Chézy friction laws, together with slope, to calculate the discharge approximately. It seems somewhat strange that slope has been used in this approximate context and not in situations where rather more is known, such as we have described in Section 3.2 above. The method is described in Chapter 7 of Herschy (1995) and in Australian Standard AS 3778.3.3 (1990). Herschy writes:

> *The most important use of the slope-area method is for the indirect determination of flood discharge, normally after the flood has passed.*

Although not mentioned in the Standard, the Slope-area method is often implemented in approximate applications where the surface slope is not even known, but is assumed to be equal to the bed slope or mean slope of the surrounding country.

It is necessary to know the mean area A and the wetted perimeter P, both properties of the cross-section in a reach of channel of known length; the slope of the water surface or the energy gradient in the same reach of channel; and the character of the streambed so that a suitable roughness factor may be chosen. Then either of the friction laws is used:

$$\text{Manning: } Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}S^{1/2}$$

$$\text{or Chézy: } Q = C\frac{A^{3/2}}{P^{1/2}}S^{1/2} .$$

The references cited above state that the slope of the energy gradient, containing kinetic energy contributions should be calculated if the section changes much over the reach. This seems an unnecessary complication, as the friction factor is approximately known at best, and the correction is of the order of the square of the Froude number, which in most flood situations will be small and its relative contribution is dwarfed by the inaccuracy of knowledge of the friction factor.

Both references cited give the same table of friction factors for channels of varying characteristics, however Chow (1959), Barnes (1967) and Hicks and Mason (1991) provide more useful data, with interesting pictures and friction factors for a variety of streams.

4. Correcting for Unsteady Effects in Obtaining Discharge from Stage

4.1 Introduction

In this section we consider the problem of calculating discharge from stage records where the effects of variable slope may be significant but it has not been measured, as is usually the case. The relationship of equation (2.4), $Q(t) = K(\eta(t))\sqrt{S_\eta(t)}$, shows mathematically how a particular flood event can have a looped trajectory as shown in Figure 1.1. As a flood wave approaches, the surface gradient (hence discharge) is larger than the steady surface slope (and discharge) for the same stage, so that on a stage-discharge diagram the trajectory lies to the right of the rating curve, and after the peak passes the gradient will be smaller and so the trajectory lies to the left. The whole trajectory will, in general, form a loop as shown in the figure. For a large flood the head of the loop may, as shown in the figure, be outside the range of the rating curve. In many cases, however, the deviation of the actual surface slope from the steady slope may not be large, such that no corrections are necessary. We need to devise a means of estimating when this is so and when corrections may have to be made.

It is interesting that the maximum discharge in a flood event does not occur when the water level is highest or when the slope, the gradient of that level, is greatest. Using equation (2.4), the greatest discharge occurs when the combined effects of the driving slope and the conveyance K increasing with stage are greatest. We could differentiate the expression with respect to time t and set the resulting expression for dQ / dt to zero to give us an equation that could be solved, but it seems simpler, given the discrete numerical nature of much of the data, simply to interpolate the maximum discharge from the sequence of calculated discharge readings. That maximum discharge corresponds to the point on the looped flood trajectory in Figure 1.1 where the tangent is vertical, while the maximum stage is where the tangent is horizontal.

4.2 Using slope and the stage-conveyance formulation

As the method described in Section 3.2 above uses the actual measured slope, together with a known relationship between stage and conveyance, the method automatically corrects for unsteadiness and the methods of this section are not necessary. We go on now to describe previous approaches to the problem where slope has not been measured, and then present a method that is of higher accuracy than previous ones. For fast-rising streams such corrections will be more accurate, but in many cases the surface slope at a station does not vary much, and there is no need to make the corrections of this section. Below in Section 4.4.2 we present formulae for estimating when this is the case.

4.3 Previous approaches to calculating unsteady corrections

In conventional hydrography the stage is measured repeatedly at a single gauging station so that the time derivative of stage can easily be obtained from records but the surface slope along the channel is not measured at all. The methods of this section are all aimed at obtaining the slope in terms of the stage and its time derivatives at a single gauging station. The simplest and most traditional method of calculating the effects of unsteadiness has been the Jones formula, derived by B. E. Jones in 1916 (see for example: Chow, 1959; Henderson, 1966). The principal assumption is that to obtain the slope, the x derivative of the free surface, we can use the time derivative of stage which we can get from a stage record, by assuming that the flood wave is moving without change as a kinematic wave (Lighthill and Whitham, 1955) such that it obeys the partial differential equation:

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0, \tag{4.1}$$

where h is the depth and c is the kinematic wave speed given by the rate of change of flow with respect to cross-sectional area, the Kleitz-Seddon law. Using $Q_r = K(\eta)\sqrt{\bar{S}}$, where Q_r is the steady rated discharge corresponding to stage η, *where the slope has been taken to be the mean slope of the stream* $\bar{S}$, this gives

$$c = \frac{1}{B}\frac{dQ_r}{d\eta} = \frac{1}{B}\frac{dK}{d\eta}\sqrt{\bar{S}}, \tag{4.2}$$

where B is the width of the surface.

The Jones method uses equation (4.1) to give an approximation for the surface slope: $\partial h/\partial x \approx -1/c \times \partial h/\partial t$. Then the simple geometric relation between surface gradient and depth gradient can be used, that $\partial\eta/\partial x = \partial h/\partial x - \bar{S}$, giving the approximation

$$S_\eta = -\frac{\partial\eta}{\partial x} = \bar{S} - \frac{\partial h}{\partial x} \approx \bar{S} + \frac{1}{c}\frac{\partial h}{\partial t} \,,$$

and recognising that the time derivative of stage and depth are the same, $\partial h / \partial t = \partial\eta / \partial t$, equation (2.4) gives

$$Q = K\sqrt{\bar{S} + \frac{1}{c}\frac{\partial\eta}{\partial t}} \,. \qquad (4.3)$$

If we divide by the steady discharge corresponding to the rating curve we obtain

$$\frac{Q}{Q_\mathrm{r}} = \sqrt{1 + \frac{1}{c\bar{S}}\frac{\partial\eta}{\partial t}} \,. \qquad \text{(Jones)}$$

In situations where the flood wave does move as a kinematic wave, with friction and gravity in balance, this theory is accurate. In general, however, there will be a certain amount of diffusion observed, where the wave crest subsides and the effects of the wave are smeared out in time. Henderson (1966) modified the Jones formula by allowing for the subsidence of the wave crest, where a parabola approximates the flood wave. He obtained

$$\frac{Q}{Q_\mathrm{r}} = \sqrt{1 + \frac{1}{c\bar{S}}\frac{\partial\eta}{\partial t} + \frac{2}{3r^2}} \qquad \text{(Henderson 9-92)}$$

where $r = \bar{S} / S_w$ in which S_w is the wave slope, equal to the elevation increase of the flood wave divided by the length over which it occurs. A similar approach was adopted by Di Silvio (1969), who used a triangular approximation to the flood wave. Both approaches assume the nature of what is being sought.

Gergov (1971) based his approach on manipulation of the long wave equations. There seem to be some problems with it, including some unjustified assumptions. A.C.T. Electricity and Water (1990) found that it gave poor results.

Fread (1975) adopted a rather more rational approach but applied it only to a wide channel. As described above, the main problem is to approximate the space derivative. Fread used the approximation for the space derivative that Henderson used in (Henderson 9-92), but he substituted into the full momentum equation (similar to equation (B.2) in the Appendix). He then made the approximation that the kinematic wave speed is 1.3 times the fluid velocity and gave an expression for r making some further approximations about the nature of the flood wave. Then he replaced partial derivatives in the dynamic long wave equation by finite difference expressions and substituted into Manning's equation to give a nonlinear transcendental equation for the discharge Q that could be solved over a number of time steps to give the discharge hydrograph. He obtained good results when compared with his observations for the Mississippi and a tributary. However the method seems to be arbitrary, particularly in the characterisation of the shape of the hydrograph. The case of multiple peaks seems to be difficult to characterise.

Weinmann (1977, p62) described the approach of Koussis (1975), who incorporated the Jones formula into a kinematic routing model corrected for dynamic effects. Weinmann (p141) stated: ''its main advantage is that it allows the approximate conversion of stage to discharge hydrographs and vice versa''. However, it was primarily concerned with the process of flood routing, and was preoccupied with incorporating a looped rating curve into the method. This seems somewhat unnecessary, as the trajectory of a flood event may be looped, but the rating curve is by definition that for a steady flow which defines implicitly the conveyance, or the relationship between flow and surface slope. Once that has been established it can be used as part of computations - it does not matter that in routing computations the discharge is not that corresponding to the rating curve.

The work of Faye and Cherry (1980) is the most rational model for the determination of the effects of unsteadiness to date. They considered the governing long wave equations ((B.6) and (B.7) in the Appendices below) and eliminated one space derivative between them, leaving the time derivatives and the space derivative of stage. To eliminate this they made the approximation of the Jones method, that the wave motion is that of a kinematic wave. This left a differential equation for the discharge that could be solved.

A.C.T. Electricity and Water (1990), written by Christopher Zoppou, provides an interesting comparison of the various methods. It was concluded that the Faye and Cherry and Fread models were the most successful. It is the writers' view that both models have a significant disadvantage, different in each case: Fread adopted Henderson's approximate model for the wave subsidence, while Faye and Cherry did not properly account for the subsidence. Zoppou noted that the problem associated with these models is that of estimating the resistance slope. As has been noted in Section 2, measuring the slope of the surface would overcome almost all problems of such methods by bypassing them.

Birkhead and James (1998) used Muskingum routing of floods from points where the rating curve is known to points where it is not known. They made some procrustean assumptions, including a power law relationship for the stage-discharge relationship, which would not seem to reveal the nature of what is going on in a waterway.

Singh, Li and Wang (1998) considered approximations to a *u-h* formulation of the long wave equations for a rectangular canal that are most valid near a flood peak. This was a modification to the Jones method similar to those obtained by Henderson and Di Silvio, but does not seem to provide a general method.

Overall, the situation as described above seems to be unsatisfactory. It is desirable to have a method that is capable of processing stage records and being able to calculate the corresponding discharges based on rational theory, and it is that which we now attempt to provide.

4.4 A new method for calculating the effects of unsteadiness

4.4.1 Presentation of the method

Appendix C provides the theoretical derivation of two methods for calculating the discharge. The derivation of both is rather lengthy and is not presented here in the body of this Report. The first method, described in Appendix C.1 uses the full long wave equations and approximates the surface slope using a method based on a linearisation of those equations. The result is a differential equation (C.5) for dQ / dt in terms of Q and stage and the derivatives of stage $d\eta / dt$ and $d^2\eta / dt^2$, which can be calculated from the record of stage with time. It can be solved numerically, and below we describe some results.

The next method, described in Appendix C.2, is rather simpler, and is based on a low-inertia approximation to the long wave equations, where inertial terms, which are of the order of the square of the Froude number, are ignored, giving an advection-diffusion equation which approximates motion in most waterways quite well. In that equation, we have expressed a second space derivative in terms of the second time derivative using the kinematic wave approximation, so that the surface slope is expressed in terms of the first two time derivatives of stage. The resulting expression is equation (C.7):

$$Q = Q_r(\eta)\sqrt{\underbrace{\underbrace{1}_{\text{Rating curve}} + \frac{1}{c\bar{S}}\frac{d\eta}{dt}}_{\text{Jones formula}} - \underbrace{\frac{D}{c^3\bar{S}}\frac{d^2\eta}{dt^2}}_{\text{Diffusion term}}}\,, \qquad (4.4)$$

where Q is the discharge at the gauging station, $Q_r(\eta)$ is the rated discharge for the station as a function of stage, $\bar{S}$ is the bed slope, c is the kinematic wave speed given by equation (4.2):

$$c = \frac{1}{B}\frac{dQ_r}{d\eta} = \frac{1}{B}\frac{dK}{d\eta}\sqrt{\bar{S}}\,,$$

in terms of the gradient of the conveyance curve or the rating curve, B is the width of the water surface, and where the coefficient D is the diffusion coefficient in advection-diffusion flood routing, given by equation (B.44), which we can re-write to give:

$$D = \frac{K}{2B\sqrt{\bar{S}}} = \frac{Q_r}{2B\bar{S}}. \qquad (4.5)$$

In equation (4.4) it is clear that the extra diffusion term is a simple correction to the Jones formula, allowing for the subsidence of the wave crest as if the flood wave were following the advection-diffusion approximation, which is a good approximation to much flood propagation. Equation (4.4) provides a means of analysing stage records and correcting for the effects of unsteadiness and variable slope. It can be used in either direction:

- If a gauging exercise has been carried out while the stage has been varying (and been recorded), the value of Q obtained can be corrected for the effects of variable slope, giving the steady-state value of discharge for the stage-discharge relation, Q_r.

- And, proceeding in the other direction, in operational practice, it can be used for the routine analysis of stage records to correct for any effects of unsteadiness.

4.4.2 Estimating when to apply the unsteady corrections

We examine the necessity of correcting for unsteadiness by considering the relative contribution of the advective (Jones) and diffusive terms in equation (4.4). As the terms will be relatively small compared with the first term, 1, we expand the square root sign using the binomial theorem to first order:

$$Q \approx Q_r(\eta)\left(1+\frac{1}{2}\frac{1}{c\bar{S}}\frac{d\eta}{dt}-\frac{1}{2}\frac{D}{c^3\bar{S}}\frac{d^2\eta}{dt^2}\right), \qquad (4.6)$$

where a factor of ½ has appeared in front of each term. This approximation is accurate to within 1% if the corrections are less than 25%.

We write equation (4.6) as $Q \approx Q_r(\eta)(1 + \Delta_A + \Delta_D)$, thereby introducing the terms *advection correction* Δ_a and *diffusion correction* Δ_d respectively defined by

$$\Delta_a = \frac{1}{2}\frac{1}{c\bar{S}}\frac{d\eta}{dt} \quad \text{and} \quad \Delta_d = -\frac{1}{2}\frac{D}{c^3\bar{S}}\frac{d^2\eta}{dt^2}. \qquad (4.7)$$

In a typical flood these corrections will contribute at different times, the advection correction contributing when the rate of change is highest, when the second derivative will usually be small, and the diffusion correction will be greatest in the vicinity of the flood peak, when the other contribution is small.

The advection correction will be positive when the stage is increasing and negative when decreasing, such that relative to the stage graph both rising and falling limbs of the hydrograph, and accordingly the peak, are always earlier than the hydrograph taken from the rating curve. The diffusion correction will be positive near a flood peak, as $d^2\eta / dt^2$ is negative there, and so the general result can be stated that the flow maximum of every flood occurs earlier and is larger than the flow computed from a rating curve.

In many situations this will not be very significant, however, and we now obtain formulae to estimate the magnitude of these corrections. The kinematic wave speed c is approximated by the wide-channel result from Manning's equation $c \approx 5/3 \times U$ where U is the mean water velocity (see Appendix B.2.5), then we have for the advection correction

$$\Delta_a \approx \frac{3}{10}\frac{d\eta/dt}{\bar{S}U}, \qquad (4.8)$$

while for the diffusion correction we have, also substituting $D = Q / 2B\bar{S} = UA / 2B\bar{S}$, and introducing the mean depth $h = A / B$:

$$\Delta_d \approx -\frac{27}{500}\frac{h}{U^2\bar{S}^2}\frac{d^2\eta}{dt^2}. \qquad (4.9)$$

These equations are in terms of the mean velocity, and might be useful in practice to evaluate the importance of these corrections, as velocity can be computed from the discharge and the cross-sectional area, both of which are usually known at a gauging station. Alternatively, an order of magnitude estimate for U could be had by assuming a value typical of river or flood flows in the region, say, 2m/s. The velocity is probably the quantity that varies least over all conditions.

From both equations (4.8) and (4.9) we see that the corrections are largest for rivers where the conditions change quickly but are otherwise slow-moving with a mild slope. In fact, these conditions are often mutually exclusive, such that slow-moving rivers are likely to be slow to rise and fall. Nevertheless, it is quite possible that there are stations where the corrections are necessary. More insight into what determines the magnitude of the corrections can be gained if we relate the flow velocity to river characteristics. We assume that the velocity is given by Manning's equation for a wide channel of mean depth h: $U = 1/n \times h^{2/3} S^{1/2}$, which in practice would require knowledge of the roughness n. Substituting into equation (4.8) gives

$$\Delta_a \approx \frac{3}{10}\frac{n}{\bar{S}^{1.5}h^{2/3}}\frac{d\eta}{dt}. \qquad (4.10)$$

Using another wide-channel approximation $d\eta/dt \approx dh/dt$ (the exact expression is actually $dh/dt = (1 - h/B \times dB/d\eta) \times d\eta/dt$), in a whimsical result the effects of the geometry of the channel can be expressed simply as the cube root of the depth:

$$\Delta_a \approx \frac{9}{10}\frac{n}{\overline{S}^{1.5}}\frac{d}{dt}\left(h^{1/3}\right). \qquad (4.11)$$

This reveals how the effect is greatest for rough (n large), gently sloping ($\overline{S}$ small) channels where the depth is large and changes quickly, as we expect.

Following a similar procedure for the diffusion correction, using the wide-channel approximation we obtain

$$\Delta_d \approx -\frac{27}{500}\frac{n^2}{\overline{S}^3}\frac{1}{h^{1/3}}\frac{d^2\eta}{dt^2}. \qquad (4.12)$$

In fact, if we seek to isolate the variation with depth, the whimsy seems to continue, for *at a flood peak*, where we have used $dh / dt = 0$, we obtain

$$\Delta_d^{\text{peak}} \approx -\frac{81}{1000}\frac{n^2}{\overline{S}^3}\frac{d^2}{dt^2}\left(h^{2/3}\right)\bigg|_{\text{peak}}, \qquad (4.13)$$

where, except for the 1000 in the denominator, everything on the right of this equation, including the differential operator, is the square of a corresponding term in the formula for the advective correction, equation (4.11). There is no obvious physical reason for this, however the result is interesting for it shows that the diffusive correction depends on the same quantities, on n, on $\overline{S}$, and on mean depth h in a roughly similar manner to the advective correction. This is not obvious from the original definitions in equation (4.7).

This suggests that we could introduce the quantity with units of time

$$T = \frac{nh^{1/3}}{\overline{S}}, \qquad (4.14)$$

such that we could estimate the effects of unsteadiness by monitoring the magnitudes of

$$\Delta_a \approx \frac{9}{10}\frac{dT}{dt} \quad \text{and} \quad \Delta_d^{\text{peak}} \approx -\frac{81}{1000}\frac{d^2T^2}{dt^2}\bigg|_{\text{peak}}, \qquad (4.15)$$

where the numerical coefficients could be rounded to 1 and 0.1 respectively, remaining in keeping with the approximate nature of the calculation.

In most applications of this work, the diffusive correction at the peak of a flood, in this form or from equation (4.13), might be the only calculation performed. It is a value that could be computed for typical floods at each gauging station, to test whether or not the unsteadiness correction should subsequently be applied to stage records there. If the value were less than a certain desired value, say 0.02 or 0.05 (2% or 5% of the maximum), then it might be decided that no further effort need be made to correct for unsteadiness. On the other hand, if the peak correction were large enough, then it might be decided to apply the unsteady corrections in the form of equation (4.4) or equation (4.6) to every stage record from that gauging station, as is now described more fully.

4.4.3 Implementation of the theory for practical problems

The theory described above could be implemented at two levels. The first would be a screening of a particular gauging station and its records simply to determine whether it is necessary to correct for unsteady effects. This will mean initially estimating the slope of the river $\overline{S}$. Then the data from a particular flood can be taken and the stage record processed by evaluating the magnitudes of the corrections, Δ_a at a point corresponding to the maximum rate of rise, and Δ_d at the peak. Equations (4.8) and (4.9) could be used in terms of a representative assumed or measured flow velocity ($U = Q / A$), as well as a representative value of mean depth. Alternatively, formulae in terms of the roughness in the form of n could be used, equations (4.10) and (4.11) (or (4.15)), where n would be obtained from observation of the site and references such as Chow (1959), Barnes (1967) and Hicks and Mason (1991), or even from measurements of discharge and slope. In these formulae it would be simplest and quite accurate enough, provided the time interval of the readings is small enough to describe the variation, especially at the crest, to use the three-point finite difference approximations for the derivatives:

$$\frac{d\eta}{dt}\bigg|_i \approx \frac{\eta_{i+1} - \eta_{i-1}}{2\delta} \quad \text{and} \quad \frac{d^2\eta}{dt^2}\bigg|_i \approx \frac{\eta_{i+1} - 2\eta_i + \eta_{i-1}}{\delta^2}, \qquad (4.16)$$

where η_{i-1} , η_i , and η_{i+1} are three successive stage readings, taken with a time interval between readings of δ. In this preliminary screening case it has only been necessary to take representative values of velocity, which might involve using a representative value of area, and mean depth.

If the calculated values of the advection corrections and diffusion corrections at a particular site are found to be large enough to be worth incorporating, then a more systematic procedure is necessary. Either the differential equation (C.5) for dQ / dt is to be solved, or the low-inertia approximation is used, equation (4.4), with quantities defined by (4.2) and (4.5). In most practical problems the square of the Froude number should be small enough that this is accurate enough.

It is interesting that the low inertia method can be applied with relatively little extra information. All that is necessary is, firstly the traditional information to hand:

1. The stage record, giving measured values of stage at equally spaced times separated by an interval δ: η_i, $i = 1, 2, \ldots$. At a particular reading the first and second derivatives can be calculated numerically using the difference approximations (4.16). If the data is not equally spaced, then different formulae for the time derivatives are necessary, and spline interpolation might be useful.
2. The rating curve in the form of a number of data pairs of stage and rated discharge: (η_j , $Q_{r,j}$), for j = 1, 2,... . In Australian hydrographic practice these might have to be converted from Megalitre/day to m^3/s by dividing by a conversion factor of 86.4. It is necessary to be able to interpolate in this data to be able to calculate a discharge for an arbitrary stage value.

Now it is necessary also to know

3. The rating curve has to be able to be differentiated to give the $dQ / d\eta$, also at an arbitrary stage. For sufficiently large numbers of data points (small intervals) simple finite difference formulae could be used, however it might be reasonable to develop a global approximation using methods described in Appendix E.1.2.
4. The mean bed slope $\overline{S}$.
5. The cross-sectional geometry simply in the form of a number of data pairs of stage and surface width: (η_k , B_k), for k = 1, 2,... . These would probably be obtained from cross-sectional data in the form of tables of readings of position and elevation, as well as judgement and knowledge of the site.

It is no more difficult to apply this method than to apply the Jones method.

4.4.4 A theoretical example

We have solved numerically the particular case of a fast-rising and falling flood in a stream of 10km length, of slope 0.001, which has a trapezoidal section 10m wide at the bottom with side slopes of 1:2, with a Manning's friction coefficient of 0.04. Above a downstream weir the depth of flow is 2m, while carrying a flow of 10 m^3/s. The incoming flow upstream is linearly increased ten-fold to 100 m^3/s over 60 mins and then reduced to the original flow. We solved the initial backwater curve problem and then solved the long wave equations in the channel over six hours to simulate the flood. At a station halfway along the waterway we recorded the computed stages, which are the data one would normally have, as well as the computed discharges so that we could test the accuracy of this work, using some of the above-mentioned methods.

With the results obtained from the simulation we solved the differential equation (C.5) corresponding to the full long wave equations. Then we applied the three levels of approximation in equation (4.4) (which is (C.7)). We found that the highest approximation, including the diffusion term, agreed very closely indeed with the results from the differential equation, better even than we expected. Results are shown on Figure 4.1, where we have not shown the differential equation results separately.

It can be seen that the application of the diffusion level of approximation has succeeded well in obtaining the actual peak discharge. The results are not exact however, as the derivation of the differential equation (C.5) and the low-inertia approximation equation (C.7) depend on the diffusion being sufficiently small that the interchange between space and time differentiation will be accurate. In the case of a stream such as the example here, diffusion is relatively large, and our results are not exact, but they are better than the Jones method at predicting the peak flow.

Nevertheless, the results from the Jones method, shown in Figure 4.1, are interesting. A widely held opinion is that the Jones formula is not accurate. Indeed, we see here that in predicting the peak flow it is not. However, over almost the entire flood it is accurate, and predicts the time of the flood peak well, which is also an important result. It shows that both before and after the peak the "discharge wave" leads the "stage wave",

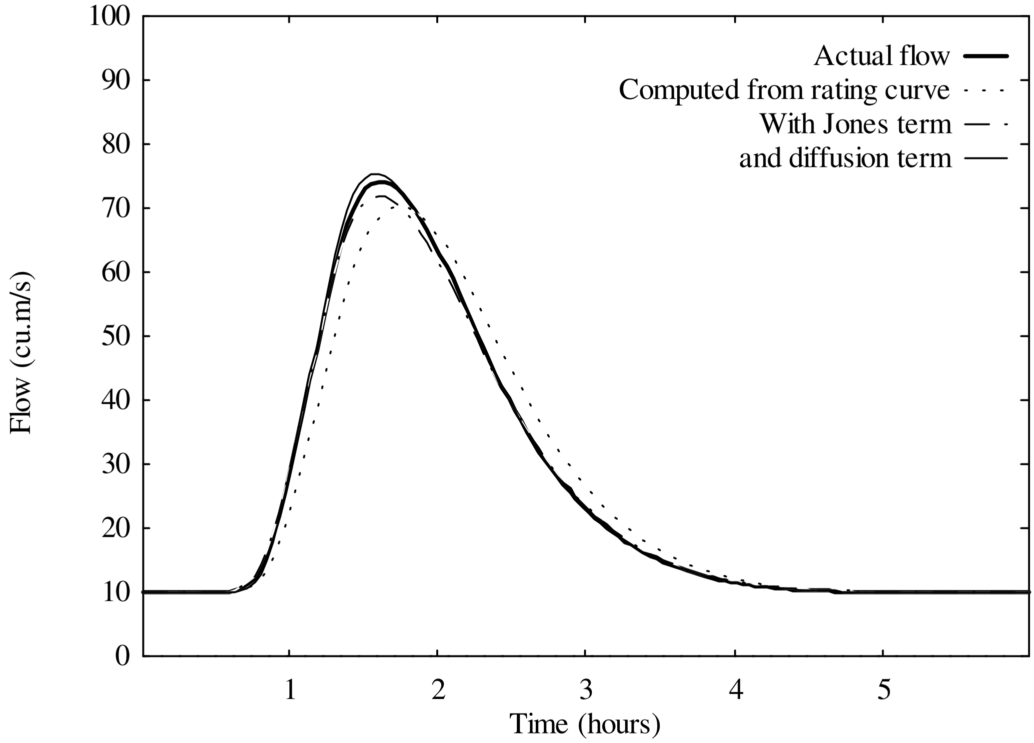

Figure 4.1 Simulated flood with hydrographs computed from stage record using the three levels of approximation in equation (4.4)

which is of course in phase with the curve showing the flow computed from the stage graph and the rating curve. As there may be applications where it is enough to know the arrival time of the flood peak, this is a useful property of the Jones formula. Near the crest, however, the rate of rise becomes small and so does the Jones correction. Now, and only now, the inclusion of the extra diffusion term in equation (4.4) gives a significant correction to the maximum flow computed, and is quite accurate in its prediction here that the real flow is some 10% greater than that which would have been calculated just from the rating curve. In this fast-rising example the application of the unsteady corrections seems to have worked well and to be justified. It is no more difficult to apply the diffusion correction than the Jones correction, both being given by derivatives of the stage record.

4.4.5 An example where unsteady effects are not important

For streams where the rate of change of the stage is gradual we expect to find less justification for correcting for unsteadiness. We processed a flood for the gauging station at Bundarra (418008) on the Gwydir River in NSW, from midnight on the 26 January 1984 to midnight on 2 February 1984. Results are shown in Figure 4.2, and it can be seen that the effects of unsteadiness are barely noticeable, although on the rising limb the advection correction was about $\Delta_a = 0.10$ (i.e. using the rating curve underestimated the flow by 10%), while on the falling limb it overestimated it by some 7%, so that the net effect was to shift the actual hydrograph earlier in time. At the crest the diffusion correction only gave $\Delta_d = 0.01$.

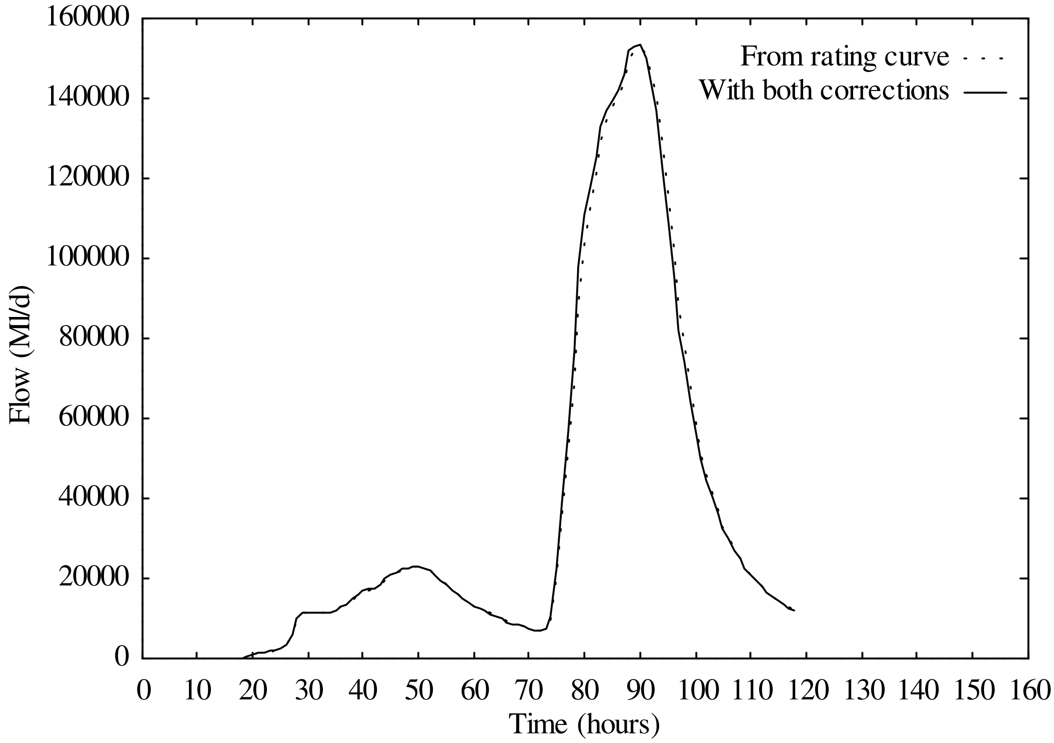

Figure 4.2 Flood at Bundarra on the Gwydir River, 27 Jan 1984 - 2 Feb 1984

5. Hydraulic Derivation of Rating Curves

5.1 An equation for rating curves

We now consider the problem of generating rating curves in the absence of flow-gauging information. An approximate theoretical equation for the rating curve is derived in Appendix D, giving stage as a function of discharge for the common case where there is a control downstream of the gauging station, but also allowing for the limit when this is drowned out. To do this the steady equation for gradually varied flow in a prismatic waterway is considered and as a first approximation we assume that the conveyance at any section varies linearly with depth in the vicinity of uniform flow, which enables an analytical solution for the free surface upstream of the control. It is convenient to present the results here in terms of a local depth - in practice the stage itself is more important, but the two are simply related. The result obtained in the Appendix is:

$$h_G(Q) = \left(h_{\text{csf}} + \bar{S}L + H_c(Q)\right) e^{Q^2 \Omega_0' L} + h_{0N}(Q)\left(1 - e^{Q^2 \Omega_0' L}\right), \qquad \text{(D.8)}$$

where $h_G(Q)$ is the depth at the gauging station, which is shown as a function of the discharge Q; h_{csf} is the cease-to-flow depth at the gauging station; $\bar{S}$ is the mean bed slope, L is the distance of the control downstream of the station; $H_c(Q)$ is the head-discharge relationship at the control; $\Omega_0' = d(K^{-2})/dh\big|_0$, the derivative of $1/K^2$ with respect to depth evaluated at some reference depth h_0; and $h_{0N}(Q)$ is the approximation to the normal (uniform) depth upstream, given by

$$h_{0N} = h_0 + \frac{\bar{S}}{Q^2 \Omega_0'} - \frac{\Omega_0}{\Omega_0'}. \qquad \text{(D.6)}$$

Equation (D.8) shows rather complicated dependence on the discharge Q, especially in the usually unknown nature of the head-discharge relationship at the control. In Appendix D.3 it is shown that if the control is a rectangular broad-crested weir or sharp-crested weir, then $H_c \sim Q^{2/3} = Q^{0.67}$. In the case of a triangular weir or V-notch, $H_c \sim Q^{0.4}$. Between these two extremes is a parabolic weir (i.e. a U-shaped cross-section), for which it can be shown that $H_c \sim Q^{0.5}$. If the weir were trapezoidal, the discharge formula would be a linear combination of the rectangular and triangular formulae, one of exponent 0.67 and the other 0.4. We can imagine that this could be approximated by a single power, like the parabolic weir, with exponent about 0.5. In a natural stream the actual nature of the control will be rather more complicated, although it may well be able to be approximated by a similar power law expression. Natural river topography at the control is rather more likely to look like a U, rather than a rectangle or V-notch, so that it could be modelled by a parabola, giving $H_c \propto Q^{0.5}$.

To implement this theory one would need some knowledge of the geometry at the control. Reference could be made to French (1985, #8.3), Bos (1978), or Ackers *et al*. (1978) for formulae relating head and discharge in idealised situations, such as if a rectangular weir is installed as a control. If the geometry were not known, or, equally likely, weir formula were thought not to be applicable, then it would be necessary for at least one or more gaugings to be undertaken to determine the form of the relationship.

5.2 Calculating the low flow end of a rating curve

5.2.1 The equation for a rating curve for low flows

In the limit $Q \to 0$ the square-root-like behaviour of the term $H_c(Q)$ will dominate other terms in equation (D.8), and in Appendix D.4 it is shown that in the low flow limit at the gauging station,

$$h_G = h_{\text{csf}} + H_c(Q) + \text{Terms of order } Q^2, \qquad \text{(D.10)}$$

showing that the stage-discharge relationship is given by that of the control, and the reach between gauging station and control is essentially a reservoir with flow through it, whose dynamical effects on the surface are negligible, and the surface is horizontal with elevation given by the control. This makes matters rather difficult in deriving a rating curve theoretically, as the complicated geometry and nature of most controls makes a theoretical determination usually not possible.

However this has shown that our knowledge of simple control formulae can be applied to approximate the behaviour of the rating curve in the low flow limit.

On the basis of this work, we expect the stage to vary like Q^v, where v will be between about 0.4 and about 0.7, and is reasonably likely to be in the vicinity of 0.5. This means that if we were to plot the rating curve on axes with the square root of discharge $\sqrt{Q}$ horizontally and stage vertically, the relationship will often approximate a straight line for low flows. This may provide a convenient method of extrapolating low-flow results to determine the cease-to-flow limit.

5.2.2 Calculating the low flow end of a rating curve

At the low-flow end there is probably not much that can be done to calculate a stage-discharge relationship from first principles, as the nature of local controls is often such that the geometry is complicated, and no flow formula could be applied. However, in situations where rating is difficult, being able to obtain a rating curve valid for typical everyday low flows at least might be useful. It is possible that, if the variation were linear on a $\left(\sqrt{Q},\eta\right)$ plot, such that the relationship between stage and discharge is the quadratic

$$Q = \alpha(\eta - \eta_0)^2 \qquad (5.1)$$

then two or more measurements could establish the unknown coefficients α and η_0, where $\eta_0 = \eta_{csf}$, the cease-to-flow stage.

5.3 Calculating the middle to high flow part of a rating curve

5.3.1 The equation for a rating curve for larger flows

If the flow is sufficiently large, another downstream control may become dominant, but now the geometrical situation is sufficiently complicated that the theory such as we have used above would contain too many variables to be useful. For sufficiently large flow or a control sufficiently distant, the effects of such local controls become negligible, and it is the channel itself that provides the dominant control. This is demonstrated by equation (D.8), where, as Ω_0^1 is negative, for distant control (L large) or large flows (Q large) the exponent in becomes large negative, the term becomes negligible, and so the flow is uniform, with the depth given by

$$h_G(Q) = h_N(Q)$$

showing that the depth at the gauge is simply the normal depth for uniform flow in that section. For this case, we do not have to use the approximation (D.6), but instead could use the normal depth as found by solving the transcendental uniform flow equation $Q = K(h_N)\sqrt{S}$.

We can explore this by examining results from Manning's and Chézy's uniform flow formulae for a family of cross-sections given by monomials, that is, where the bed elevation z_b is given by $z_b = a\left|y\right|^v$, Where a is a constant at a section and y is the co-ordinate across the channel. A V-shaped waterway would have $v = 1$, while U-shaped valleys would be approximated by larger values of v. One with a large value would have a flat bottom, with suddenly steeply rising sides. We can imagine, that as this is a rough approximation, the convenience of a parabola ($v = 2$) might be useful.

If we calculate the area of cross-section for water surface a height d above the bottom of the section, we obtain $A = v/(v+1) \times Bd$, where B is the surface width, which we can express in terms of the constant a but it is simpler here to retain it. For a V-shaped valley we obtain $a = 1/2 \times Bd$, and for quadratic variation we obtain $a = 2/3 \times Bd$, both familiar results. When we go on to compute the wetted perimeter P, however, we obtain an integral that we cannot express in simple terms. It is within the approximate nature of this analysis to adopt the approximation $P \approx b$, which has an error of magnitude $(d/B)^2$, which is usually small.

We now use the conventional uniform flow formulae, where the dependence of discharge on the cross-sectional characteristics is expressed by being proportional to the geometrical quantity $(A/P)^\mu \times A$, where for Manning's relation $\mu = 2/3$ and for Chézy $\mu = 1/2$. Evaluating this for the general monomial bottom topography, after substituting the rather awkward expression for the surface width $B = 2(d/a)^{1/v}$, we find that discharge varies with surface elevation above the bottom of the stream like

$$Q \sim d^{1+\mu+1/v} \qquad (5.2)$$

In the case of Manning's formula the exponent becomes $5/3 + 1/v$, and so we see that for a V-shaped valley the exponent is about 2.67, while if the topography is a U-shaped valley approximated by a parabola, which should be rather more common, the exponent is 2.17. If Chézy's law were used, the general expression for the exponent is $1.5 + 1/v$. For a parabolic valley, $v = 2$, and a value of 2 for the exponent is obtained. For a cubic, with Manning's law, the exponent is also 2.

As we have to express the relationship in terms of stage η, linearly related to surface height above the thalweg, we write (5.2) for d as a function of Q, giving

$$\eta \sim Q^{1/(1+\mu+1/v)}. \tag{5.3}$$

This means that for typical U-shaped waterways, the theoretical stage-discharge curve will be approximated by a power law relationship where the numerical value of the exponent will be about 0.5 or a little below that, so that η will vary like $Q^{0.5} = \sqrt{Q}$. On $\left(\sqrt{Q}, \eta\right)$ axes the stage-discharge relationship would appear almost as a straight line.

This, in fact, is similar to the variation we suggested would apply for many local controls in the limit of low flow. The coefficients in each case will be different, but the manner of variation is the same. This suggests the utility of plotting the square root of discharge, $\sqrt{Q}$ against stage η, when we might see the useful result that at both low and high flows, the relationship should plot as nearly straight lines. At the lower end, this would be convenient in extending the stage-discharge relationship to the cease-to-flow point, and in fact determining that point, and, at the higher flow end providing a semi-rational way of extrapolating the results, if that were necessary.

5.3.2 Calculating the middle to high flow part of a rating curve

For higher flows we have suggested that the uniform flow formulae could be used. There are two levels at which we could operate:

1. Assumed monomial variation

The assumption of monomial variation could be made, using equation (5.2), giving

$$Q = \beta(\eta - \eta_0)^{1+\mu+1/v}, \tag{5.4}$$

where β is a constant and the exponent could be assumed to have a value of 2, or another corresponding to the actual value of the geometrical parameter μ from the cross-section, obtained using the method set out in Appendix E.1.3. A value of η_0 corresponding to the bottom of the section might be made, as in this case the control is drowned out. Then, a single rating point could be obtained from measurements, or a single assumed value of the slope-roughness parameter made, so as to generate the curve.

2. Semi-rational method

Here we use the geometrical information from a cross-section. We write the stage-discharge relationship in the form based on Manning's expression

$$Q = \frac{\sqrt{S}}{n}\theta(\eta), \text{ where } \theta(\eta) = \frac{A^{5/3}}{P^{2/3}}, \tag{5.5}$$

and will refer to θ as the geometrical friction parameter. To provide a calculated stage-discharge relationship we would need to know the value of the friction-slope parameter $\sqrt{S}/n$. There has been some work on the variation of n with stage, however in the absence of any other measurements this would probably have to be assumed constant for a given station. Then, from a cross-sectional survey, tabulated values of θ as a function of stage could be obtained, and the stage-discharge relationship calculated. A single measured rating point could be used to provide a value for the slope-roughness parameter in equation (5.5).

A problem here is that if we required the calculated rating curve to extend to low flows we would require the flow to be zero at the cease-to-flow stage. In the absence of any other information the flow below the cease-to-flow level could be subtracted:

$$Q = \frac{\sqrt{S}}{n}\left(\theta(\eta) - \theta(\eta_0)\right). \tag{5.6}$$

However, in higher flow situations the effects of the local control should be drowned out and the whole cross-section should be contributing, so that it might be preferable just to use equation (5.5) and not expect it to apply to low flows.

We will see below that this method did not work very well compared with the more simplistic method of assumed monomial variation presented here above.

6. Representation, Approximation and Calculation of Rating Curves

In this Section we bring together some of the theory described above, common practice, and some of the suggestions we have made above, to test them and to make recommendations for the representation, approximation and calculation of rating curves.

6.1 From data to rating curve: global and local, approximation and interpolation

At various stages in the description below some terminology is introduced which it is worth explaining here. *Interpolation* is the process by which, from a series of discrete data points, intermediate values may be calculated. For this to be reasonable the data points should lie on a reasonably smooth and continuous path. If the data points are scattered, such as a collection of actual stages and flow gaugings at a station, then a process of *approximation* is used to pass a reasonable curve or series of curves through them. Subsequently, values could be interpolated from that approximation. There are two main alternative ways of implementing these processes. The first is *global*, where a single mathematical function is used, which is valid over the entire region required; the other is local, where a sequence of different functions may be used. The most common version of this is *piecewise linear*, where a series of straight-line segments are used.

In the process that often takes place at a particular gauging station, a set of data points of stage and corresponding discharge are taken and the main problem is to obtain the rating curve by approximation. Usually this is done by eye, often on a log-log plot, by plotting a sequence of points, each of which is an approximation to the data points in the neighbourhood, and then if necessary, further points added by piecewise-linear interpolation. The sequence of such points and the presumed straight-line interpolation between them, make up a rating curve.

In the case of gauging data, there may be some justification, possibly using the representation of the data described below, to automate the procedure and to use a global approximation method to generate the rating curve, for which a method is given in Appendix E.1.2.

Where data is discontinuous, however, piecewise-linear methods are robust and simple to implement. In some of the computational procedures described in this Report there are several quantities which are functions of stage, notably geometric quantities such as area A, surface width B, and wetted perimeter P which can show rapid or irregular variation with stage. Especially in this case, but also as a general-purpose tool, piecewise linear interpolation is both simple and powerful.

6.2 Logarithmic scales

A problem with rating curves is that they have to represent a relationship between stage, which might vary by 10 metres or so, and discharge which can vary by several orders of magnitude, from 0 to hundreds of thousands of discharge units. A traditional solution is to plot the discharge using a logarithmic scale, effectively expanding the region for small flows and contracting that for large. In many books and standards (for example, Herschy, 1995, and Australian Standard AS 3778.2.3, 1990) it is shown how it is convenient to use a logarithmic scale for the stage η as well, in that often it is found that that by subtracting of some arbitrary value η_0 such that if one plots the logarithm of the discharge against the logarithm of $\eta - \eta_0$, points on the rating curve approximately fall on a straight line. The implication of this is that the discharge obeys a law of the form

$$Q = C(\eta - \eta_0)^n, \qquad (6.1)$$

where C and n are constants. A typical means of doing this is to take three points on the rating curve at which values for η and Q are known and substitute into (6.1) and solve the resulting three equations for C, η_0 and n (see, for example #4.4 of Herschy, 1995). While this is a relatively simple method, it is not clear that it is safe to advise it as a technique for routine practice, as in many cases the rating curve does not show simple variation of this form *over the whole range*. Above, we have shown that such a relationship might be valid for low flow, with a different relationship for high flows. To handle this more complicated situation, the curve could be broken up into a small number of segments, each of which is a straight line on the log-log plot, such as performed by Herschy, (1995, #4.5). This does seem a rather arbitrary procedure, however. Rather better would be to approximate it with a larger number of such segments, when the use of straight-line approximation would be sound. This is widely done in practice.

There are different stages of analysis of a rating curve, however. The first is the conversion from raw data, showing scatter, into a representation with a unique relation between stage and discharge. There are two ways which this can be done, (1) placing points by eye and assuming that the variation between such points is linear on the plot which has been chosen, and (2) fitting a multivariate function to the data points using least-squares methods.

In either of these cases it becomes necessary to decide on the manner in which the data are to be represented and approximated. The log-log plot is visually convenient, and to represent it mathematically one could use a polynomial for $\log Q$ in terms of $\log \eta$. A disadvantage of the log-log plot is, however, that the point corresponding to cease-to-flow discharge cannot be plotted as it occurs at $-\infty$ on the horizontal axis, and special attention has to be given to this part of the plot. A simple polynomial representation for Q in terms of η could be used, but as the function is required to vary over several orders of magnitude, in general the approximation for high flows would mean that for low flows some inaccuracies would occur.

Some problems of log-log scales are demonstrated by an example here, from Pallamallawa on the Gwydir River. Rating curves for this and other stations on the same river are given in Table F.7, and in Figure 6.1 a set of data points are plotted and a global straight-line approximation fitted to all but the two points of highest discharge, where, corresponding to overbank flow occurring, a significant discontinuity occurred. On the axes shown the straight line does not seem unreasonable. To obtain the linear fit, however, a least-squares procedure has been used in log-log space that means that as this opens out the points very much at the low-flow end, they contribute more than their real importance. This is illustrated by plotting on linear axes, as shown in Figure 6.2, using both the same data and line of best fit, now curved. It becomes obvious how the wide-spacing of data points at the low-flow end on the logarithmic plot has distorted the result considerably, and in reality, the plausibly-satisfactory results on log scales are not acceptable. Even if a single straight line were not fitted, the shrinking of the scale at the upper end is such as to render apparently small changes or errors innocuous, whereas in reality they are important, as revealed by Figure 6.2.

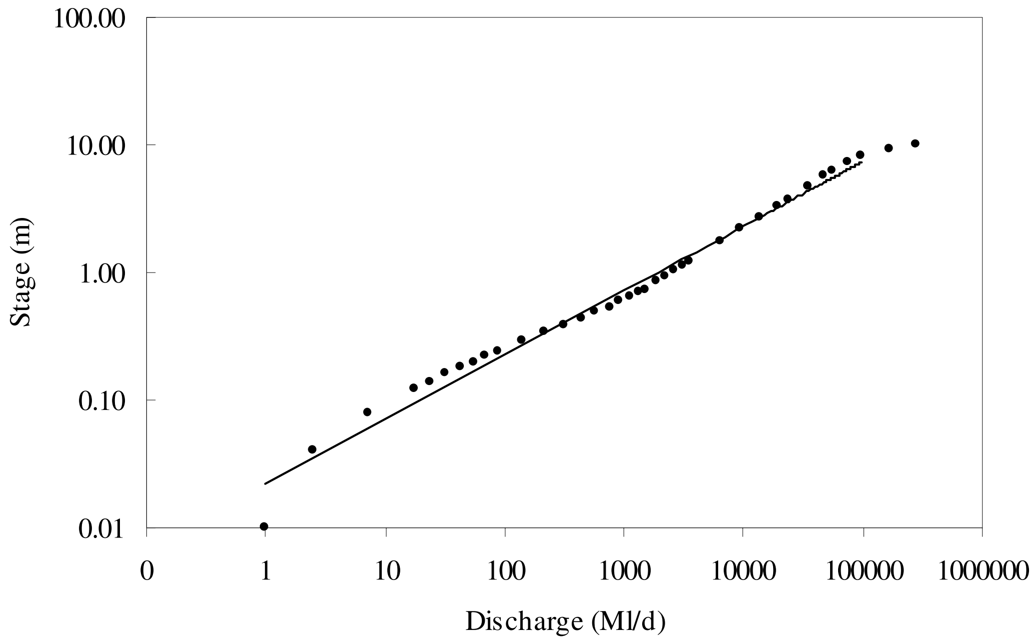

Figure 6.1 Stage-discharge relationship using log-log axes for Pallamallawa on the Gwydir River, with a line of best fit on these axes to all but the last two points

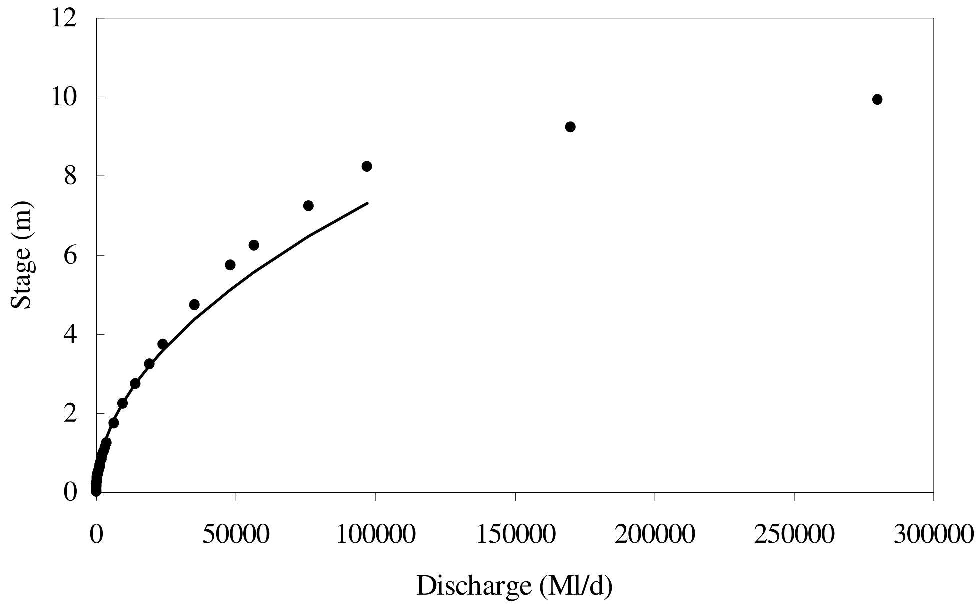

Figure 6.2 The same data and approximation as Figure 6.1 but using natural axes.

In practice there is no physical justification for such global log-log approximation and it should not really be used, despite the apparent tacit acceptance of it in standard textbooks and both International and Australian Standards.

6.3 Use of $(\sqrt{Q},\eta)$ scales for representing stage-discharge relationships

6.3.1 Discussion and justification

Plotting rating curves on simple linear axes could be used to overcome some of the problems of logarithmic scales. However, the requirement to include large flows means that the region of small flows becomes graphically insignificant, as shown in Figure 6.2. That figure strongly resembles a plot of the square root function (or indeed, it could be a plot of the function Q^n, where n is any number between 0 and 1). This suggests an alternative scale to use. Indeed, such a suggestion has been made by Chester (1986), who suggested that $Q^{2/5}$ should be used. The choice for the power 2/5 was made by assuming the discharge formula for a V-shaped section control. This seems to us a rather procrustean assumption, and it is unlikely that such an assumption is generally valid. However, the assumption of a power law scale seems to be useful. We wish to make the simplest such assumption, and if one is to adopt a number between 0 and 1, then 1/2 seems the obvious choice. More importantly, in Section 5.1 above we have shown that for both low flows across a U-shaped weir control and high flows down a U-shaped waterway, the stage-discharge relationship will tend to show stage varying approximately like $\eta \sim Q^{1/2} = \sqrt{Q}$, such that both parts of the relationship would plot as straight lines on ($\sqrt{Q}$, η) axes. Alternatively, in both cases, for a V-shaped weir and for a V-shaped waterway, stage should vary like $\eta \sim Q^{0.4}$, and results would be straight lines on the axes recommended by Chester.

The data from the previous example are shown plotted on $\sqrt{Q}$, η axes in Figure 6.3. It can be seen that the low-flow points still collapse into a relatively small region, but they do locally form a straight line of finite gradient. It can be seen that if we were to take the data points and represent them by a sequence of straight lines, a piecewise-linear approximation, it could be done adequately with only about four or so segments. The low flow region is shown, the behaviour is clear, but unlike the log-log plot it does not dominate the

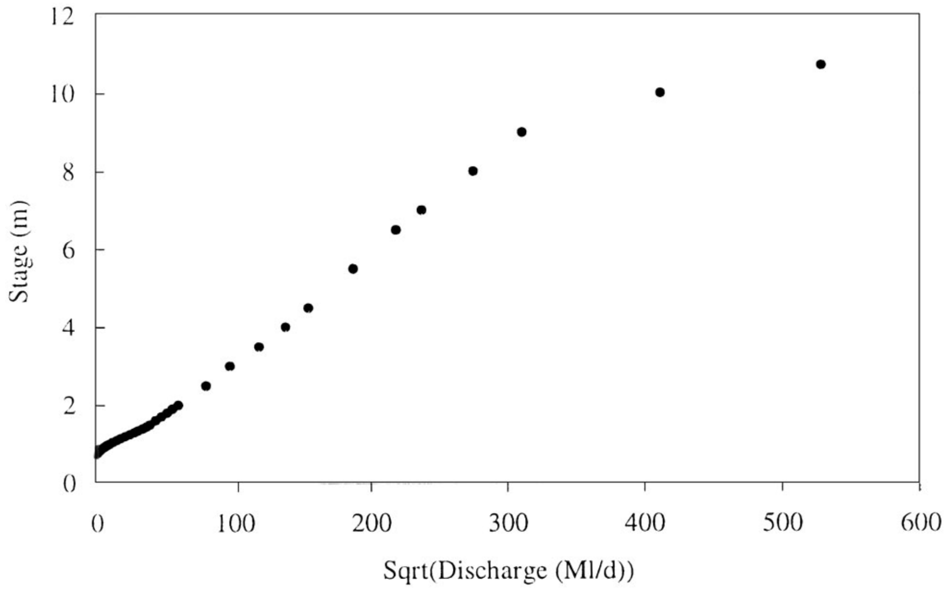

Figure 6.3 The same data as the previous two figures but plotted on ($\sqrt{Q}$, η) axes

plot. One can still extract low-flow information, but in that region the results resume their real importance, and plotting stage to within 1cm vertically would be satisfactory without moving a point an apparently large distance as on the log-log plot.

An advantage that the power plots have, for both 2/5 and 1/2 exponents, is that the treatment of the cease-to-flow point is more satisfactory than with a log-log scale. If the cease-to-flow point is known, it can be plotted without any special treatment. If it is not known, it seems that the tendency of the points to lie on straight or very-nearly straight lines provides a reasonable way of extrapolating the low-flow data to determine the cease-to-flow stage. On the other hand, using a log-log scale, if the cease-to-flow point is not known it has to be found by using a global approximation, such as equation (6.1) and finding η_0 such that the expression plots as a straight line, all requiring non-trivial operations.

At the other end of the scale, the problem of extrapolation arises if one needs to estimate a flood for a higher stage. As the log-log scale condenses data so very much at that end (*e.g.* Figure 6.1), the results obtained are extremely sensitive to the choice of method or data. Using a square-root/linear scale would seem to give plausible results. This is discussed below.

The use of a 2/5 power scale for discharge has been implemented as an option in the widely used HYDSYS package. Here we have to justify our suggestion that a square root (1/2 power scale) is to be preferred. There is not a lot of distinction between the two, but some points that spring to mind are:

- The square root function is slightly more familiar and standard than the other.
- Its inverse, the square function, is slightly easier to handle and make interpretations from. For example, on Figure 6.3 the point where there is an apparent change of behaviour is at a value of a little more than 300, which one would easily calculate to be about 90,000 Ml/d. In comparison, calculating a value of $300^{2.5}$ is not as easy using mental arithmetic. If special plotting paper were drawn up, with tick marks at major values as with a log scale, this would not matter, but in practice with non-specialist standard software as we used to produce the figures, with equally-spaced tick marks, the square root scale is simpler.
- For low flows it is more likely that stage varies like the square root of discharge, as discussed above, where the sill is horizontal in cross-section, giving a U-section, rather than if the control were a V-section.

- For higher flows where the local hard control is drowned the stream cross-section is more likely to provide a channel or frictional control. Streams with an approximate U-shape of cross-section are likely to predominate over those with a V-shape. Above we have shown that in this case too, there is a tendency for discharge to vary quadratically with stage.

6.3.2 Some examples of $(\sqrt{Q}, \eta)$ scales for representing stage-discharge relationships

We consider here some examples, providing some circumstantial evidence for and against the choice of the square root scale. Initially data are taken from a couple of standard sources, and are probably of a somewhat idealised nature. Subsequently we use data from three stations on the Gwydir River.

- The first we consider is the data from Table 4.1 of Herschy (1995), reproduced in Table F.5 here in the Appendix. It shows smooth behaviour, and, after calculation and subtraction of a certain value of stage, it was able to be approximated quite well by a straight line on a log-log scale. Here, on the square root of the discharge scale, shown in Figure 6.4, we find that the straight line $\sqrt{Q} = 4.959 + 8.210\eta$ reasonably approximates it globally. Adding a η^2 term and fitting a quadratic in η gave very close agreement indeed. This data gave one example where an even more linear graph was obtained by plotting using a $Q^{2/5}$ scale (using a log-log scale, Herschy found the exponent to be 0.394). Even though we have considered global approximation here, it is generally not going to be feasible to approximate globally a rating curve by a low-order function, and it will in general be better to use piecewise-linear approximation.
- The next case is one where the data comes from Table 3 of Australian Standard AS 2360.7.2 - (1993), identical to International Standard 7066 - 2: 1988, given in Table F.6 here. It is also presented in #4.9 of Herschy (1995) as an example of polynomial curve fitting. It is for that reason that we attempted also to perform a global approximation, but to $\sqrt{Q}$. A quadratic in η was used to fit by least-squares, giving

$$\sqrt{Q} = 51.55 - 11.178\,\eta + 1.7737\,\eta^2,$$

and, of course, (6.2)

$$Q = \left(51.55 - 11.178\,\eta + 1.7737\,\eta^2\right)^2.$$

Results are shown in Figure 6.5, and it can be seen that the representation is good. In the Standard from whence this data was obtained a 4th degree polynomial was fitted to the data in $Q = f(\eta)$ form, where $f(\eta)$ represents some function of η. The result obtained in the Standard (with a slightly different result reported by Herschy) was

$$Q = 4800 - 3742\,\eta + 1073\,\eta^2 - 122.28\,\eta^3 + 6.079\,\eta^4. \quad (6.3)$$

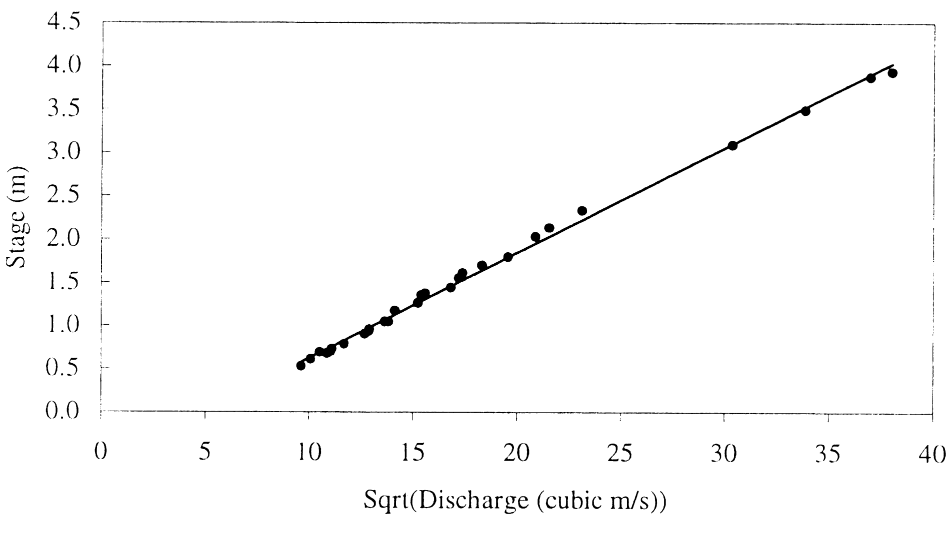

Figure 6.4 Data from Table 4.1 of Herschy (1995) plotted on $(\sqrt{Q}, \eta)$ axes, with a global straight line fit

We repeated the calculation from the Standard but using a 2nd degree polynomial, as we did for equation (6.2) using the square root scaling, and the results were very poor indeed, suggesting the superior nature of the use of $\sqrt{Q}$.

Equation (6.3) from the Standard reveals some of the problems of using higher-order polynomial approximation. The coefficients as presented are insufficiently accurate, and they alternate in sign. In calculating Q for a typical value of stage, say $\eta = 10$, the contributions of the successive terms become progressively larger such that the number of significant figures presented is not enough. This is part of a fundamental problem to do with the problem of interpolation and approximation in civil engineering, described in Fenton (1994). If the independent variable takes on large values, such as local stage or even worse, elevation above a maritime datum, inaccuracies can result. These problems can easily be overcome, as described in Appendix E.1.2.

Now we consider three gauging stations on the Gwydir River in northern New South Wales. The first station is at Bundarra, in hilly country. The cross-section is shown in Figure 6.6. The horizontal lines show 20 equally-

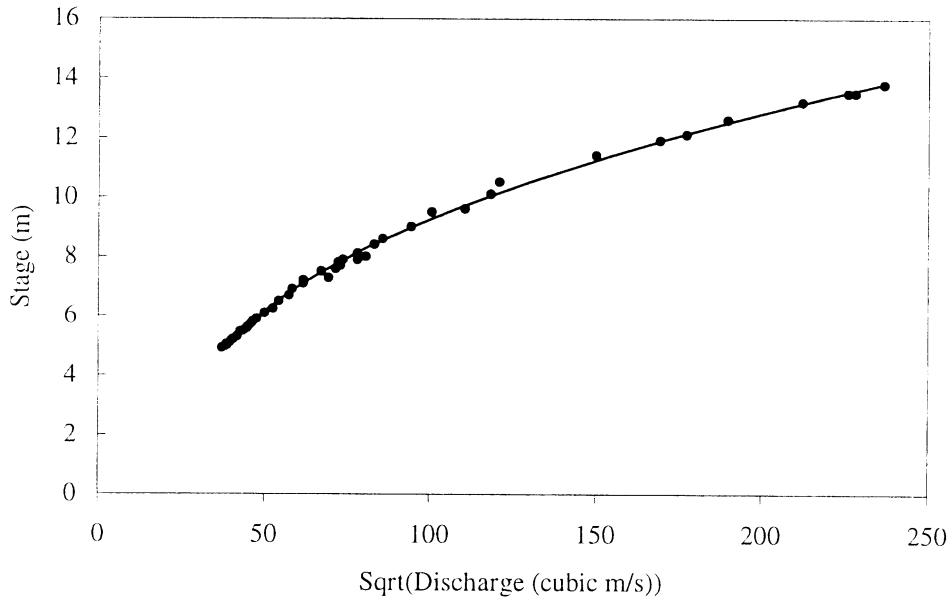

Figure 6.5 From Table 3 of Australian Standard AS 2360.7.2. Axes are $(\sqrt{Q}, \eta)$, showing data and second degree polynomial fit

spaced water levels, from which the geometric data of area and wetted perimeter were calculated, and the semi-rational method of equation (5.6) used to calculate an approximate rating curve from such geometric data.

Figure 6.7 is a plot at the low-flow end of stage against discharge using $(\sqrt{Q}, \eta)$ axes. Points from the published rating curve are shown as squares, while the dashed line shows a least squares approximation to the first six points using a straight line on these axes. It can be seen that the points do approximately fall on a straight line and the line could be used to determine the cease-to-flow stage.

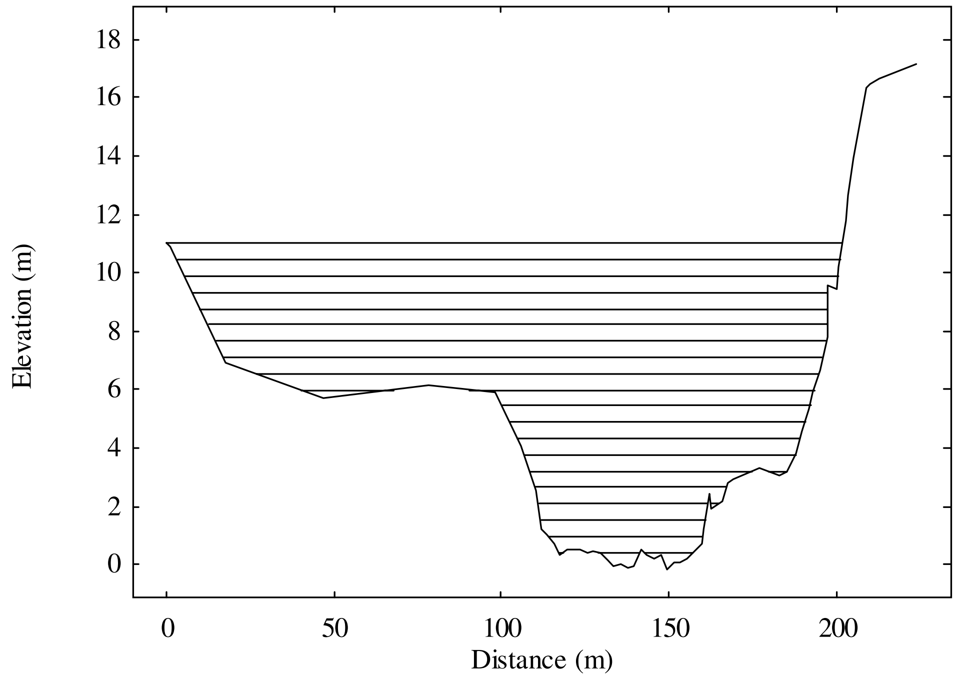

Figure 6.6 River cross-section at the gauging station at Bundarra

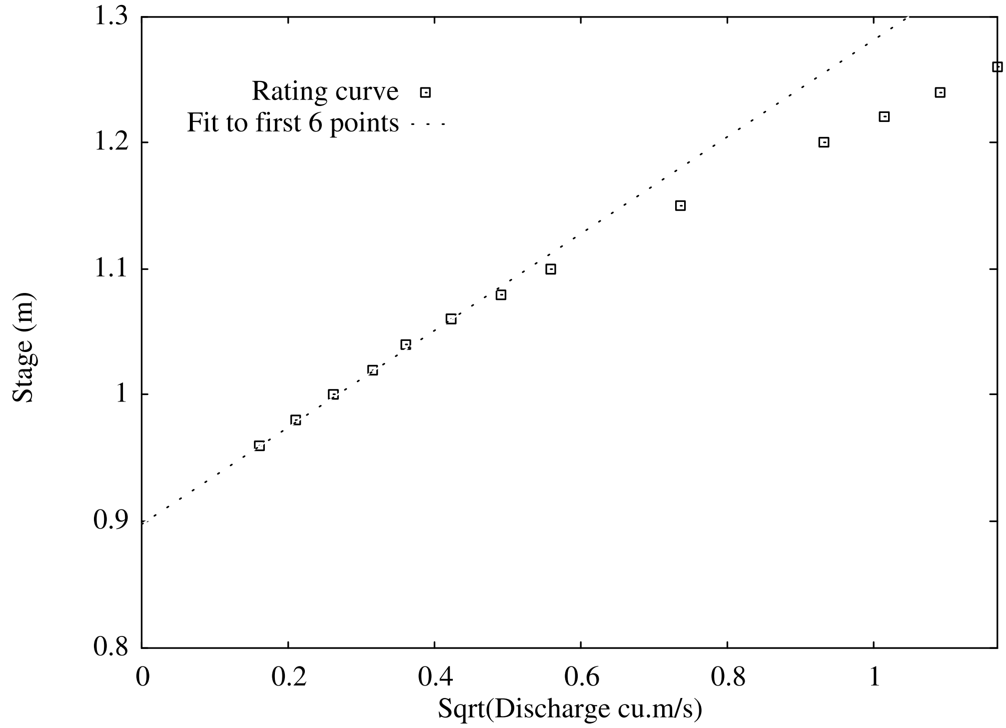

Figure 6.7 Stage-discharge plot for Bundarra using $(\sqrt{Q}, \eta)$ axes, showing the first few points and a linear approximation using least-squares to the first six points

Figure 6.8 is a plot of all the rating curve points on the same axes, with some other results. On this curve the dashed line is the same one as on the previous figure, and it can be seen that it has a quite different gradient from the solid line, which is a least-squares approximation to almost all the points using a linear relationship on these axes. It can be seen that the latter relationship gives a good fit to almost the whole rating curve, although we do not advocate the use of such low-order global approximation. The cross-section shows a pronounced widening at a stage of 6m, but which seems to have no effect on the rating curve. This is shown on Figure 6.8 by the dashed and dotted line, which is a graph of the stage-discharge relationship as obtained using equations (5.5) and (5.6) from uniform flow formulae, using the measured cross-sectional data plus an assumed value for the slope-friction quantity $\sqrt{S}/n$ which the authors chose arbitrarily so that the results approximated the data as closely as possible. It is interesting that this method, of rationally incorporating the details of the geometry, does not seem to describe the actual rating curve well. Somehow the flow in the river seems relatively unaffected by the sudden widening, and over almost all the range of the rating curve points the single linear relationship $\sqrt{Q} = az + b$, where a and b are constants, describes the relationship well. We do not necessarily want to use such a global relationship here. We include it here to test the efficacy of the $(\sqrt{Q}, \eta)$ plot and to provide some justification that a linear extrapolation to higher stages and flows would be quite justified on this evidence. It is also circumstantial evidence that for some other station where there were few data points available, the linear relationship between $\sqrt{Q}$ and η could be used if necessary.

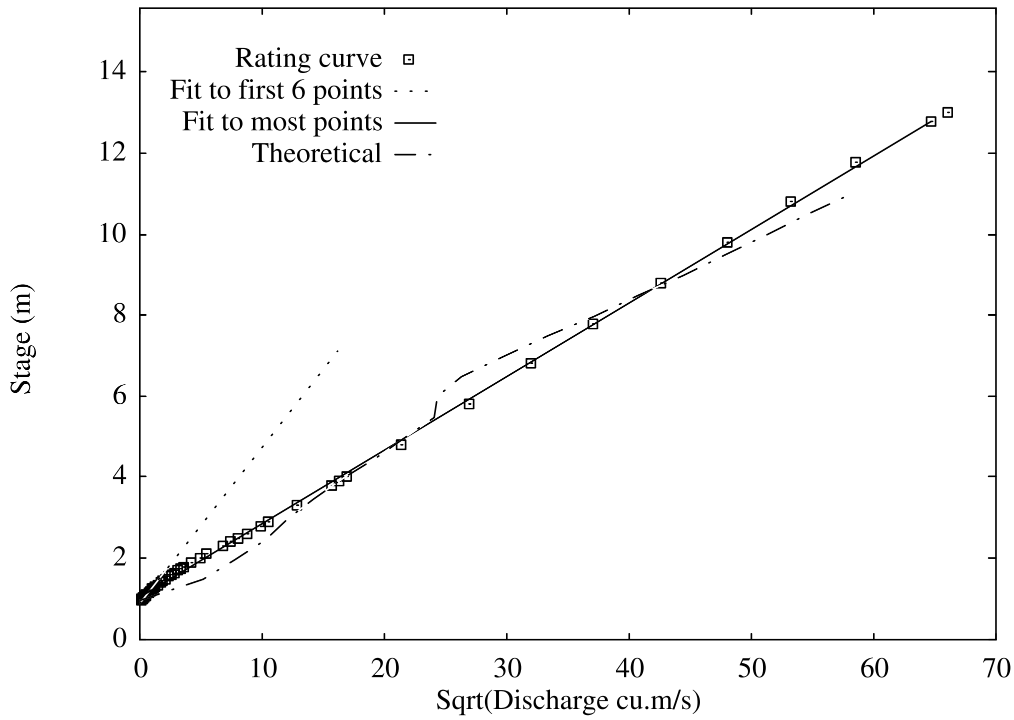

Figure 6.8 Stage-discharge plot for Bundarra using $(\sqrt{Q}, \eta)$ axes, showing all data points, two linear approximations, and theoretical results calculated from details of the cross-section

The next station we consider is at Gravesend, some distance downstream on the same river, in gently undulating countryside. The cross-section is shown in Figure 6.9, and contains three additional channels into which the river can flow when sufficiently high. The two linear features terminating in sharp crests are road embankments, leading up to a high bridge.

Figure 6.10 shows a plot similar to Figure 6.7 for low flow, and it can be seen that, once again, the rating curve points for low flow fall on a straight line on these axes.

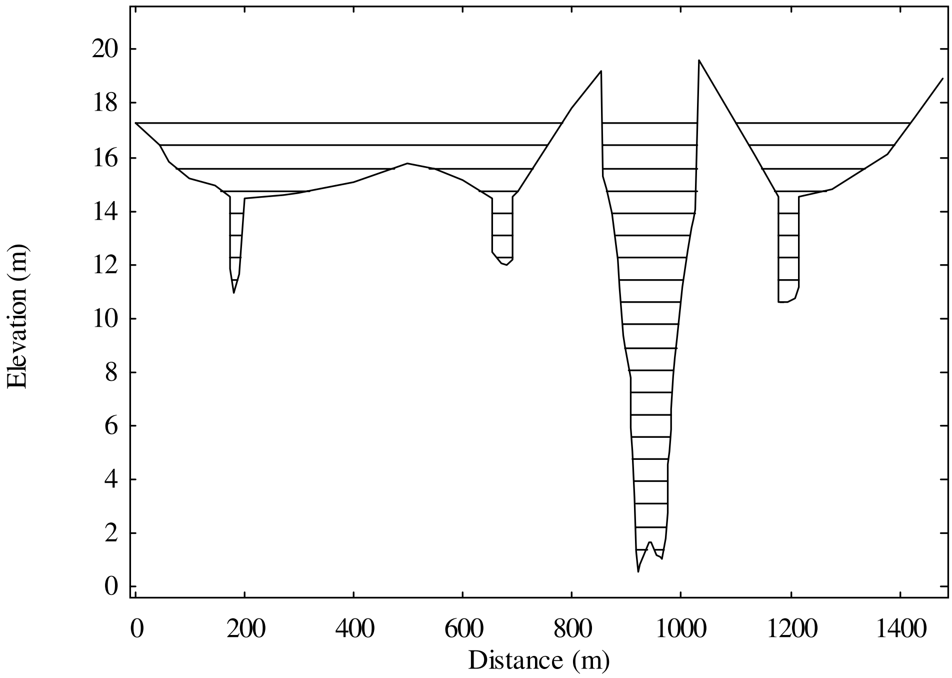

Figure 6.9 River cross-section at the gauging station at Gravesend

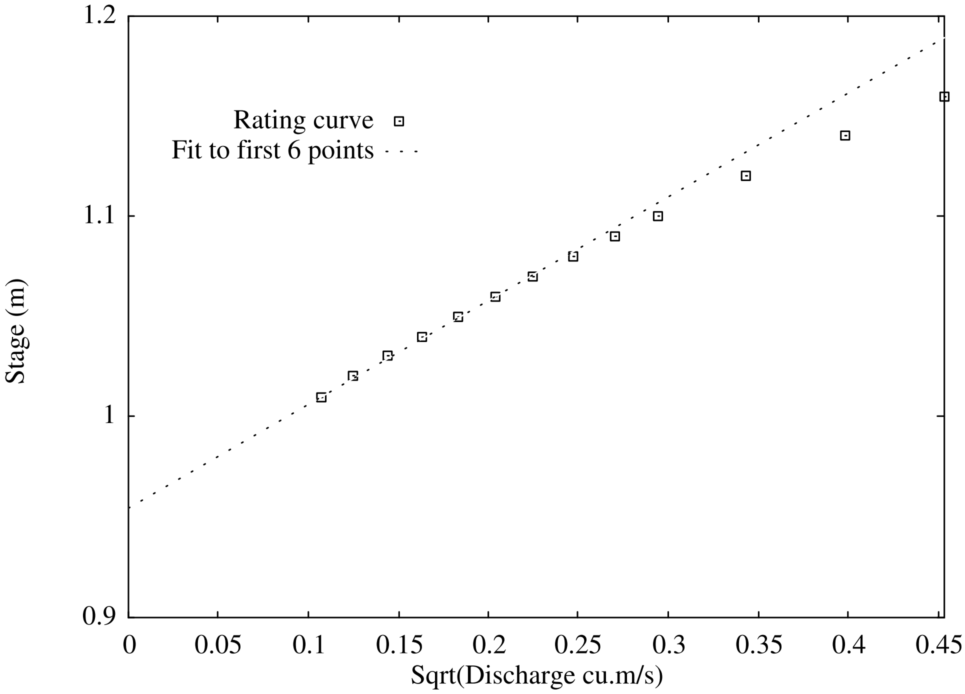

Figure 6.10 Stage-discharge plot for Gravesend using ($\sqrt{Q}$, η) axes, showing the first few points and a linear approximation to the first six points

Figure 6.11 shows a plot similar to Figure 6.8, containing all the points from the rating curve. This time not all the results for high flows fall on a straight line; above a stage of 12m there is a gradual deviation from the solid straight line shown, obtained from a least-squares fit to the points that appeared to form a straight line. Examination of Figure 6.9 shows that this is when flow into any or all of the three subsidiary channels is possible, and it is no surprise that the results for moderate flows cannot be simply extended to higher flows. Again the results for low flow form a different linear relationship. Again the data points for high flows could plausibly be extended by linear extrapolation if that were absolutely necessary. And, again, theoretical results from the detailed geometry do not agree particularly well, although the trend for higher stages is mimicked. The problem seems too complicated to expect very much from such simple approaches.

Further downstream where the river has come out onto plain country is the station of Pallamallawa, where the cross-section is shown in Figure 6.12 and which is rather regular except for a widening near 4m.

Figure 6.13 again shows rating curve points for low flow, and again they fall almost on a straight line, however here there is some curvature, suggesting that the control has a different nature. These axes still allow the behaviour of the points to be shown clearly, and the cease-to-flow point could still be obtained by a continuation of the trend.

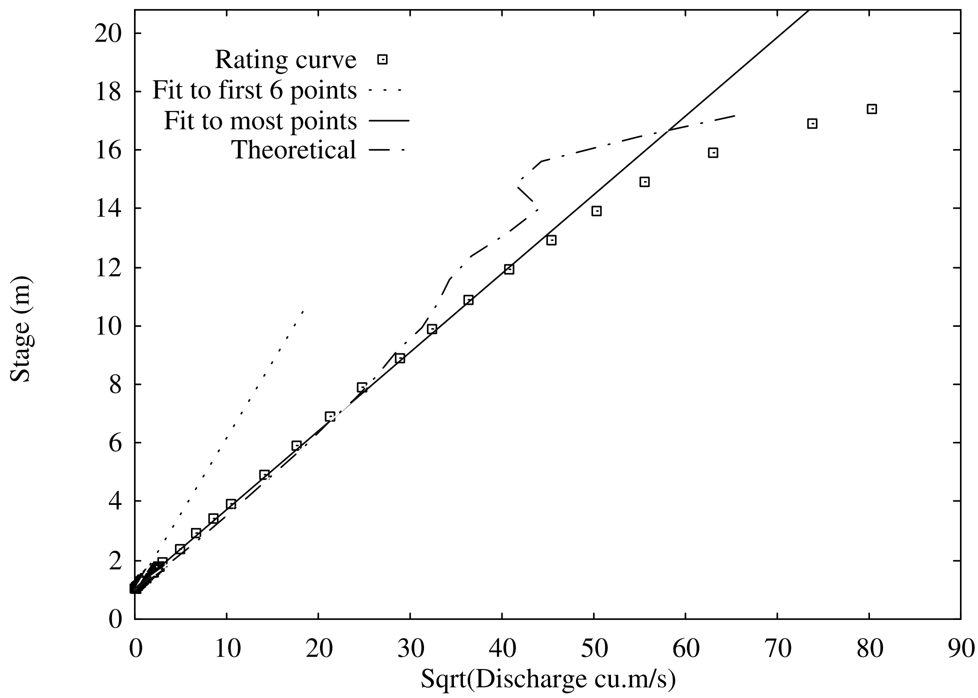

Figure 6.11 Stage-discharge plot for Gravesend using ($\sqrt{Q}$, η) axes, showing all data points, two linear approximations, and theoretical results calculated from a knowledge of the cross-section

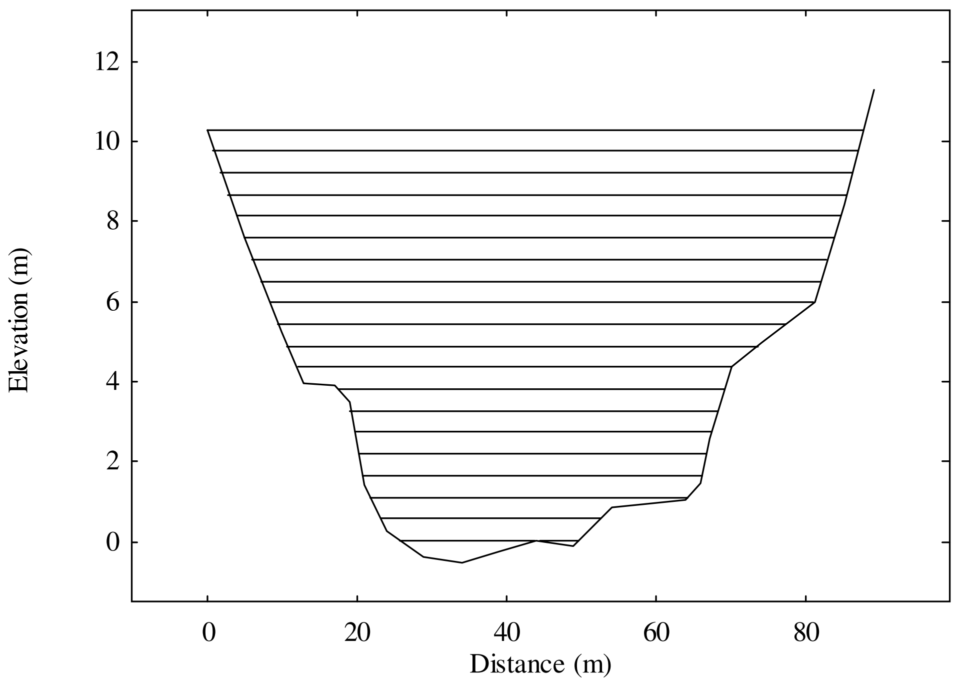

Figure 6.12 River cross-section at the gauging station at Pallamallawa

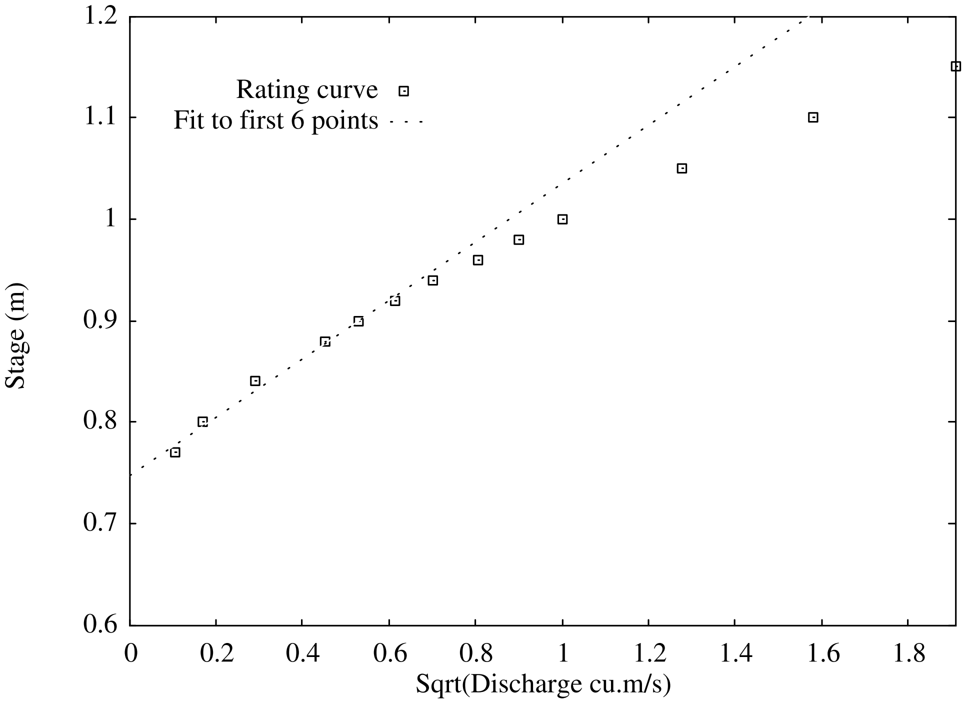

Figure 6.13 Stage-discharge plot for Pallamallawa using ($\sqrt{Q}$, η) axes, showing the first few points and a linear approximation to the first six points

Figure 6.14 shows all the data points. There are two rated points that were obtained for very high flows, requiring the operation of a boat over flooded farmland. These two points, almost indistinguishable on the log-log plot of Figure 6.1 do not follow the previous fairly consistent trend when plotted on these axes. We did not include them in the linear fit shown by the solid line. It is clear that the bulk of the points for moderate to high flows almost lie on a straight line. Also, for this rather regular geometry the theoretical curve using the actual geometry worked quite well, compared with the previous cases, and there is little obvious effect of the widening at 6m, both in these results and in the actual data.

Considering Figure 6.14, on the $(\sqrt{Q},\eta)$ plot it is clear that a piecewise-linear representation of the rating curve would be adequate. The two data points for high overbank flow, unsurprisingly, do not seem to follow the trend of the previous data. In general, for such a discontinuity it might be better to use something like piecewise linear approximation, However, here we show the possible power of a global approximation in *approximating* the actual rating data so as to automatically generate data for the rating curve at Pallamallawa. We experiment with a global approximation, expressing $\sqrt{Q}$ as a sixth-degree polynomial in stage, using the methods described in Appendix E.1.2. The results are shown in Figure 6.15, and are encouraging. At the cease-to-flow end of the data, the polynomial was able to describe the region dominated by the local control. Possibly even more usefully, it seems able to make a plausible continuous relationship that incorporates the two high flow points.

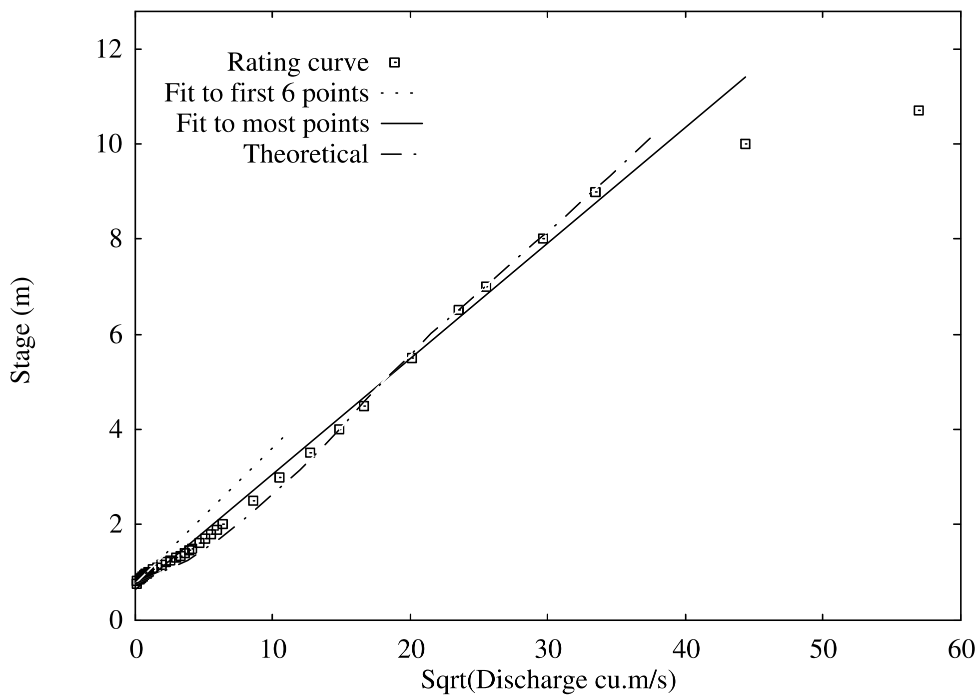

Figure 6.14 Stage-discharge plot for Pallamallawa using $(\sqrt{Q}, \eta)$ axes, showing all data points, two linear approximations, and theoretical results calculated from a knowledge of the cross-section

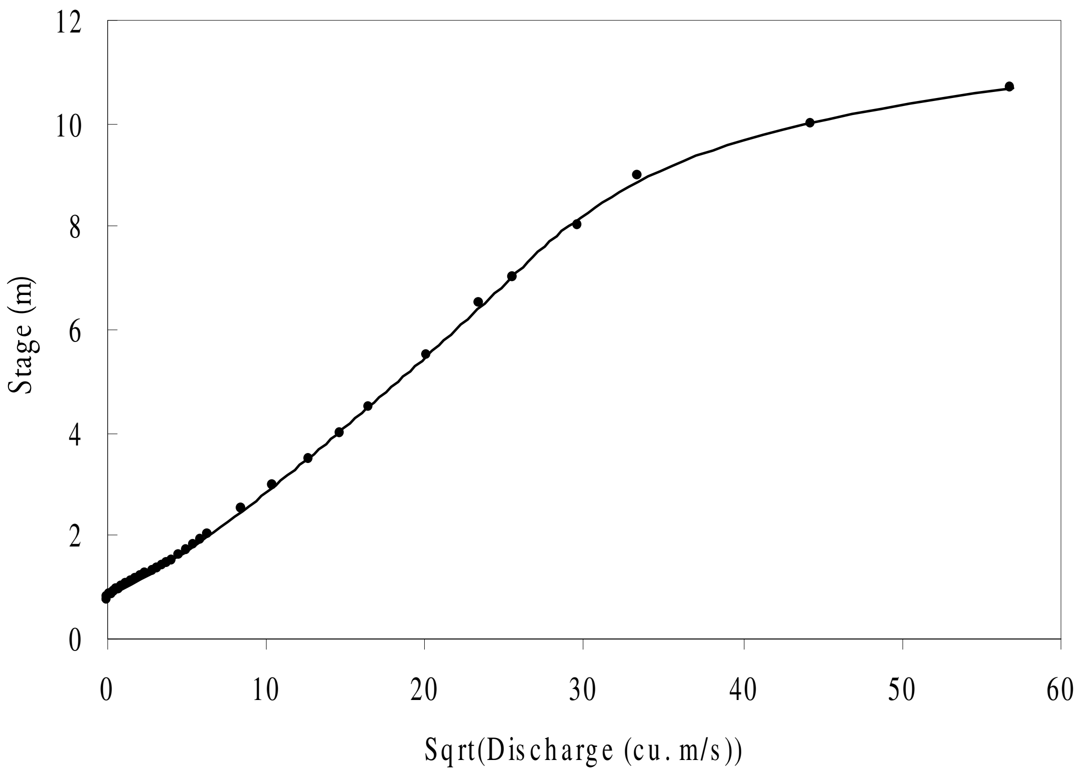

Figure 6.15 Rating curve for Pallamallawa, showing the data and a 6th degree polynomial fit

6.4 Extrapolation

An important feature of all the figures we have produced is that on the ($\sqrt{Q}$, η) axes, the examples usually showed a rating curve which did not depart much from a straight line at the upper end of the curve (although the immediately-preceding example of Pallamallawa provides a warning). This suggests that generally, if it were essential for a rating curve to be extrapolated, a straight line on a ($\sqrt{Q}$, η) plot, from the penultimate point, through the last point and continued, seems usually not to be far wrong. This can be tested by going back through the previous figures, pretending that the last point does not exist in each, and seeing how a line from the third-last point through the second last point would approximate the remainder of the curve. The result is quite good in all cases. This might be the best way to proceed with extrapolation. Global approximation like that shown in Figure 6.15, however well it might perform in the range of data points used to approximate it, is likely to behave highly irregularly outside that and should not be used for extrapolation. Linear extrapolation using the last two points could be adopted with caution.

7. Conclusions and Implications for Management

We have considered a number of aspects of the use of stage measurements and rating curves to calculate flows in rivers. A number of detailed conclusions have been drawn, which we now summarise and indicate what the implications are for river management:

- It might be expected by practitioners that good instructions and guidance would be provided by the International and Australian Standards; however, the Standards relevant to this subject are somewhat inadequate. As mentioned in Section 1 of this report, in Appendix A a detailed review of the Standards is presented. They state the obvious in many places, but they give almost no serious guidance for practical implementation of sophisticated methods for approximating and representing rating curves.
- Consideration of the hydraulics governing rivers in Section 2 shows that in general the slope of the surface, as well as stage, is a determinant of flow. This means that the concept of a unique rating curve is flawed in principle, but in practice it is often accurate enough. Ideally then, as shown in Section 3, it would be best always to use two gauges and to measure the slope, which would automatically correct for backwater effects from downstream and for unsteady effects at the time of flood propagation. This would lead to the development of a *Stage-Conveyance Relationship* which could then be used in conjunction with the measured slope, rather then the conventional use of a *Stage-Discharge Relationship.*
- Then, allowing for the usual case where the slope is not measured, Section 4 presents previous approaches to correcting for varying slope, including the well-known Jones method for allowing for effects of the variation of stage with time. A new method is developed and presented in Section 4.4, which gives a further correction to the flow calculated from a rating curve. The method is no more difficult to apply than the Jones method, but is more accurate, as it allows for the subsidence of a flood wave as it propagates. Formulae are given to estimate when these unsteady effects are worth correcting for, and some examples are presented. For most rivers the effects are small, and results from conventional rating curves are quite acceptable. A number of details of practical implementation are given.
- In Section 5, an attempt is made to provide means of deriving rating curves when little flow information is available. A theoretical model is developed for a reach of river with a gauging station and local control, which is used to predict the rating curve for low flows and, when the control washes out, for high flows. In practice, for natural controls, the nature of the local control will be too complicated to be able to calculate the low flow end of the rating curve. The use of theory for both low and mid- to high flows shows that, in many cases, the stage will vary like the square root of discharge, both at low flows and for mid to high flows. This can be used to calculate a rating curve in the absence of other information, or preferably, together with one or two ratings, to calibrate the model.
- In Section 6 a critique is made of the widespread use of log-log plots in representing rating curves; it is suggested that plotting the square root of discharge against the stage has some practical advantages. In particular, both for low and high flows, many data points from gaugings should plot roughly as a straight line, which can help the determination of the cease-to-flow point, as well as the possible extrapolation of the curve at high flows. Finally, in that section, reference is made to Appendix E, where it is shown how global approximation of the rating data can be implemented *via* a robust numerical method.

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Appendix A. Review of the International and Australian Standards for the determination of the stage-discharge relation

This appendix contains a critical review of the International and Australian Standards (the two are identical) for the determination of the stage-discharge relation. Australian Standard AS 3778.2.3 - *Measurement of water flow in open channels – Part 2.3: General – Determination of the stage-discharge relation* (1990) contains a number of details concerning the use of rating curves. It has been reproduced from an International Standard. Parts of the document contain wise advice, but in some parts it is rather inadequate and poorly explained. In view of some of the developments described in this report, it could really be upgraded.

6 Calibration of a gauging station

This is the first substantial section, after some relatively unimportant initial remarks.

6.1 General preparation of stage-discharge relation states that

> ***6.1.1*** *The primary object of a gauging station is to provide a record of the discharge of the open channel or river at which it is sited. This is achieved by measuring stage and converting this to discharge by means of a stage-discharge relation which correlates discharges to either the water level at a section of the channel or the measurements of water level at each end of a reach. In the latter case, a twin gauge station is necessary and a different procedure for establishing the stage-discharge relation is used.*

The inclusion here of a twin gauge station is, as we shall see, a far-sighted but only partial implementation of hydraulic theory that this report asserts could be routinely used. In the later detail, in fact, the presentation is quite inadequate. A use of twin gauges is described in Section 3.2 of this report.

> ***6.1.2*** *Depending on the stability of the channel, the stage-discharge relation may be classed as stable or unstable. By accepted definition, a stable channel is one wherein the physical form and frictional properties of the bed and sides remain constant with respect to time. Conversely therefore, an unstable channel is one wherein the physical form and frictional properties vary with respect to time; that is, the channel itself is mobile. However in any reach, whether stable or unstable, any transient natural or artificial phenomenon may affect the actual relationship between stage and discharge at that time. Thus a stable channel may exist as an unstable reach. In all cases instability in a stage-discharge relation arises from variable conditions of back-water for a given discharge at the gauging site concerned and this is true, also, for changes in mean velocity of approach at a given stage due to unstable conditions occurring upstream from the gauging site, although the effect so experienced is a secondary rather than primary manifestation.*

This long-winded statement appears to include the definition that an "unstable reach" is one where effects of backwater and unsteadiness are present. That is not a definition of unstable that we would have used.

However, once again anticipating the real nature of the actual hydraulics, but never subsequently delivering it in substance, the Standard goes on:

> *In all circumstances of variable backwater at a given discharge, a stable channel will provide a stable stage-discharge relation when the energy slope of the water in the reach concerned can be included as a parameter in the discharge relation.*

6.2 Stable stage-discharge relation

The Standard spends some time on the basics of presentation of a rating curve, and in so doing it takes a laudable overview of the arbitrariness of presentation:

> ***6.2.2*** *The simplest expression of the stage-discharge relation is a plot on arithmetically divided graph paper with discharges plotted as abscissae against corresponding stages as ordinates. Since discharge often ranges over several orders of magnitude, it is sometimes more convenient to plot the relation on single or double logarithmically divided paper.*

In this way the Standard has emphasised that there is nothing special about log-log or any other plots, and considers linear axes as a reasonable alternative. Then we read

> ***6.2.3*** *A smooth curve should be drawn by eye through the array of data points to detect points which may be in error. In the case of a single control, the relation may generally be expressed by a power curve of the form* $Q = C(h + a)^{\beta}$ *(see annex A). In the case of a multiple control, the relation may have to be expressed by several power curves with inflexions and reversals in curvature indicating changes from the influence of one control to another. If the stage measurements have been adjusted to relate to a zero which coincides with the level at which discharge becomes zero; then on double logarithmic paper the relation between (h + a) and Q will plot as a straight line or a series of straight lines, otherwise the relation between h and Q will plot as a curve.*

The primitiveness of the "smooth curve … drawn by eye" is surprising, as there are general techniques available for least-squares approximation.

The presentation of the formula for the power law is misleading here, as it is not the single control that would determine a single power-law relationship. It would be a very special stream indeed that would exhibit such simple behaviour. Rather, there will almost always be a more general relationship between stage and discharge, even if there is a single control. The Standard seems to recognise this at the end of the paragraph, and to take the quite reasonable approach that it is legitimate to represent the rating curve by a series of straight lines on a log-log plot, provided enough data points are provided in regions where the gradient changes relatively quickly.

Next, the Standard turns to a phenomenon that is often ignored, and addresses it fairly and squarely, emphasising that ideally measurements should be made at steady stage. It notes that at most gauging stations there will be a difference between the discharge for a given stage on the rising and falling limbs. It does not explain this in terms of the simple fact that the driving surface slope is greater for the rising limb, but it states the phenomenon succinctly:

> ***6.2.5*** *The curve should be examined for hysteresis (see annex B). Where possible the measurements should have been made at steady stage, but if not, those taken at rising or falling stages should be indicated by distinguishing symbols. At most gauging stations it will be found that there is a tendency for a gauging taken on a rising stage to indicate a higher discharge than one taken at the same water level on a falling stage. Because of this, rising stage measurements generally plot below a curve and falling measurements above a curve established under steady stage conditions. In the case of stable channels, a mean curve can generally be adopted.*

Unfortunately in the last sentence the poor terminology of "stable channels" is used for those where the effects of unsteadiness are unimportant.

> ***6.2.6*** *The equation for the curve should be obtained, or the curve may be treated as a purely graphic record. The equation may be computed mathematically using the least squares procedure (see annex A). Alternatively stage-discharge equations may be prepared by computer direct from the gauging data but it is still advisable to plot the gaugings for preliminary examination to determine if they should be split into ranges for separate treatment. Computer techniques should be used only when discharge measurements can be said to have known weight.*

The Standard seems to approve of obtaining an equation for the curve, but later in Annex A provides no systematic procedure other than a presentation of the standard elementary theory for a least-squares fit using a straight line on a log-log plot. It notes that the data should at least be plotted first for purposes of checking and for subdivision into ranges.

6.3 Unstable stage-discharge relation

The first section (#6.3.1) deals with unstable channels, where changes of conditions are prevalent, and then, strangely, in #6.3.2 it deals with "unstable reaches", in which it mentions weed growth and ice, both more pertinent to the previous section, and then by other conditions "due to abnormal conditions of backwater" including tributary inflow, artificial regulation downstream, and tidal influence:

> *Although the geometry and friction properties of the channel in a measuring reach may, in themselves, be stable, a simple invariable relationship between stage and discharge will not exist if those channel properties are not in overall control of the depth of flow for any given discharge. Such loss of overall control due to abnormal conditions of backwater can be caused by weed growth, by tributary inflow downstream, by artificial regulation downstream, by tidal influence, by ice, or by some combination of these. It may prove possible to deal effectively with weed growth in the manner described in 6.3.1 but the problem of tributary inflow causing variable backwater requires measurement of water surface slopes concurrently with stage measurement at the gauging site.*

In a masterstroke it is noted that all backwater effects can be overcome by measuring the water slope, and it goes on to outline in the next section how to use that, described more fully in an Annex:

> ***6.3.3*** *Evaluation involving surface slope (fall)*
>
> *In the case of unstable reaches, the evaluation of the stage-discharge relation requires additionally the use of the value of fall between two reference gauges located within the reach concerned, one of which is the gauging station reference gauge.*
>
> *The plotting of the stage-discharge observations with the value of fall against each observation will reveal whether the relationship is affected by variable slope at all stages or is affected only when the fall reduces below a particular value. In the absence of any channel control, the discharge would be affected by the fall at all times, and the correction is applied as indicated in the constant fall method (see annex C). When the discharge is affected only when the fall reduces below a particular value, the normal-fall method is applied (see annex C).*

The Standard has revealed how to overcome the problem, but explains it rather poorly. Throughout it refers to the "fall" between two gauges, when what is really governing the flow is the *slope*, the fall divided by the length over which it occurs. In fact, the slope affects the discharge at all times, regardless of whether there is a channel control or a hard control. For an explanation of this, see Section 2.

Importantly the Standard goes on to consider in #6.4 the extrapolation of the stage-discharge relation:

> *A stage-discharge curve should not be applied outside the range of observations upon which it is based. If estimates of flow, however, are required, they should be so identified having regard to the range, number and quality of the observations which have been made, to the natural features of the gauging station and to the conditions of flow with respect to time. Little reliance shall be placed on extrapolation below the lowest observed value (see annex D).*

The warning is well judged and succinctly written. However the extra warning of the last sentence would seem to be just as, if not more, apposite for the highest observed value.

Next in #6.5 the preparation of a rating table is described, however there is recognition of modern practice in the last sentence when it notes that a rating table may not be as useful as the actual equations for the stage-discharge curve:

> *A rating table can be prepared directly from the stage-discharge curve(s) or from the equation(s) of the curve(s), showing the discharges corresponding to stages in ascending order, and at intervals suited to the desired degree of interpolation. This can be conveniently performed by a computer program using the stage-discharge relation. However it may be useful to program the data for computer evaluation using the stage-discharge equations without resorting to a rating table.*

Next, in *#7 Methods of testing stage-discharge curves* considerable effort is expended in discussing the statistics of the data and their relation to the rating curve. In fact the statistical jargon rather obscures the underlying hydraulic aspects of the problem. Here we briefly note *#7.1.4 Tests for bias for unstable channels (tests 2 and 3)*, which states:

> *For stable channels where the control is uniform and remains unchanged, it will be possible to fit a mathematical curve as explained in clause 6. More frequently, even in a stable channel, where the curve has to be drawn by visual estimation, for example when the section is not uniform, the tests described for unstable channels become equally necessary.*

It is not clear why, if a section is not uniform or for any other reason, that the curve has to be drawn by visual estimation, especially if the "unstable channel" tests are to be used, when numerical computations need to be performed.

The next section contains several poorly expressed passages and apparent contradictions to what has been written before:

> ***7.1.5** In the case of natural unstable channels, different controls come into operation at different stages in different years, so that not only are the curves for the rising and falling stages different from each other and from year to year, but there are also inflexions and discontinuities due to shifts in control within a stage.*

It is not just different controls coming into operation at different stages or the strangely expressed "shifts in control within a stage" which give apparent discontinuities in the rating curve, it is the whole geometry of the reach, where the conveyance characteristics of the section may change smoothly with stage or suddenly at a given stage if overbank flow suddenly occurs. Immediately following:

> *The inordinate labour involved in fitting the high-degree composite curves rules them out in practice. The best-fitting rising and falling curves have, therefore, to be drawn by visual estimation and shall therefore be tested, for absence from bias and for goodness of fit, separately for the individual portions between shifts in control.*

The first sentence may have been correct before electronic computers, but fitting high-degree curves is no longer a problem, and standard methods exist. It would be simplest to use global approximation methods for such curves. It is interesting that, once again, the necessity for treating rising and falling limbs has been emphasised, but the claim that they should be drawn by "visual estimation" seems archaic, as does the claim that it is necessary to apply statistical tests "between shifts in control" when often those shifts are continuous and it may not be possible to distinguish them.

In any case, simple piecewise linear approximation using log-log scales is a perfectly viable alternative.

Following this are several appendices, known as Annexes, which describe details of the methods in more detail, but which are still quite skimpy.

Annex A – Stage-discharge curve

This deals with several aspects which we have described above. *#A.1 Stage-discharge equation* describes methods for determining the coefficients in power law approximations to the stage-discharge curve. It includes the advice

> *For many purposes the graphical record obtained by plotting the measured discharges on arithmetically divided graph paper may be sufficient, but plotting logarithmically is sometimes advantageous.*

Also, however, is the practically useless and inaccurate statement that:

> *Usually the stage-discharge relation at a station may be expressed by an equation of the form $Q = Ch^{\beta}$ (where Q is the discharge, h is the gauge height and C and β are coefficients) over the whole range of discharges, or more often by two or more similar equations each relating to a portion of the range.*

If the zero of the gauge does not coincide with zero discharge a correction a must be applied to h, making the equation $Q = C(h + a)^{\beta}$.

A couple of methods for determining a follow, including a graphical method. Then a simple linear least squares fit on log-log scales is described.

Section *A.2 Tests for bias* commences with *A.2.1 Stable channels* and the well-judged remark:

In the case of stable channels, it is generally found that one curve for the rising and falling stages is adequate, unless the river has a steep slope or is flashy; sometimes, one curve is applicable with a narrow belt of dispersion for some years.

Subsequently some over-arching statements are made describing a number of statistical tests which should be applied. Following this are recommendations for what the Standard refers to as *A.2.2 Unstable channels*:

In addition to the shift in control at some particular stage due to a different control coming into operation as exhibited by some stable channels, the unstable channels manifest another peculiarity, that is, a shift of control with reference to time or intensity of floods, and the consequential seasonal scour and fill phenomena. There are significant differences in the gauge-heights for the same discharge in the rising and falling stages, and a complete change in regime occurs after the maximum flood of the year. Minor freshets, specially during the clear-water seasons, may also cause shifts in control.

Once again we see the words "change of control" being used as a catch-all phrase. The comments about scour and fill are appropriate, but the subsequent generalisations are rather meaningless.

Subsequently *A.2.3 Tests for absence from bias and goodness of fit* describes the statistical tests in some detail, but they seem to be rather arbitrary and reveal nothing of the underlying hydraulics or numerical approximation of the problem. Then *A.2.4 Smoothness of curve* states

Smoothness of a stage-discharge curve is also important, but it is a property which cannot be exactly defined. The criterion frequently accepted is that the first and second order differences should progress smoothly, and higher order differences should become very small. But acceptance of such a criterion is tantamount to accepting that the curve should approximately be a third or higher order polynomial. The higher order derivatives should show a tendency to diminish, but no test can be imposed in regard to smoothness for stage-discharge relation curves, as irregularities at some stages due to changes in control and irregularities of cross-section are inherent features of these relation curves.

This contains some curious statements amongst some other well-judged ones, and is typical of the apparently contradictory nature of much of the whole document. The initial statement about smoothness being important is contradicted by the last sentence. No definitions of first and second differences are provided so that no guidance is given to practitioners, but technically the statement is quite correct, that requiring them to diminish is to impose a cubic approximation. Once again, there would seem to be room for the application of rather more sophisticated numerical methods.

Subsequent sections *A.2.5 Methods for locating shift in control* and *A.2.6 Check on subsequent shifts in control* describe briefly some procedures, but once again the actual identification of a "shift in control" seems totally unnecessary, as the rating curve contains all the information necessary without any arbitrary breaking up into sections when the physical justification for that is unknown anyway.

The following Section *A.3 Uncertainty in the stage-discharge relation and in a continuous measurement* contains details of a number of statistical tests which assume a power law relation valid over the whole range, and as such, are considered to be of little or no use.

Annex B – Hysteresis in the stage-discharge relation

This describes the situation where effects of unsteadiness are important but contains no practical details, which follow in the subsequent Annexes.

> *In general, the sites chosen for gauging stations have channel characteristics (a gradient sufficiently steep, and a downstream channel of sufficient capacity) which ensure that, except for occasional changes in control which may affect calibration, the relationship between stage and discharge is substantially consistent in that one particular gauge height will indicate one corresponding discharge.*

That statement is partly true, but for flashy streams the effects of unsteadiness may be important even for steep waterways. The Standard has not quantified or explained when these effects may be important. Then it states with admirable clarity the reason for the looped nature of some rating curves:

> *Under certain conditions (flatter gradients and constricted channels), the phenomenon known as hysteresis (the effect on the stage-discharge relation at a gauging station subject to variable slope where, for the same gauge height, the discharge on a rising stage differs from that on the falling stage) can occur where a looped stage-discharge curve is obtained for floods with differing stage-discharge relations for rising and falling levels. The shape of loop rating curves can vary from station to station, and also at the same station, with the height of flood, but generally the curve for the rising stage will plot to the right of that for the falling stage, indicating a higher discharge for the same water level. This is usually due to the fact that during rising flood the slope of the flood wave-front is significantly steeper than the steady state hydraulic gradient of the river, the reverse applying during the recession. Difference in discharge caused by this effect can be significant.*

> *If discharge measurements are made equally on rising and falling stages, an average rating curve falling between the two is obtained, which in most cases is usually of sufficient accuracy. In practice, however, there is a tendency for flood gaugings to be made on the falling stage only, especially on rivers which rise quickly and carry quantities of debris on the rising flood. On stations, therefore, where channel conditions are favourable for hysteresis, precautions should be taken to check the extent of the effect before a decision is made on whether to use an average rating curve or a series of looped curves.*

Despite the appropriateness of the warning, no advice is given as to how this should be done.

Annex C – Twin gauge station fall-discharge method

As the title suggests, this introduces the use of two gauge stations. Unfortunately, despite an excellent justification for this, the Standard does not introduce it. The use of twin stations even overcomes the problems associated with unsteadiness, but this is not mentioned.

The Standard suggests

> *The plotting of the stage-discharge measurements, with the value of fall $\bar{z}_0 - \bar{z}_1$, where $\bar{z}_0$ (and) $\bar{z}_1$ are the upstream and downstream stages, respectively, marked against each measurement, will reveal whether the relationship is affected by variable slopes at all elevations or is affected only when the fall reduces below a particular value. In the absence of any channel control, the discharge would be affected by the fall at all times and a correction is applied by the fixed or the constant fall method (see below); on the other hand, however, when the discharge is affected only when the fall reduces below a particular value, the normal fall method is applied (see below).*

In fact, the discharge is governed by the fall (actually the slope) at all times and at all stages, according to Manning's or Chézy's laws assumed to extend to unsteady conditions.

Next in *C.2 Constant fall method* a rather *ad hoc* procedure is outlined where the fall can be included:

> *… for the same elevation $\bar{z}_0$ on the upstream gauge, the discharge is a function of the fall $H = \bar{z}_0 - \bar{z}_1$ that is: $Q = Q_n\, f\left(H / H_n\right)$ where Q is the measured discharge; Q_n is the constant fall discharge at the same stage; H is the fall corresponding to the measured stage; H_n is the selected constant fall (arbitrary) known as the reference fall.*

The determination of the functional relationship $f(H / H_n)$ is supposed to be able to be determined from a trial and error examination of the gauging values. To the writers, this two-dimensional approach would seem to be an optimistic approach indeed, given the difficulties even of identifying the rating curve for discharge as a function of stage alone.

Next, in *C.3 Normal fall method* a procedure is introduced where one is supposed to be able to examine a rating curve and determine at which points "backwater has no effect". The writers had great difficulty understanding the procedure, described in a single paragraph, and presumes that many others would have the same problem. There seems to be no real justification for it.

Annex D – Extrapolation of stage-discharge relation

This section shows some good judgement:

> *Stage-discharge relation curves are primarily intended for interpolation, and their extrapolation beyond the highest recorded high stage or the lowest recorded low stage may be subject to error. Physical factors such as overbank spills at high stages, shifts in controls at very low and very high stages, or changes in rugosity coefficients at different stages, materially affect the nature of the relationship at the extreme ends and must be taken into account. Extrapolation should be avoided as far as possible, but where this is necessary, the results obtained should be checked by more than one method. The physical conditions of the channel, that is whether the channel has defined banks over the entire range, or only up to a certain stage and over-bank spill above that stage, as well as whether the channel has fixed or shifting controls, should govern the methods to be used in the extrapolation. Consideration should also be given to the phenomenon of the kinematic effect of open channel flow when there may be a reduction in the mean velocity in the main channel during inundation of the flood plain.*

The Standard goes on to present some methods suitable for a channel with defined banks ("and fixed controls", whatever that means), as well as "for a channel with spill" but there seems no justification for that.

> ***D.2*** *If the control does not change beyond a particular stage, it may be possible to fit a mathematical curve as indicated in annex A and obtain the values in the range at the upper or lower end of the stage-discharge curve to be extrapolated.*

Which, as warned in the previous quotation, should be done very carefully indeed. The following procedure may be less prone to error:

> ***D.3*** *Another simple method would be the separate extensions of the stage area curve and the stage-mean velocity curve. The latter has little curvature under normal conditions and can be extended without significant error. The product of the corresponding values of A and $\bar{v}$ may be used for extending the discharge (Q) curve.*

An even more rational method is then described (#D.4):

> *The hydraulic mean radius (R_h) can be found for all stages from the cross-section. A logarithmic plot of ($\bar{v}$) against (R_h) generally shows a linear relationship, for the higher measurements and the values of $\bar{v}$ for the range to be extrapolated may be obtained therefrom.*
>
> ***D.5*** *A variation of the last method is by the use of Manning's formula*
>
> $$Q = A\bar{v} = \frac{AR_h^{2/3} S^{1/2}}{n}$$
>
> *Assuming $S^{1/2} / n$ remains constant and substituting (mean depth) $\bar{d}$ for R_h, a curve can be prepared for Q against $AR_h^{2/3} \approx A\bar{d}^{2/3}$. After the bank-full stage, the discharge of the spill portion will have to be worked out separately by assuming an appropriate value of n. If accurate gauges do not exist for computing the slope, it may roughly be estimated from the flood marks.*

A similar procedure is outlined using the Chézy friction formula.

Annex E – Correction for discharge in unsteady flow

This contains a presentation of the Jones method, described elsewhere in this report: "under certain conditions, it is possible to compute approximately the true discharge … of an unsteady flow from the normal discharge … obtained from the stage-discharge curve …". The Jones formula is presented without theoretical justification but with a couple of misleading qualifications. To conclude, the optimistic statement is made:

> *When a sufficient number of field measurements of discharge are available, it may be possible to establish an acceptable family of calibration curves empirically by evaluating the effect of the rate of change in stage measured at one gauge as a parameter.*

To conclude our review, we found the Standard to be a mixture of good judgement and bad presentation of methods for which there was little justification. Few details are presented of definitive numerical methods. The overall feeling is of a document written with little understanding of the processes actually at work, but with a great deal of emphasis on issues such as controls, whose effects are never explicit. The heavy emphasis on controls detracts considerably from the ability to understand the Standard.

Appendix B. The long wave equations and the nature of flood propagation

B.1 The long wave equations

B.1.1 The long wave equations for straight waterways

The flow of water and the propagation of long waves in waterways are described well by the long wave equations, which we present here as obtained by Fenton (2001). Consider the mass conservation equation:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=q\,, \tag{B.1}$$

where A is the cross-sectional area, t is time, Q is discharge, x is distance down the channel, which is assumed straight for this work, and q is the inflow per unit length. This equation is exact for waterways that are not curved in plan. The momentum equation written in terms of area and discharge is:

$$\frac{\partial Q}{\partial t}+\left(\frac{gA}{B}-\beta\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}=gA\left(\bar{S}-S_f\right)+qu_q-\frac{Q^2}{A}\frac{d\beta}{dx}\,, \tag{B.2}$$

where g is gravitational acceleration, S_f is the friction slope, $\bar{S}$ is the mean bed slope at a section, and the inflow has a velocity of u_q before mixing. The coefficient β is the Boussinesq momentum coefficient that is the correction that must be applied such that the integral of the square of the velocity over the section can be approximated in terms of discharge and area:

$$\int_A u^2\,dA=\beta\frac{Q^2}{A}\,. \tag{B.3}$$

We use an empirical friction law for the friction slope S_f, in terms of a conveyance function K, so that we write

$$S_f=\frac{Q^2}{K^2}\,, \tag{B.4}$$

where the dependence of conveyance K on stage may be determined empirically, or by a friction law, such as Manning's or Chézy's law:

$$\text{Manning's law: } K = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}} \text{ or Chézy: } K = C\frac{A^{3/2}}{P^{1/2}}, \tag{B.5}$$

where n and C are Manning's and Chézy's coefficients respectively, and P is the wetted perimeter of the section, such that as both A and P are functions of surface elevation at a section, it could be expressed in terms of A if that formulation were chosen.

Rather than the formulation in terms of surface area, it is usually more convenient to work in terms of stage (surface elevation) η, and if we assume that the water surface is horizontal across the stream, then $\partial A/\partial t = B\,\partial\eta/\partial t$, where B is the width of the water surface, so that the equations become (Fenton, 2001):

$$\frac{\partial\eta}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = \frac{q}{B}. \tag{B.6}$$

and

$$\frac{\partial Q}{\partial t} + \left(gA - \beta\frac{Q^2 B}{A^2}\right)\frac{\partial\eta}{\partial x} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} = \beta\frac{Q^2 B}{A^2}\bar{S} - gAS_f + qu_q - \frac{Q^2}{A}\frac{d\beta}{dx}. \tag{B.7}$$

Each pair of equations (B.1) and (B.2) and (B.6) and (B.7) is a pair of partial differential equations which express the evolution with time of disturbances in waterways. They are at the core of much hydraulic modelling, especially where rapid transients are required to be computed, and several commercial programs are available. There have been some incorrect interpretations of the nature of these equations, which will be described below.

Fenton and Nalder (1995) have obtained the equations for curved waterways, as shown below in Appendix B.1.3. They showed that the effects of curvature are proportional to B/R, the ratio of the breadth of the river to its radius of curvature in plan.

The mass conservation equations (B.1) and (B.6) are exact, within the approximation that the waterway is not curved. The derivation of the dynamic equation (B.2) and subsequently (B.7) requires the additional assumptions that the pressure distribution is hydrostatic and that the non-uniformity of velocity over a section can be expressed by the Boussinesq momentum coefficient. It is not necessary to assume that velocity is uniform over each cross section, as almost all results can be presented in terms of discharge. In the only term where an assumption has to be made as to the velocity distribution, the Boussinesq momentum coefficient has been introduced. We will show below that the terms containing this quantity are of the order of the Froude number squared, which is usually small, such that the precise value of β is usually not important, and in many applications terms containing it can be neglected.

It is not necessary to assume that the bed slope is small, as in the derivation a control surface is used where the important faces are perpendicular to cartesian coordinate axes, and a postulate is made that an approximation to the *horizontal* component of shear stresses is used.

B.1.2 Steady flow – equation for backwater curves

Here we consider an important special case of the equations, where the flow is steady, such that there is no variation with time. In this case, integrating equation (B.6) gives

$$Q(x) = Q_0 + \int_{x_0}^{x} q(x')\,dx', \tag{B.8}$$

where Q_0 is the flow at some point x_0, and we see that the flow at point x is modified by integrating the inflow between the points, in which x' is a dummy variable of integration. In many applications, where there is no inflow, Q is constant.

Now considering the momentum equation (B.7) for steady flow we have, using (B.8) where necessary, but generally retaining the symbol Q, understanding that (B.8) will be substituted, we have

$$\left(gA - \beta \frac{Q^2 B}{A^2} \right) \frac{d\eta}{dx} = \beta \frac{Q^2 B}{A^2} \bar{S} - gAS_f + q\left(u_q - 2\beta \frac{Q}{A} \right), \tag{B.9}$$

which is an ordinary differential equation for $d\eta / dx$, which may be solved using standard numerical methods for differential equations, to give the *backwater curve*. Equation (B.9) is rather more general than usual presentations of the equation. Further below we will see how this relates to those. The geometric quantities such as area and breadth are usually more conveniently expressed in terms of a local elevation, which in the case of prismatic channels might be the depth. Here a depth-like quantity is introduced which will be referred to as the *local elevation*, *h*, which is the local surface elevation above an arbitrary reference curve running the length of the waterway. In the case of a natural stream the reference curve might simply be a line inclined at the mean bed slope over a reach of the stream, or in the case of a prismatic canal with a flat bottom it might be a line coincident with the canal bottom, when *h* actually is the depth of water over that bottom. If the elevation of the reference curve is $z_0(x)$, then $\eta = z_0 + h$, and to relate the gradients we have to introduce the (possibly local) slope of the reference curve $S_0 = -dz_0 / dx$ following the usual sign convention in hydraulics that a downward sloping bed has a positive slope. Now, $d\eta / dx = dh / dx - S_0$, and substituting into equation (B.9) gives the equation in terms of *h*:

$$\left(gA - \beta \frac{Q^2 B}{A^2} \right) \frac{dh}{dx} = \beta \frac{Q^2 B}{A^2} \left(\bar{S} - S_0 \right) + gA\left(S_0 - S_f \right) + q\left(u_q - 2\beta \frac{Q}{A} \right). \tag{B.10}$$

In practice, one is likely to choose the local slope of the reference axis to be the mean slope of the bed such that $S_0 = \bar{S}$, and the differential equation becomes

$$\frac{dh}{dx} = \frac{S_0 - S_f + \dfrac{q}{gA}\left(u_q - 2\beta \dfrac{Q}{A} \right)}{1 - \beta \dfrac{Q^2 B}{gA^3}}. \tag{B.11}$$

Equation (B.11) is a generalisation of the usual presentation of the differential equation for backwater curve computations (*e.g.* Henderson, 1966, p125), which is for the case of no inflow, $q = 0$, and where the implicit approximation has been made that $\beta = 1$. In fact, the actual value of β does not matter very much, as in the denominator of (B.11) it multiplies a term $Q^2 B / gA^3$, which is the square of the Froude number, small in many applications and could be neglected.

B.1.3 The long wave equations for curved waterways

The equations corresponding to (B.1) and (B.2) but where the curvature of the waterway is included have been obtained by Fenton and Nalder (1995):

$$\left(1 - \kappa n_m \right) \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = q, \tag{B.12}$$

$$\begin{aligned} &\left(1 - \kappa \bar{n} \right) \frac{\partial Q}{\partial t} + \frac{Q}{A}\left(2 + 3\left(\kappa n_m - \kappa \bar{n} \right) \right) \frac{\partial Q}{\partial s} + \left(\frac{gA}{B} - \frac{Q^2}{A^2}\left(1 + 2\left(\kappa n_m - \kappa \bar{n} \right) \right) \right) \frac{\partial A}{\partial s} \\ &\qquad + gAS_f \left(1 - \kappa \bar{n} \right) - gA\bar{S} + \frac{Q^2}{A}\left(\kappa' n_m - \kappa' \bar{n} \right) = qu_q, \end{aligned} \tag{B.13}$$

where *s* is the distance co-ordinate along an axis down the river, $\kappa = \kappa(s)$ is the curvature of the river, equal to $1/r$, where r is the radius of curvature of the axis, taken positive for a river turning to the left and negative for turning to the right; n_m is the transverse offset of the mid-point of the surface from the curved axis down the river; and $\bar{n}$ is the transverse offset of the centroid from that axis. These equations have exactly the same structure as (B.1) and (B.2), the only real differences being that the coefficients of the various derivatives include terms in $\kappa n_m = n_m / r$, the ratio of the offset of the surface mid-point to the radius of curvature, and $\kappa\bar{n} = \bar{n} / r$, the ratio of the offset of the centroid to the radius of curvature. If $\kappa = 0$, equations (B.12) and (B.13) reduce to (B.1) and (B.2), although here they are presented for the case $\beta = 1$. In streams (such as canals) where the cross section is symmetric about the centreline, both $n_m = \bar{n} = 0$ and the curvature has no effect. Consideration of curvature complicates the equations, and in this work we will maintain the traditional approximation that all streams are straight.

B.1.4 The characteristic formulation and a misleading result for wave speed

The corresponding characteristic form of the equations will now be obtained, and it will be suggested that this has led to a traditional interpretation that is quite wrong. That interpretation is that disturbances in waterways travel at a speed given by the square root of gravitational acceleration times the mean depth, that is, $\sqrt{gA/B}$, where A/B is easily interpreted as the mean depth. In fact, although information in the form of characteristics does travel at that speed, physical quantities such as surface elevation do not. It is possible that many simple calculations based on the supposed propagation speed of long waves have been wrong in the past.

Either of the equation pairs (B.1) and (B.2) or (B.6) and (B.7) may be converted to a set of four ordinary differential equations, leading to the *characteristic formulation*. In this case there are paths known as characteristics along which information flows. The equations of the characteristics are given by the solution of the two differential equations (taking both sign alternatives $\pm$):

$$\frac{dx}{dt} = \beta\frac{Q}{A} \pm C, \tag{B.14}$$

where C is the quantity

$$C = \sqrt{\frac{gA}{B} + \frac{Q^2}{A^2}\left(\beta^2 - \beta\right)}, \tag{B.15}$$

whose significance will be discussed below. On the characteristics that are solutions of the differential equations in equation (B.14), using the formulation in terms of stage, the stage and discharge satisfy the differential equation

$$B\left(-\beta\frac{Q}{A} \pm C\right)\frac{d\eta}{dt} + \frac{dQ}{dt} = \beta\frac{Q^2 B}{A^2}\bar{S} - gAS_f + q\left(u_q - \beta\frac{Q}{A} \pm C\right) - \frac{Q^2}{A}\frac{d\beta}{dx}. \tag{B.16}$$

A similar equation exists for the area-discharge formulation. These results contain some interesting physical significance. The quantity dx/dt is the local gradient of the characteristics on a plot of distance x against time t; which could be termed the velocity of the characteristics. It contains two parts: the first, $\beta Q/A$ is simply β times the mean fluid velocity in the waterway at that section and shows the contribution of the fluid velocity to the velocity of the characteristics. It is interesting that the effect of β is expressed so simply, but possibly surprising that it is not like $\sqrt{\beta}$, as β multiplies Q^2/A in the original equations.

The next term in equation (B.14) is more noteworthy. The two solutions corresponding to the positive and the negative signs, $\pm C$, are the "wave" velocities of the characteristics relative to the water corresponding to both upstream and downstream propagation of information. Their definition, equation (B.15), is perhaps surprisingly complicated as obtained here, where we have included the momentum coefficient β. The familiar traditional result is that $C = \sqrt{gA/B}$, where A/B is the mean depth. Here we have generalised the formulation to allow for different values of β. The traditional result holds if we assume that velocity is constant over the section such that $\beta = 1$ or if we ignore the quadratic velocity terms altogether, $\beta = 0$.

The traditional result that $C = \sqrt{gA/B}$ has led to a widespread misconception in hydraulics such that C has often been referred to as the speed of propagation of waves. It is not – it is the speed of the *characteristics*. If surface elevation were constant on a characteristic there would be some justification in using the term "wave speed" for the quantity C, as disturbances travelling at that speed could be observed. However as equation (B.16) holds in general, neither surface elevation η, nor discharge Q is constant on the characteristics and one does not have observable disturbances or discharge fluctuations travelling at C relative to the water. While C may be the speed of propagation of information in the waterway relative to the water, it cannot properly be termed the wave speed as it would usually be understood.

B.1.5 The effects of velocity distribution

The coefficient β has been discussed in many places. Chow (1959, Chapter 2) wrote: "For channels of regular cross section and fairly straight alignment, the effect of non-uniform velocity distribution ... is small, especially in comparison with other uncertainties involved in the computation. Therefore the coefficients are often assumed to be unity". Xia and Yen (1994) examined results from numerical solutions of the long wave equations with pressure-correction coefficients and the momentum coefficient β. They found that the effects of β were small. Equation (B.3) points to this, as the coefficient multiplies a term proportional to the square of the discharge. This is examined below by non-dimensionalising the equations and showing that its relative magnitude is the square of the Froude number. In many flow situations this will be small. While large values of β are commonly encountered in a compound waterway such as a river and a floodplain, in this case the flow will often be of such a small Froude number that the term could be neglected altogether, which could be accomplished by setting $\beta = 0$ in computations. In general, the effect of β on wave propagation in waterways is expected to be small and no great effort need go into its evaluation.

B.1.6 The nature of wave propagation in waterways and the Telegrapher's equation

Either pair of equations (B.1) and (B.2), or (B.6) and (B.7), describe the motion of long waves in waterways. However they do not reveal the essential nature of the propagation of waves. Indeed, interpretations of them such as the method of characteristics in Appendix B.1.4 can lead to misleading deductions about the nature of wave propagation. More insight is provided if we approximate the equations by considering small disturbances about a uniform flow.

We linearise the equations by considering disturbances about a uniform steady state to be small, which gives a single linear equation. It is convenient to replace temporarily the stage (surface elevation) by a local depth co-ordinate h such that $\eta = z_b + h$, where z_b is the elevation of the bed co-ordinate axis, such that $\partial\eta/\partial x = \partial z_b/\partial x + \partial h/\partial x = -S_0 + \partial h/\partial x$. We consider the case where there is no inflow into the river $q = 0$ and we assume that the momentum coefficient β is constant. Let the steady uniform flow in the waterway be of depth h_0, discharge Q_0, area A_0 and conveyance K_0 such that $\bar{S} = Q_0^2 / K_0^2$. We write $h = h_0 + \varepsilon h_1 + 2$ and $Q = Q_0 + \varepsilon Q_1 + 2$, where ε is a small quantity expressing the magnitude of the disturbance. Substituting these expansions into equations (B.6) and (B.7), and taking only first order terms in ε gives a pair of linear equations. After cross-differentiation and back-substitution between them, the equations can be reduced to a single equation, the Telegrapher's equation, well known in electrical engineering for describing transients on lines with losses:

$$2\alpha_0\left(\frac{\partial\phi}{\partial t} + c_0\frac{\partial\phi}{\partial x}\right) + \frac{\partial^2\phi}{\partial t^2} + 2\beta U_0\frac{\partial^2\phi}{\partial t\partial x} + \left(\beta U_0^2 - C_0^2\right)\frac{\partial^2\phi}{\partial x^2} = 0, \qquad \text{(B.17)}$$

where ϕ could stand for the actual depth or discharge, h or Q. Various velocities appear here. The quantity c_0 is a velocity such that

$$c_0 = U_0 + V_0 = \frac{\sqrt{\bar{S}}}{B_0} \left.\frac{dK}{dh}\right|_0 , \tag{B.18}$$

which is simply the Kleitz-Seddon law for kinematic wave speed, which can be obtained rather more simply in the limited context of kinematic wave theory. We have introduced the symbol V_0 here as it makes subsequent expressions simpler to write. Other velocities appearing in the equation are: $C_0 = \sqrt{gA_0 / B_0}$, the traditional expression for the speed of long waves on still water; and $U_0 = Q_0 / A_0$, the mean fluid velocity in the stream.

Another quantity which has been introduced here is $\alpha_0 = gA_0 \sqrt{\bar{S}} / K_0$, which is a friction parameter. It can be written in any of the equivalent forms

$$\alpha_0 = \frac{gA_0\sqrt{\bar{S}}}{K_0} = \frac{gA_0\bar{S}}{Q_0} = \frac{g\bar{S}}{U_0} , \tag{B.19}$$

such that it increases with slope but decreases with conveyance/discharge/velocity.

Equation (B.17) has two parts: the first, involving first derivatives only, is the frictional or decay part, while the second is the wave propagation part. Such an equation has been obtained by Deymie and by Lighthill and Whitham (1955) and several others, often for special cases such as a rectangular waterway. Surprisingly little attention has been given to this formulation, yet it can convey some insight into how waves propagate in waterways.

Here we consider some approximations which can be used.

Friction negligible – horizontal stream

If we consider a horizontal waterway, with zero slope, then $\alpha_0 = 0$, the friction term in equation (B.17) disappears, and as for this case we also have $U_0 = 0$, no underlying flow, the equation becomes

$$\frac{\partial^2\phi}{\partial t^2} - C_0^2 \frac{\partial^2\phi}{\partial x^2} = 0$$

which is the Wave Equation, for which reference can be made to any book on engineering mathematics which deals with partial differential equations. Solutions of this equation are composed of arbitrary disturbances travelling at velocities of $\pm C_0$, the conventional solution for long waves on a frictionless fluid, travelling in both directions relative to the water, where $C_0 = \sqrt{gA_0 / B_0}$. This is unlikely to be important in river hydraulics.

Friction dominant - kinematic wave theory

Generally in the case of floods in rivers the friction is a dominant effect. If the frictional first term in equation (B.17) only is retained, then we have

$$\frac{\partial\phi}{\partial t} + c_0 \frac{\partial\phi}{\partial x} = 0 ,$$

with the solution that of an arbitrary disturbance travelling without change at a velocity of c_0, which is the kinematic wave solution. In the case of a wide river, the relative variation of perimeter with stage is small, so that if Manning friction is used, $c_0 \approx 1.7 \times U_0$. Waves in rivers are much more likely to travel at this kinematic wave speed (about 1.7 times the mean fluid velocity) than at the speed $C_0 = \sqrt{gA_0 / B_0}$, almost universally described as the speed of long waves in waterways.

Intermediate case – moderate friction

We would now like to examine the intermediate case of equation (B.17). Unfortunately simple deductions from the form of the partial differential equation seem not to be possible. However, we obtain useful insight by assuming that, as any disturbance can be written as a Fourier series, we can consider just one term of that Fourier series and examine how it propagates, namely what is its speed of propagation and how does it decay as it travels. This is only possible as the equation is linear in ϕ. Ponce and Simons (1977) have done this.

It is simpler to write the variation in *x* in complex form, such that we represent all variation along the waterway as $\exp(ikx)$, where $i = \sqrt{-1}$ and $k = 2\pi / L$, where *L* is the wavelength we are considering. The exponential could be expanded: $\exp(ikx) = \cos kx + i \sin kx$, thereby revealing more its periodic form. We write for the general solution:

$$\phi = \exp(ikx + \mu t), \tag{B.20}$$

showing the periodic behaviour in *x*, and where the behaviour in time *t* is contained in the term μ. The real part of μ shows how the solution decays or grows in time, the imaginary part will determine how the solution oscillates in time as the periodic waves pass a point. We substitute (B.20) into (B.17) and find that it does satisfy the equation exactly, giving a quadratic equation for μ which can be solved. The solution is rather shorter if we make the sensible approximation that the momentum coefficient $\beta = 1$ in equation (B.17), and it is:

$$\mu = -\alpha - ikU_0 \pm \sqrt{\alpha^2 - k^2 C_0^2 - 2i\alpha k V_0} \,.$$

This is a complex expression. Extracting both real and imaginary parts gives quite complicated expressions. We can see that part of the variation in time is simply that of a wave being carried downstream at the underlying velocity in the channel of U_0, as shown by part of the behaviour being like $x - U_0 t$, were we to substitute this back into equation (B.20). The remainder of the imaginary part gives us an expression for the speed of propagation of a wave relative to the flow:

$$\frac{C}{C_0} = \frac{1}{\sqrt{2}} \sqrt{1 - \left(\frac{\alpha}{kC_0}\right)^2 + \sqrt{\left(1 - \left(\frac{\alpha}{kC_0}\right)^2\right)^2 + \left(2 \frac{V_0}{C_0} \frac{\alpha}{kC_0}\right)^2}} \,, \tag{B.21}$$

where *C* is the general expression for the speed of propagation, and we can see that in general it depends on the wavelength *via* $k = 2\pi / L$. Thus, contrary to popular belief, the long wave equations *with friction* show wave propagation where the wave speed depends on the wavelength, and is not a constant. In fact, if we expand equation (B.21) as a power series in α / kC_0 we obtain

$$\frac{C}{C_0} = 1 - \frac{1}{2}\left(1 - \left(\frac{V_0}{C_0}\right)^2\right)\left(\frac{\alpha}{kC_0}\right)^2 + 2 \,, \tag{B.22}$$

and we see that in the limit of no friction, $\alpha \to 0$, we obtain $C \to C_0$, as we obtained above.

Possibly of greater interest is the case where α / kC_0 is large. To explore this we expand (B.21) as a power series in $(\alpha / kC_0)^{-1}$, the inverse of what we used previously. The result is

$$C = U_0 + V_0 \left(1 + \frac{1}{2}\left(1 - \left(\frac{V_0}{C_0}\right)^2\right)\left(\frac{kC_0}{\alpha}\right)^2 + 2\right), \tag{B.23}$$

This shows that in the limit of strong friction, where kC_0 / α is small, the solution propagates at a velocity of $U_0 + V_0$, which is c_0, the kinematic wave speed, governed by the velocity of the water in the channel, rather than the speed of long waves on still water!

B.2 Low-inertia approximation to the equations for rivers

B.2.1 Non-dimensionalisation of the equations

To examine the importance of the various terms we non-dimensionalise the equations, temporarily neglecting the inflow terms. We use four different length scales: L_x is the length scale along the channel, L_y is the width scale, L_d is the depth scale, and L_z is the drop scale, the amount by which the channel loses elevation in a distance L_x such that slope is of the order of L_z / L_x. We non-dimensionalise area *A* with respect to $L_y L_d$; breadth *B* with L_y; discharge Q with $UL_y L_d$, where *U* is a characteristic mean flow velocity; slopes with respect to L_z / L_x; *x* with L_x; and *t* with respect to a time scale *T*. Substituting the non-dimensionalising expressions into equations (B.6) and (B.7) the equations become, *where all quantities are the dimensionless equivalents of those in the original equations, for example A here actually means $A / L_y L_d$, and so on*:

$$\theta \frac{\partial \eta}{\partial t} + \frac{1}{B} \frac{\partial Q}{\partial x} = 0, \tag{B.24}$$

$$F^2 \left(\theta \frac{\partial Q}{\partial t} - \beta \frac{Q^2 B}{A^2} \frac{\partial \eta}{\partial x} + 2\beta \frac{Q}{A} \frac{\partial Q}{\partial x} - \beta \frac{Q^2 B}{A^2} \overline{S} \right) + gA \left(\frac{\partial \eta}{\partial x} + \frac{L_z}{L_d} S_f \right) = 0, \tag{B.25}$$

where we have ignored the derivative of β in the last equation.

The quantity θ in (B.24) and (B.25) is the dimensionless ratio L_x / UT, expressing the relative time scale of motion, the ratio of the physical length scale L_x to the length scale given by the time scale of change multiplied by the velocity at which it is carried. Usually in flows in natural waterways the velocity of motion is finite and the time scale of motion is dictated by the velocity at which disturbances are carried by the flow. In this case the time scale $T \sim L_x / U$, in which case θ is of order one, which will be the case described in this work. Where it might be greater is, for example, in situations such as where a level channel connects two bodies of water, and waves might be generated in one of them. We will not consider that here.

The quantity F^2 in (B.24) and (B.25) is the square of the Froude number scale of the flow, $F^2 = U^2 / gL_d$ and it can be seen that most of the terms in the momentum equation are of this order. Not only are the momentum flux terms involving β of order F^2 but so is the $\partial Q / \partial t$ term. In most river flows, even in floods, the Froude numbers are still sufficiently small that we can ignore these terms. For example, a flow of 1 m/s with a depth of 2m has $F^2 \approx 0.05$.

There is another scale in equation (B.25), and that is the ratio of the vertical scales L_z / L_d, which is the ratio of the drop in river bed or surface level, over a reach of interest, to the water depth. In the case of an irrigation canal, this might be of order of magnitude about 1, whereas in a mountain stream it will be small. It seems that in general no approximations can be made as to its magnitude.

Price (1985) also performed a non-dimensionalisation, as did Sivapalan, Bates, and Larsen (1997), and they both obtained complicated advection-diffusion equations, where there were mixed derivatives in *x* and *t*. Price obtained the results in terms of F^2 and the depth/drop parameter $\varepsilon = L_d / L_z$, and effectively obtained the same results as equation (B.25). He stated that for rivers $F^2 \le \varepsilon \le 1$, which seems slightly strange. In the lower reaches of a river it is possible that the depth-drop parameter could easily be greater than one. However, as he wanted to model rivers where the depth was much smaller than the drop, he developed an approximation which was $O(\varepsilon)$, and made no approximation as to F^2, which is strange in view of the claimed size hierarchy. This is effectively a "steep slope" approximation that is complicated.

B.2.2 Low-inertia routing – a new approximation and the old advection-diffusion formulation

Now if we ignore the terms of the order of F^2 as revealed in equation (B.25), its dimensional counterpart which is equation (B.7), the momentum equation, becomes simply

$$\frac{\partial \eta}{\partial x} + S_f = 0, \tag{B.26}$$

which expresses the fact that, even in a generally unsteady situation, the surface slope and the friction slope are the same magnitude, remembering that the free surface will slope downwards so that its gradient is negative. Although expressions such as equation (B.26) are known in the literature, it has not always been known that it is such a good approximation to the dynamic equation, errors only being of order $O(F^2)$.

Now we use an empirical friction law for the friction slope S_f in terms of a conveyance function *K*, so that we write

$$S_f = \frac{Q^2}{K^2}, \tag{B.4}$$

where the dependence of *K* on stage at a section may be determined empirically, or by a standard friction law, such as Manning's or Chézy's law.

Substituting (B.4) into (B.26) gives us an accurate expression for the discharge in terms of the slope:

$$Q = K\sqrt{-\frac{\partial \eta}{\partial x}}, \tag{B.27}$$

even in a generally unsteady flow situation, provided the Froude number is sufficiently small. This good approximation to the dynamic equation has a number of implications for flow measurement that are examined in the body of this report.

Now we eliminate the discharge *Q* from the equations by simply substituting equation (B.27) into the mass conservation equation (B.6) to give the single partial differential equation in the single variable η:

$$\frac{\partial \eta}{\partial t} + \frac{1}{B}\frac{\partial}{\partial x}\left(K\sqrt{-\frac{\partial \eta}{\partial x}}\right) = \frac{q}{B}. \tag{B.28}$$

The conveyance is usually expressed as a function of roughness and of the geometry. We will not show the dependence on roughness explicitly, but we show it as depending on both *x* and η so that we write $K(x,\eta)$, where of course η is a function in general of *x* and *t*. Hence we can perform the differentiation in equation (B.28) to give

$$\frac{\partial \eta}{\partial t} + \frac{1}{B}\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \eta}\frac{\partial \eta}{\partial x}\right)\sqrt{-\frac{\partial \eta}{\partial x}} = \frac{K}{2B\sqrt{-\frac{\partial \eta}{\partial x}}}\frac{\partial^2 \eta}{\partial x^2} + \frac{q}{B}. \tag{B.29}$$

The equation is a single partial differential equation in a single variable η. Usually the upstream boundary condition is where the discharge is given as a function of time, and from equation (B.27)

$$Q(0,t) = K(0, \eta(0,t))\sqrt{-\frac{\partial \eta}{\partial x}(0,t)}, \tag{B.30}$$

and so if the discharge is specified then the equation is a rather messy equation connecting the surface elevation and its spatial derivative at the upstream end, which we have presumed to be at $x = 0$.

At the downstream end, if there is a control, where discharge is some known function of surface elevation, then an even more complicated situation arises. The simplicity of derivation of this equation may not be useful in practice. Below we present an alternative form which is probably more practical.

As the conveyance is more likely to be able to be expressed in terms of the local elevation of the water surface, we introduce again the local depth *h* above a reference curve $z_0(x)$ such that $\eta = z_0 + h$, and $\partial \eta / \partial x = \partial h / \partial x - S_0$, and substituting into equation (B.29) gives the equation in terms of *h*, where we now have $K(x,h)$:

$$\frac{\partial h}{\partial t} + \frac{1}{B}\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial h}\frac{\partial h}{\partial x}\right)\sqrt{S_0 - \frac{\partial h}{\partial x}} = \frac{K}{2B\sqrt{S_0 - \frac{\partial h}{\partial x}}}\left(\frac{\partial^2 h}{\partial x^2} - \frac{\partial S_0}{\partial x}\right) + \frac{q}{B}. \tag{B.31}$$

This equation is essentially of the same form, but contains an extra term in the derivative of the slope of the reference curve. In many situations this will be zero. Still, this equation is of considerable generality, and the only approximation still is that the square of the Froude number be small. In the common situation where the channel is prismatic and its properties do not change along its length such that we can write $K = K(h)$ only, and it is of constant slope, we can write

$$\frac{\partial h}{\partial t} + \frac{1}{B}\frac{dK}{dh}\frac{\partial h}{\partial x}\sqrt{S_0 - \frac{\partial h}{\partial x}} = \frac{K}{2B\sqrt{S_0 - \frac{\partial h}{\partial x}}}\frac{\partial^2 h}{\partial x^2} + \frac{q}{B}. \tag{B.32}$$

Further, if the variation of the local depth is small compared with the overall slope so that $|\partial h / \partial x| << S_0$ we can write

$$\frac{\partial h}{\partial t} + \frac{\sqrt{S_0}}{B}\frac{dK}{dh}\frac{\partial h}{\partial x} = \frac{K}{2B\sqrt{S_0}}\frac{\partial^2 h}{\partial x^2} + \frac{q}{B}. \tag{B.33}$$

We can see that the coefficient of $\partial h / \partial x$ is simply c_0 from equation (B.18), which is the advective velocity at which solutions move, the kinematic wave speed. The coefficient of $\partial h^2 / \partial x^2$ is the diffusion coefficient

$$D_0 = \frac{K}{2B\sqrt{S_0}}. \tag{B.34}$$

The equation then can be written

$$\frac{\partial h}{\partial t} + c_0\frac{\partial h}{\partial x} = D_0\frac{\partial^2 h}{\partial x^2} + \frac{q}{B}, \qquad \text{(Advection-diffusion equation)}$$

which reveals its essential nature more. It is in this form that the equation is usually written and is quite well-known. It can be seen that rather more approximations have been made in its derivation.

B.2.3 Volume routing

Here we use a transformation of variables which enables us to use a single dependent variable in low-inertia routing and to introduce an equation which makes fewer approximations. Consider the mass conservation equation (B.1), which we repeat here:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=q\,. \tag{B.1}$$

Now consider the volume of fluid upstream of a point x at a time t, denoted by $V(x,t)$. From simple calculus, the derivative of volume with respect to distance x gives the cross-sectional area: $\partial V/\partial x=A$, and as the time rate of change of V at a point is equal to the total rate upstream at which the volume is increasing, which is $\int_x q\,dx$ less Q, the volume rate which is passing the point, we have $\partial V/\partial t=\int_x q\,dx-Q$. Hence, substituting the relations

$$A=\partial V/\partial x \text{ and } Q=\int_x q\,dx-\partial V/\partial t \tag{B.35}$$

into equation (B.1) shows that it is identically satisfied! This might have been expected, as the equation is a mass conservation equation, and hence for an incompressible fluid it is a volume conservation equation.

Now, with this ability to use upstream volume we go on to use it in the simplified momentum equation. Firstly, it can be shown (Fenton, 2001) that the derivative of the cross-sectional area can be related to the derivative of the stage by

$$\frac{\partial A}{\partial x}=B\left(\frac{\partial \eta}{\partial x}+\overline{S}\right),$$

so that the simplified momentum equation (B.26) becomes

$$\frac{1}{B}\frac{\partial A}{\partial x}=\overline{S}-S_f\,, \tag{B.36}$$

such that if we use the frictional law (B.27):

$$\frac{1}{B}\frac{\partial A}{\partial x}+\frac{Q^2}{K^2}-\overline{S}=0\,, \tag{B.37}$$

and substituting for Q and A in terms of V, from equation (B.35) and as both breadth B and conveyance K can be written as functions of area we obtain the single equation in the single variable

$$\frac{\partial V}{\partial t}+K(V_x)\sqrt{\overline{S}-\frac{1}{B(V_x)}\frac{\partial^2 V}{\partial x^2}}=\int_x q\,dx\,, \tag{B.38}$$

which we will term the *Volume Routing Equation*, in which the *only* approximation relative to the long wave equations has been that we ignore terms of $O(F^2)$, such that it will be accurate for $F^2 \ll 1$. Usually the inflow term on the right hand side will be zero.

The Volume Routing Equation might be useful in a range of hydrologic and hydraulic computations, replacing the solution of the long wave equations. It is a nonlinear partial differential equation that is a single equation in a single variable. From it, deductions can be made about the nature of wave propagation in waterways, which are not as misleading as those from the characteristic formulation of the long wave equations.

Here we consider what boundary conditions might be specified. At an upstream boundary x_0 we might have a given inflow as a function of time $Q(x_0,t)$, which, from equation (B.35) gives us what dV/dt is there, such that we have also to solve the ordinary differential equation $dV(x_0,t)/dt=-Q(x_0,t)$ there as part of the solution, which is relatively simple. At control points, we will usually have some relationship between Q and A such as provided by weir formulae, which gives $\partial V/\partial t$ as a function of $\partial V/\partial x$ there. As part of the solution we will have to differentiate numerically to give the latter and then integrate to give the updated value of V. At open boundaries, where there is no control point, we may simply be able to apply the partial differential equation as if it were an interior point, although if finite differences were being used to evaluate the spatial derivatives a different formula in terms of points to one side would have to be used.

B.2.4 A linear approximation to the volume routing equation

Unfortunately the Volume Routing Equation (B.38) does not reveal simply its essential nature as an advection diffusion equation, and we have to show this mathematically. If we consider small perturbations of this equation about a steady flow of area $A_0(x)$ and discharge $Q_0 + \int_{x_0}^{x} q(x')\,dx'$, then we write

$$V = \int_{x_0}^{x} A_0(x')\,dx' - Q_0(x)\,t + \varepsilon v(x,t)\,,$$

where ε is a small quantity which expresses the magnitude of perturbations about the base flow and v is the perturbation volume, then we have

$$\frac{\partial V}{\partial x} = A_0(x) + \varepsilon\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial V}{\partial t} = -Q_0(x) + \varepsilon\frac{\partial v}{\partial t}\,,$$

and using series expansions for K and B about the steady flow, such that, for example, $K(V_x) = K_0 + \varepsilon K_0^{/}\,\partial v/\partial x$, where $K_0 = K(A_0)$ and $K_0^{/} = dK/dA\big|_{A_0}$, substituting into equation (B.38) taking series where necessary and ignoring all products ε^2 and higher, we obtain at zeroth order (the coefficient of ε^0):

$$\frac{dA_0}{dx} + B(A_0)\left(\frac{Q_0 + \int_{x_0}^{x} q(x')\,dx'}{K(A_0)}\right)^2 - \bar{S}\,B(A_0) = 0\,. \qquad \text{(B.39)}$$

This is an ordinary differential equation to be solved for $A_0(x)$, written showing where the dependent variable A_0 occurs. It is more familiar if it is written in terms of the steady local elevation h_0 such that $dA_0/dx = B(A_0)\,dh_0/dx$ as well as in terms of the friction slope S_f without showing the dependencies, giving:

$$\frac{dh_0}{dx} = \bar{S} - S_f\,. \qquad \text{(B.40)}$$

This can be compared with the differential equation (B.11) for backwater curve computations obtained above from the full long wave equations, presented here for the case of no inflow:

$$\frac{dh}{dx} = \frac{\bar{S} - S_f}{1 - \beta F^2}\,, \qquad \text{(B.41)}$$

where $F^2 = Q^2 B / gA^3$ is the square of the Froude number, which in this section we have assumed small and are neglecting it. As numerical solution of (B.41) is no more difficult than (B.40), there might be some small gain in using it to determine the initial steady solution.

Now we extract the coefficient of ε^1 from the substitution into equation (B.38)and obtain

$$\frac{\partial v}{\partial t} + u_0(x)\frac{\partial v}{\partial x} = D_0(x)\frac{\partial^2 v}{\partial x^2} \qquad \text{(B.42)}$$

where the coefficients $u_0(x)$ and $D_0(x)$ are functions of the underlying steady state solution:

$$u_0(x) = \frac{K_0^{/} Q}{K_0} + \frac{B_0^{/}\left(\bar{S}K_0^2 - Q^2\right)}{2QB_0}, \text{ and} \qquad \text{(B.43)}$$

$$D_0(x) = \frac{K_0^2}{2QB_0}. \tag{B.44}$$

We have used the symbol $Q = Q_0 + \int_{x_0}^{x} q(x')\,dx'$ to denote the flow at any section. If there is no inflow then $Q = Q_0 = \text{constant}$. In the form given by (B.42) the equation governing the propagation of floods and long waves is an Advection-Diffusion Equation with variable coefficients. In fact, the advection-diffusion approximation has been known since 1951. Here, however, we have derived it without making any approximation that the underlying flow is uniform – we have allowed it to be a general backwater curve. Usually Q is constant, but if the flow is not uniform, then B_0 and K_0 and their derivatives are functions of x.

B.2.5 The nature of flood and long wave propagation in waterways

In the common approximate case where it is assumed that the underlying flow is uniform such that $Q = K_0\sqrt{S}$, then the coefficients are constant, and we obtain $u_0 = K_0'\sqrt{S} = dQ/dA_0$, the Kleitz-Seddon law, such that u_0 is constant and is known as the kinematic wave speed. We have shown that it is in general variable and have given it a symbol that is more suggestive of it being a local advection velocity. Also in this uniform flow case, $D_0 = K_0/2B_0\sqrt{S} = Q/2B_0\overline{S}$, which has been obtained by several people.

Previously we have suggested that the long wave equations themselves do not reveal the nature of wave propagation at all. If one recasts them in a characteristic formulation then some information as to the nature of the transmission of information is extracted, but some misleading results can also be inferred, such as the deduction that disturbances travel at a speed $c = \sqrt{g \times \text{Mean depth}}$. The present work has shown that, provided terms proportional to the square of the Froude number are ignored, that floods and long waves obey an advection-diffusion equation. This means that the waves are carried along locally at a velocity $u_0(x)$ given by equation (B.43), such that there is no wave motion back up the waterway. Disturbances can travel up the waterway, but only by a process of diffusion as described below. The magnitude of u_0 can be estimated by using the uniform flow approximation $u_0 = K_0'\sqrt{S} = dQ/dA_0$. In the case where Manning's friction law is assumed we have $K = 1/n \times A^{5/3}/P^{2/3}$, and differentiation with respect to area gives

$$u_0 = \frac{Q_0}{A_0}\left(\frac{5}{3} - \frac{2}{3}\frac{A_0}{P_0}\frac{dP_0}{dA_0}\right).$$

It is more convenient to express the derivative in terms of the local height, and so using the fact that $dA_0/dh_0 = B_0$ we have

$$u_0 = \frac{Q_0}{A_0}\left(\frac{5}{3} - \frac{2}{3}\frac{A_0}{B_0 P_0}\frac{dP_0}{dh_0}\right).$$

The quantity dP_0/dh_0 is easily shown to be related to $1/\sin\theta$, where θ is the angle of the banks, so that this might have a value of say, $2\times 1/(1/2) = 4$, for $\theta = 30^{\text{A}}$. This finite value means that the relative contribution due to the perimeter changing is of the order of A_0/B_0P_0, which is the mean depth divided by the wetted perimeter, which will be small for wide channels. Hence, ignoring this contribution we see that if Manning friction is used, the advection velocity, the velocity with which disturbances are transported, is about $5/3$ times the mean velocity in the waterway, Q_0/A_0. If we had used Chézy friction this factor would have been $3/2$. The actual numerical value is perhaps not as important as the knowledge that in waterways where the square of the Froude number is small, that the above theory accurately represents the actual behaviour of disturbances in the waterway, and that the velocity with which those disturbances are transported is of the order of the mean velocity in the water, and has nothing to do with the square root of gravity times depth.

That discussion has, however, ignored the effects of the remaining diffusion term containing the second derivative with respect to *x*. It is well known in physics and mathematics that the behaviour of diffusion is to eliminate any discontinuities, to reduce the amplitudes of disturbances and to smear them out. This effect is proportional to the diffusion coefficient.

Now let us consider the relative importance of various terms in the advection diffusion equation, (B.33), but where we assume that we have the simpler case of underlying uniform flow so that it becomes

$$\frac{\partial v}{\partial t} + K_0' \sqrt{S_0} \frac{\partial v}{\partial x} = \frac{K_0}{2B_0\sqrt{S_0}} \frac{\partial^2 v}{\partial x^2}. \qquad \text{(B.45)}$$

The time scale of events is the length scale divided by the velocity with which they are transported, so that we can say that the time derivative is of relative magnitude $\partial / \partial t \sim U_0 / L_x$. We have shown above that the advection velocity is of the order of the fluid velocity, so that we can say $u_0 \partial / \partial x \sim U_0 / L_x$ also, while the diffusion term varies like

$$\frac{K_0}{2B_0\sqrt{S_0}} \frac{\partial^2}{\partial x^2} \sim \frac{U_0 L_y L_d}{L_y \times L_z / L_x} \times \frac{1}{L_x^2} \sim \frac{U_0 L_d}{L_x L_z} ,$$

and we see that relative to both the time and advection terms, the diffusion term has a magnitude of L_d / L_z, which is the ratio of the depth to drop scales. This shows a very surprising result, which is contained in the formulae above, that for streams which are steep and the water depth relatively shallow, such as steep mountain streams, the gravity and friction terms are in balance, and in this high friction limit, apparently paradoxically, the flood wave moves as a kinematic wave with little diminution. On the other hand, where slopes are shallow and friction less, such as in irrigation channels, the effects of diffusion are stronger, in apparent contradiction to what one might expect. This is shown by equation (B.45), where the slope is in the denominator of the diffusion term. This is an unusual and unexpected result, and opposes the intuitive practical interpretation of the behaviour of waves in rivers and channels that they travel without a great deal of diminution due to friction. In fact, waves in a waterway with a mild slope may be markedly diminished in height and spread out much more in space and time.

We have shown for situations where the Froude number is small, that waves propagate only in the downstream direction, at a velocity $u_0(x)$, but that they also show diffusion, with coefficient $D_0(x)$. For constant coefficients and based on a supposed perturbation about a uniform flow, this is well known. However, there still seems to be some confusion because of the continued presentation of the ubiquitous expression for the speed of characteristics depending on the square root of the mean depth.

We have shown that it is possible to formulate a low-inertia model such that it is a good approximation in most cases of rivers and canals, and hence is worthy of further examination, as it is considerably simpler than the full equations, both in presentation and numerical properties. It shows that waves in waterways with finite friction travel downstream at a speed roughly equal to $5/3 \times \text{Water speed}$, and *not* upstream and downstream at a speed given by $c = \sqrt{g \times \text{Mean depth}}$, widely used for back of the envelope calculations. In reality, however, the diffusion coefficient can be large, and the equation behaves more like the diffusion equation, where disturbances are diminished and spread out in space as time passes. In this case, *there is no such thing as a wave speed*. It can be shown that

$$\text{Relative importance of diffusion} = \frac{\text{Depth of water}}{\text{Drop of waterway}} .$$

Appendix C. Calculating the discharge hydrograph from stage records

C.1 Using the full long wave equations

The following has been given in more detail in Fenton (1999). Our task in this work is to eliminate spatial derivatives from the equations in favour of time derivatives so that we can take data measured at a point in a waterway, such as at a gauging station, and extract as much information as possible using the data available in the form of a number of readings over time at a single point. If we eliminate the $\partial Q / \partial x$ term between the two equations (B.6) and (B.7), and using the general quadratic friction law (B.4), we have

$$\frac{\partial Q}{\partial t} - 2\beta \frac{QB}{A}\frac{\partial \eta}{\partial t} + \left(gA - \beta \frac{Q^2 B}{A^2} \right)\frac{\partial \eta}{\partial x} + Q^2 \left(\frac{gA}{K^2} - \beta \frac{B\overline{S}}{A^2} + \frac{1}{A}\frac{d\beta}{dx} \right) = q\left(u_q - 2\beta \frac{Q}{A} \right). \qquad \text{(C.1)}$$

Such a procedure was adopted by Faye and Cherry (1980). To determine the discharge hydrograph from stage records the problem remains to eliminate the $\partial \eta / \partial x$ term, which would give us an ordinary differential equation for $Q(t)$, provided we knew the stage $\eta(t)$ at all times at a particular point, from which we could calculate the quantities *A*, *B*, and *K* at any time. Faye and Cherry used the kinematic wave equation to eliminate the space derivative. However we can use a higher level approximation than that.

A tradition in river hydraulics has been to eliminate higher time derivatives in favour of space derivatives, giving kinematic and diffusion theories. The latter was originally obtained by Hayami (see, for example, Henderson, 1966, #9.6), and involves a single time derivative. Here we proceed in the other direction by eliminating all but a single space derivative. We obtain this automatically by writing

$$\frac{\partial}{\partial x} = a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial^2}{\partial t^2} + a_3 \frac{\partial^3}{\partial t^3} + 2\ . \qquad \text{(C.2)}$$

such that we assume that we can replace *x* differentiation with a series of space differentiations. We will substitute this expression into the Telegrapher's equation (B.17), which we reproduce here

$$2\alpha_0 \left(\frac{\partial \phi}{\partial t} + c_0 \frac{\partial \phi}{\partial x} \right) + \frac{\partial^2 \phi}{\partial t^2} + 2\beta U_0 \frac{\partial^2 \phi}{\partial t \partial x} + \left(\beta U_0^2 - C_0^2 \right) \frac{\partial^2 \phi}{\partial x^2} = 0\,. \qquad \text{(B.17)}$$

This equation is a linearised approximation to the full long wave equations, its only approximation being that deviations of flow and depth from those of a uniform flow are small.

After substitution of (C.2), we treat the equation as a polynomial in the differential operator $\partial / \partial t$, and each power of the operator gives an equation, to give a sequence of linear equations in the a_1 *etc*. These can be solved to give the equation which is an approximation to the Telegrapher's equation and hence to the long wave equations:

$$\frac{\partial \phi}{\partial x} = -\frac{1}{c_0}\frac{\partial \phi}{\partial t} + \frac{D_0}{c_0^3}\frac{\partial^2 \phi}{\partial t^2} + \frac{G_0}{c_0^5}\frac{\partial^3 \phi}{\partial t^3} + \text{Higher order terms}\,, \qquad \text{(C.3)}$$

where $D_0 = \dfrac{C_0^2 - \beta V_0^2 + (\beta - 1)c_0^2}{2\alpha_0}$, and $G_0 = -\dfrac{\left(C_0^2 - \beta V_0^2 + (\beta - 1)c_0^2\right)\left(C_0^2 + \beta U_0 V_0\right)}{2\alpha_0^2}$.

Equation (C.3) is an equation for the space derivative of discharge or area or depth at a point in terms of the time derivatives there, which is what we want.

If we had proceeded in the other direction, of eliminating higher *time* derivatives, we would have obtained

$$\frac{\partial \phi}{\partial t} + c_0 \frac{\partial \phi}{\partial x} = D_0 \frac{\partial^2 \phi}{\partial x^2} + 2 \ , \qquad \text{(C.4)}$$

which is the advection-diffusion equation. If we had truncated after the first term such that there were no terms on the right it is the kinematic wave equation, whose solutions are simply waves which translate without change at a velocity of c_0, as obtained by Lighthill and Whitham (1955). This presentation is a generalisation of previous derivations, for we have included the momentum coefficient β as well as terms in the definition of D_0 and G_0 which are of a magnitude which is the square of the Froude number, which have been reasonably neglected in previous presentations.

We, who thus far have made no approximations, now eliminate the $\partial \eta / \partial x$ term from equation (C.1) by using the approximation to the equations of motion, equation (C.3), written for depth *h*. Using the generic friction law (B.4) and with the identity connecting the gradients of stage and depth, $\partial \eta / \partial x = \partial h / \partial x - \overline{S}$, we obtain the ordinary differential equation for $Q(t)$:

$$\frac{dQ}{dt} = \left(gA - \frac{\beta B}{A^2} Q^2 \right) \left(\frac{1}{c} \frac{d\eta}{dt} - \frac{D}{c^3} \frac{d^2\eta}{dt^2} - \frac{G}{c^5} \frac{d^3\eta}{dt^3} \right) + 2\beta \frac{B}{A} \frac{d\eta}{dt} Q + gA\overline{S} - \left(\frac{gA}{K^2} + \frac{1}{A} \frac{d\beta}{dx} \right) Q^2 . \qquad \text{(C.5)}$$

We have replaced all partial derivatives with ordinary derivatives, as we are evaluating them at a fixed point. We have also dropped the subscripts 0 pertaining to the steady uniform flow about which we linearised, as there may be some gain in accuracy in using the actual local values of all the flow quantities. For a particular value of η, the geometric quantities *A*, *B*, $\overline{S}$ and the conveyance *K* are known, as are *c*, *D*, and *G*. From the stage record we can calculate all the necessary time derivatives of η, so that (C.5) is an ordinary differential equation in $Q(t)$ which we can solve numerically.

There are a number of approximations that could be introduced in solving the differential equation (C.5), such as neglecting the third derivative term. The computation of a third derivative from field data may well not be an accurate procedure, anyway. One could neglect momentum flux terms quadratic in *Q*, however this is not strictly necessary. Generally one could set $\beta = 1$, which is the sensible and common approximation used elsewhere in hydraulics, however one may not have to introduce it so readily, because usually at gauging stations the conveyance *K* is obtained from detailed measurements of the velocity distribution at a site. Unlike in most areas of channel hydraulics, one might be able to use a meaningful value of β. Naturally, it would be sensible to drop the term in $d\beta / dx$.

C.2 A simplified approach using the low-inertia approximation

A considerable simplification can be had by incorporating the low-inertia approximations justified in Appendix B.2.1. It was shown that the time derivative term dQ/dt in (C.5) and the quadratic momentum flux terms are of the order of the square of the Froude number. The latter are recognised by wherever a β appears, and so we set this to zero. The result is the explicit solution

$$Q = K \sqrt{\overline{S} + \frac{1}{c} \frac{d\eta}{dt} - \frac{D}{c^3} \frac{d^2\eta}{dt^2} - \frac{G}{c^5} \frac{d^3\eta}{dt^3}} \ , \qquad \text{(C.6)}$$

where *Q* is the discharge at the gauging station, *c* is the kinematic wave speed, given by equation (B.18) in terms of the gradient of the conveyance curve, $\overline{S}$ is the bed slope, and *D* is the diffusion coefficient, given by equation (B.34), where we have dropped the zero subscripts for convenience.

We can write this in terms of the rated discharge, dropping the third derivative term as being unreasonable to include in view of all the other approximations:

$$Q = Q_r(\eta)\sqrt{\underbrace{\underbrace{1}_{\text{Rating curve}} + \frac{1}{c\bar{S}}\frac{d\eta}{dt}}_{\text{Jones formula}} - \underbrace{\frac{D}{c^3\bar{S}}\frac{d^2\eta}{dt^2}}_{\text{Diffusion term}}}\,, \qquad \text{(C.7)}$$

where $Q_r(\eta) = K(\eta)\sqrt{\bar{S}}$ is the rated discharge for the station as a function of stage. This is an extension to Jones' method for correcting for the effects of unsteadiness. That method assumed that the flood wave moved as a kinematic wave without diminution. The extra diffusion term here has been obtained by allowing for diminution in the form of using an advection-diffusion level of approximation.

Appendix D The hydraulics of a gauging station

D.1 Introduction

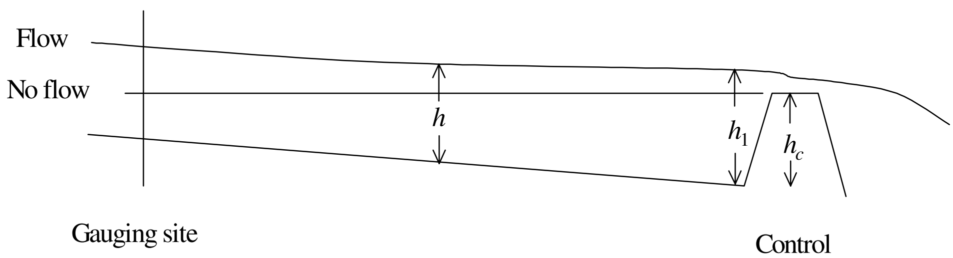

Figure D.1. Section of river between gauging station and control section showing water surface for a typical flow and for no flow.

Here we consider a gauging station, initially where there is a local control downstream, and then for higher flows, where the local control "drowns" out and control is applied by the channel itself. We obtain a theoretical expression for the rating curve in a stream where there is a local downstream control, which is often the situation for low flows, but whose effect may be negligible for higher flows. Consider a situation as shown in Figure D.1, where there is a gauging site upstream of some form of local control. If there is flow, the free surface is a backwater curve as shown, while at the cease-to-flow limit it is a horizontal line. The free surface profile is given by the steady flow (or “backwater") equation

$$\frac{dh}{dx} = \frac{\bar{S} - S_f}{1 - \beta F^2}\,, \qquad \text{(D.1)}$$

where h is the depth, $\bar{S}$ is the bed slope, S_f is the friction slope, β is the Boussinesq momentum coefficient, F^2 is the square of the Froude number $F^2 = Q^2 B / gA^3$, where Q is discharge, B is surface width, A is cross-sectional area, and g is gravitational acceleration. We have used the term “depth” as a convenient alternative to “local elevation” which is what it really is in this general case of arbitrary section – it is the height of the water surface above an inclined line running the length of the channel. We are free to choose this to be the thalweg if necessary. Equation (D.1) is presented in all books on open channel hydraulics (such as Henderson, 1966) – here we have generalised slightly by including the momentum coefficient β.

For most flows we can neglect the Froude-squared term in the denominator of (D.1). Using the general flow resistance formula $S_f = Q^2/K^2$, where $K(x,h)$ is the conveyance, in general a function of both position and depth, the differential equation becomes

$$\frac{dh}{dx} = \bar{S} - \frac{Q^2}{K^2(x,h)}. \qquad \text{(D.2)}$$

D.2 Approximate analytical solution for water surface

In the case of prismatic channels K is a function only of h, which we will assume here as we are introducing a first approximation to the problem. It is convenient to introduce the symbol $\Omega = 1/K^2$ for a quantity which we will term the "resistance", suggested by this definition, as the inverse of a power of the conveyance which is analogous to electrical conductance.

We now obtain an analytical solution by approximating the inverse of the square of the conveyance by a series expansion about an arbitrary reference depth h_0, following the method used by Samuels (1989). We write an expansion for $\Omega = 1/K^2$ about this point:

$$\Omega = \frac{1}{K^2} = \Omega_0 + (h - h_0)\Omega_0' + 2 \qquad \text{(D.3)}$$

where $\Omega_0 = 1/K^2(h_0) = 1/K_0^2$, and $\Omega_0' = d(K^{-2})/dh\big|_0$. In view of the poorly-known nature of the friction coefficient n and possibly of the cross-section, and the assumption that the section is prismatic, use of the next higher approximation in terms of Ω_0'' is not really justified, so that equation (D.3) is truncated after the linear term shown and the resistance is expressed as a linear function of depth over the operating range.

If we use Manning's law for friction, $\Omega = 1/K^2 = n^2 P^{4/3} / A^{10/3}$, we obtain, using $\Omega_0' = d\Omega/dh\big|_0$ and $dA/dh = B$,

$$\text{Manning: } \Omega_0' = \Omega_0\left(\frac{4}{3}\frac{P_0'}{P_0} - \frac{10}{3}\frac{B_0}{A_0}\right), \text{ or for Chézy friction, } \Omega_0' = \Omega_0\left(\frac{P_0'}{P_0} - 3\frac{B_0}{A_0}\right), \qquad \text{(D.4)}$$

where $P_0' = dP/dh\big|_0$. As K is an increasing function of depth, Ω is a decreasing function of depth, and Ω_0' should always be negative. We can quantify this by showing that the second (negative) term in either form of equation (D.4) is larger than the first term. The derivative P_0' can be related to the slopes of the banks, with a typical value of say 4, and P_0 is roughly equal to the width while A_0/B_0 is the mean depth, and we can see that the importance of the first term relative to the second is roughly in the ratio depth/width.

Substituting equation (D.3) truncated after the first order term into the differential equation gives

$$\frac{dh}{dx} = S_0 - Q^2\left(\Omega_0 + (h - h_0)\Omega_0'\right).$$

This has an analytical solution obtained by separating variables or using an integrating factor method. The solution is

$$h = h_1 - \left(h_1 - h_0 - \frac{\bar{S}}{Q^2\Omega_0'} + \frac{\Omega_0}{\Omega_0'}\right)\left(1 - \exp\left(-Q^2\Omega_0'(x - x_1)\right)\right) \qquad \text{(D.5)}$$

where at the control $x = x_1$ we have applied the boundary condition $h = h_1$, the depth in the unrestricted waterway just upstream of the control, as shown in Figure D.1. This equation gives the water depth at any point as a function of x, and shows that the backwater curve has simple exponential behaviour. As Ω_0' is negative, the coefficient of x in the exponential function is positive. The solution increases exponentially downstream, or, for our application here the result is that it decreases increases exponentially downstream, or, for our application here the result is that it decreases exponentially as we go upstream. For points sufficiently far upstream of the control, the solution becomes a flow of constant depth, $h = h_{0N}$, the approximation to the normal depth in the stream, given by

$$h_{0N} = h_0 + \frac{\overline{S}}{Q^2 \Omega_0'} - \frac{\Omega_0}{\Omega_0'} . \tag{D.6}$$

This enables us to write equation (D.5) as

$$h = h_1 \exp\left(-Q^2 \Omega_0'(x - x_1)\right) + h_{0N} \left(1 - \exp\left(-Q^2 \Omega_0'(x - x_1)\right)\right) \tag{D.7}$$

We now have an explicit formula for the free surface, showing that it is an exponential, and quantifying how far upstream the effects of the control exist, via the scale of motion given by the inverse of the coefficient of x in the exponential. The solution approaches uniform flow varying like $\exp\left(-Q^2 \Omega_0'(x - x_1)\right)$. The distance upstream X_ε over which the solution decreases to within ε of uniform flow is given by $\exp\left(Q^2 \Omega_0' X_\varepsilon\right) = 1/\varepsilon$, and solving this equation we see that X_ε is proportional to $1/Q^2 \Omega_0'$.

This equation still contains as a parameter the arbitrary reference depth h_0 used both in Ω_0' and h_{0N} . Results will depend on the value we use, reminding us that we have obtained an approximate solution. A typical value used might be the depth at the control. As this is usually the maximum depth we might prefer to use a smaller value, possibly the mean of the depth at the control and at the gauge, obtained iteratively after initially taking the maximum depth to allow the computation to start.

The solution given by equation (D.7) can be used in a more general and accurate sense to solve backwater curve problems. Other than the assumptions that led to the differential equation (D.2), the only assumption that has been made is that the variation in depth was sufficiently small that the conveyance could be expressed by a linear function about the uniform value. This will not be so accurate if the depth varies substantially. It should be remembered, however, that if this kind of accuracy were necessary, for more important projects, numerical solution of the differential equation (D.1) is quite straightforward anyway.

D.3 At the gauging station - the rating curve

At the gauging station, which we presume to be a distance L upstream of the control, $x = x_1 - L$, equation (D.7) gives an expression for the depth as a function of discharge, $h_G(Q)$:

$$h_G(Q) = h_1(Q) e^{Q^2 \Omega_0' L} + h_{0N}(Q) \left(1 - e^{Q^2 \Omega_0' L}\right),$$

where we have shown explicitly that the depth at the control h_1 is a function of Q, which will be given by a weir or control formula, and the normal depth h_{0N} is also a function of Q as defined in equation (D.6). In general, we can write $h_1(Q) = h_c + H_c(Q)$, where h_c is the local depth just above the control at which flow ceases (see Figure D.1), and $H_c(Q)$ is the head over the control. Hence

$$h_G(Q) = \left(h_c + H_c(Q)\right) e^{Q^2 \Omega_0' L} + h_{0N}(Q) \left(1 - e^{Q^2 \Omega_0' L}\right),$$

There is a simple geometric relation connecting h_c and the cease-to-flow depth at the gauge h_{csf} : $h_c = h_{\text{csf}} + \overline{S}L$, and so we have the general expression for the shape of the rating curve:

$$h_G(Q) = \left(h_{\text{csf}} + \overline{S}L + H_c(Q)\right) e^{Q^2 \Omega_0' L} + h_{0N}(Q) \left(1 - e^{Q^2 \Omega_0' L}\right) \tag{D.8}$$

Equation (D.8) contains L the distance between control and gauging station and the head over the control $H_c(Q)$, which in practice will be a difficult quantity to determine. As an indication of the form it might take, we might consider the case of a rectangular broad-crested weir, after re-working equation (1-35) of (Bos, 1978):

$$H_c(Q) = \frac{3}{2} B^{-2/3} g^{-1/3} Q^{2/3},$$

where B is the breadth. Alternatively, if the control were a rectangular sharp-crested weir, the expression is obtained from the traditional one

$$Q = \frac{2}{3} C_D B \sqrt{2g} H_c^{3/2},$$

also showing that $H_c \sim Q^{2/3} = Q^{0.67}$. In the case of a triangular weir or V-notch, elementary application of dimensional analysis gives a formula that shows that $H_c \sim Q^{0.4}$. Between these two extremes – a rectangular weir of fixed width and a triangular weir whose width is zero for no head – is a parabolic weir (*i.e.* a U-shaped cross-section), for which it can be shown that $H_c \sim Q^{0.5}$. Also, if the weir were trapezoidal, then the discharge formula would be a linear combination of the rectangular and triangular formulae, such that one is of exponent 0.67 and the other 0.4, and we can imagine that this could be approximated by a single power, like the parabolic weir, with exponent 0.5.

In a natural stream the actual nature of the control will be rather more complicated, although it may well be able to be approximated by a similar power law expression. One can observe that natural topography, when looking up- or down-stream, is rather more likely to look like a U, which could be modelled by a parabola, giving $H_c \propto Q^{0.5}$, rather than a rectangle or V-notch. In general, we incline to the power 0.5 being that more likely to occur in practice. To implement this theory one really needs some knowledge of the geometry at the control, and reference could be made to French (1985, #8.3), Bos (1978) or Ackers *et al.* (1978) for formulae relating head and discharge.

D.4 The rating curve for low flow

One of our tasks here is to obtain the behaviour of the water level at the gauging station as $Q \to 0$. Equation (D.7) contains Q in several places and the behaviour is difficult to extract simply. As $Q \to 0$ the normal depth as calculated from equation (D.6) becomes large and the term which multiplies it becomes small. To determine the overall behaviour we take a power series expansion of equation (D.5) in terms of Q, but for the moment neglect the fact that h_1 is actually a function of Q. The result is:

$$h = h_1 + \bar{S}(x - x_1) + Q^2\left(-\left((h_1 - h_0)\Omega_0' + \Omega_0\right)(x - x_1) - \frac{1}{2}\Omega_0'\bar{S}(x - x_1)^2 \right) + \text{Terms in } Q^4 \qquad \text{(D.9)}$$

showing that in this limit the free surface is a parabola. To do this we have expanded the expression $\exp\left(-Q^2\Omega_0'(x - x_1)\right)$, assuming that the argument of the exponential is small enough to truncate after a couple of terms, so that there will be a limit to the value of $x - x_1$ which we could use.

We have already shown that $h_1(Q) = h_{\text{csf}} + \bar{S}L + H_c(Q)$, where in the weir term $H_c(Q) \sim Q^{1/2}$ and the limit $Q \to 0$ the square root term will dominate the quadratic term in equation (D.9) and so from equation (D.8) we have in the cease-to-flow limit at the gauging station

$$h_G = h_{\text{csf}} + H_c(Q) + \text{Terms of order } Q^2, \qquad \text{(D.10)}$$

showing that the reach between gauging station and control is essentially a reservoir with flow through it, whose dynamical effects on the surface are negligible, and the surface is horizontal with elevation given by the control. At the gauging station the behaviour of the rating curve is determined entirely by the control.

Appendix E. Numerical algorithms

E.1 Interpolating or approximating data which are functions of stage

E.1.1 Piecewise linear interpolation

If a table of values of say area A_i are known for corresponding values of stage η_i, then a table look-up procedure with linear interpolation seems to be most reasonable, such as:

$$\text{If } \eta_i \le \eta \le \eta_{i+1} \text{ then } A = A_i + (\eta - \eta_i)\frac{A_{i+1} - A_i}{\eta_{i+1} - \eta_i}.$$

While that is easily written down and implemented, an important preliminary step is the determining of the interval in which we want to calculate. If data are equally spaced then, given a particular value of η the corresponding value of i is easily obtained. If they are given at irregular intervals then a simple procedure is to find i by stepping through all the data points until $\eta_i \le \eta \le \eta_{i+1}$. A computer function for doing this more efficiently is provided in Table E.1 below, as well as the subsequent interpolation.

E.1.2 Approximating the rating curve globally

If a higher-order interpolation were considered desirable, then polynomial interpolation by divided differences is a good robust way of doing it. This is presented in a scholarly context in Conte and de Boor (1980), and in a more practical civil engineering context in Fenton (1994), where it was shown that with large values of the independent variable, such as chainage, or elevation in this work, that inaccuracies can result unless simple scalings are introduced, such as will be described below.

In the context of the calculation of a rating curve, however, where different ratings may give scattered points, and one doesn't want to interpolate the points, least-squares approximation might be used, giving a *global* approximation of the stage-discharge relationship. In river hydrology this is not necessarily ideal, as the relationship may have some sort of discontinuity where the river exceeds bank-full flow. However, if a sufficiently high degree of approximation is used, this should be described perfectly adequately for practical purposes.

Consider that there are N known stage discharge pairs, (η_i, Q_i) for $i = 1, 2, N$. We choose to approximate these by the function

$$Q^{\nu}(\eta) = \sum_{j=0}^{M} b_j \phi_j(\eta), \tag{E.1}$$

where the $\phi_j(z)$ are a sequence of functions of η and ν is an exponent, both of which the user is also free to choose. In Section 6 we showed that there is some physical justification for choosing $\nu = 1/2$, when a relatively low order of approximation M can be used. In choosing the functions $\phi_j(\eta)$ the simplest choice might be the monomial functions $\phi_0(z) = 1$, $\phi_1(\eta) = \eta$, $\phi_2(z) = \eta^2$, …, when the right side of equation (E.1) is simply a polynomial of degree M. By using a standard least-squares procedure (for example, Conte and de Boor, 1980, #6.2) it can be shown that the normal equations for the unknown coefficients b_j can be written as the $(M+1)\times(M+1)$ matrix equation

$$\begin{bmatrix} S_{00} & S_{01} & S_{02} & 2 & S_{0M} \\ S_{10} & S_{11} & S_{12} & 2 & S_{1,M-1} \\ S_{20} & S_{21} & S_{22} & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ S_{M0} & S_{M1} & 2 & 2 & S_{MM} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ 2 \\ b_M \end{bmatrix} = \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ 2 \\ T_M \end{bmatrix}, \qquad \text{(E.2)}$$

where the S_{ij} are the sums of the products of the ϕ functions over all the N data points:

$$S_{ij} = \sum_{k=1}^{N} \phi_i(\eta_k)\, \phi_j(\eta_k),$$

and the coefficients T_i are weighted sums of the discharge readings taken to the power ν :

$$T_i = \sum_{k=1}^{N} \phi_i(\eta_k)\, Q_k^{\nu}.$$

These operations only have to be performed once for a particular rating curve, and both the setting up of the matrix equation and the numerical solution to give the coefficients b_j are conveniently performed on a spreadsheet. Having obtained the b_j, equation (E.1) provides a convenient and global means of computing the steady rating curve discharge for any given stage.

The obvious solution which one adopt would be to use polynomial approximation, and to use for the basis functions $\phi_j(\eta)$ simply the monomials η^j. However, over typical values of stage η over which a rating curve applies this can give rise to catastrophically poorly-posed problem (*e.g.* Fenton, 1994). If one were, as might be done, to be using actual surface elevation above datum with values of η of say 200, the problem is surprisingly poorly posed and inaccurate (Conte and de Boor, 1980, Example 6.5, present a remarkable example of how such calculations can lose accuracy). The solution, as proposed in numerical textbooks such as Conte and de Boor, and which is suggested in the Australian Standard 2360.7.2 (1993) is to use a family of orthogonal polynomials such as Chebyshev polynomials. However here we present a practical procedure that is relatively simple and robust without going to the complication of orthogonal polynomials.

As explained by Conte and de Boor, the problem of approximating numerical data by the monomials η^j is that over a typical value of stage, say, 4 – 10m, those functions all look similar (*e.g.* η^3 looks very much the same as η^4 and so on – and even more over an interval such as 204 – 210m). If one has to approximate data that shows variation, as a rating curve does, then because the basis functions all look a bit the same, then the coefficients tend to become large and fluctuating in sign. This is expressed mathematically by the fact that all rows of the matrix in equation (E.2) are similar if one uses such functions. If one uses orthogonal functions, then one has to normalise the range of stage over which the approximation is done, to the interval $(-1,+1)$, and over this range the orthogonal polynomials all look different, each crossing zero one more time than the previous one, and so on. However, at least for the first few, so do the simple monomials, and provided one didn't use too many, it would be quite enough to use them. This suggests taking the range of stage over which the approximation is to be done, and introducing the scaled stage Z:

$$Z = \frac{\eta - \eta_{\text{mid}}}{\eta_{\text{half-range}}}, \qquad \text{(E.3)}$$

where $\eta_{\text{mid}} = (\eta_{\max} + \eta_{\min})/2$ is the stage halfway between the maximum and the minimum over which approximation is to take place, and $\eta_{\text{half-range}} = (\eta_{\max} - \eta_{\min})/2$ is half that interval. It can easily be verified that when $\eta = \eta_{\min}$, $Z = -1$, and when $\eta = \eta_{\max}$, $Z = +1$. Then, if we use $\phi_j = Z^j$, and the matrix in (E.2) tends to have alternating signs and to be quite well-conditioned.

A FORTRAN program to use orthogonal polynomials is provided in Australian Standard 2360.7.2 (1993). However, of possible greater use is the fact that standard computer spreadsheets now contain, not only matrix inversion capabilities, which could be used to solve (E.2), but also there are capabilities of adding "trendlines" to graphs, which are really just doing the above. However, one might require some more flexibility, such as we have shown here, and it might be better to use customised software. If the trendline facility of a spreadsheet were used, experience with the product Excel suggests that it is computationally robust, and almost certainly does use orthogonal functions or the simple transform to $(-1,+1)$ suggested here. However, the default is for the equation, when it is shown on a graph in the spreadsheet, to be presented in terms of the raw variable it has been asked to approximate, namely the stage here, and for equations to be presented with few significant figures, looking very much like equation (6.2), so that transcribing it in that form would be dangerous, unless the option were taken for it to be displayed with at least 7 figures in the coefficients. It would be better to give the spreadsheet, not values of stage, but the transformed stage (equation (E.3)), so that the equation it produces would then be much more robust computationally. It would probably be better to incorporate the mathematics we have presented here and to set up the actual matrix equation and solve it in the spreadsheet, so that one has more control over the process. Ideally, the rating curve should then be calculated *and presented* in the form, following equation (E.3):

$$Q^{\nu}(\eta) = \sum_{j=0}^{M} b_j \left(\frac{\eta - \eta_{\text{mid}}}{\eta_{\text{half-range}}} \right)^j ,$$

and *not*, as the trendline facility would in Excel, expand the brackets to give a polynomial in η. A simple way around this is only to use the transformed quantities Z_k in the spreadsheet calculations and to use as the approximating series

$$Q^{\nu} = \sum_{j=0}^{M} b_j Z^j . \tag{E.4}$$

To reiterate, the procedure then is:

Calculate the individual Z_k at data points from equation (E.3), which will all fall in the range.

Calculate the value of each of the $M+1$ functions ϕ_j at each of the N data points.

Calculate each of the $2M+2$ quantities $S_m = \sum_{k=1}^{N} Z_k^m$ for $m = 0, 2 ,2M$ (using monomials we do not have to calculate a full two-dimensional array).

Calculate the $M+1$ weighted sums $T_m = \sum_{k=1}^{N} Z_k^m Q_k^{\nu}$.

Use (E.1) to give the matrix equation

$$\begin{bmatrix} S_0 & S_1 & S_2 & 2 & S_M \\ S_1 & S_2 & S_3 & 2 & S_{M+1} \\ S_2 & S_3 & S_4 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ S_M & S_{M+1} & 2 & 2 & S_{2M} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ 2 \\ b_M \end{bmatrix} = \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ 2 \\ T_M \end{bmatrix},$$

which can then be solved for the b_j, and equation (E.4) gives an equation for the rating curve.

E.1.3 Fitting a monomial to data

We attempt to fit a monomial function $y = ax^{\nu}$ to a set of N data pairs (x_i, y_i) for $i = 1, 2, N$. The problem is to determine the parameters a and ν by a least-squares procedure. The conventional way of doing this is to take logarithms of both sides to give the equation $\log y = \log a + \nu \log x$, which is linear in the parameters $\log a$ and ν. This procedure, however, gives undue weighting to points such as those for low flows when a rating curve was being approximated. Here we use a nonlinear procedure to solve the problem in x and y space so that such undue weighting is not introduced.

We define the error of the fit to be the sum of the squares of the deviations from the monomial over all the data pairs:

$$\varepsilon = \sum_{i=1}^{N} \left(a x_i^{\nu} - y_i \right)^2$$

and to obtain the best fit we wish to minimise ε, so that we calculate the parameters a and ν such that $\partial\varepsilon / \partial a = 0$ and $\partial\varepsilon / \partial\nu = 0$. Performing the differentiation:

$$\frac{\partial\varepsilon}{\partial a} = \sum_{i=1}^{N} 2 x_i^{\nu} \left(a x_i^{\nu} - y_i \right),$$

setting equal to zero, splitting the sum into two, and taking the constant a outside gives:

$$a \sum_{i=1}^{N} x_i^{2\nu} = \sum_{i=1}^{N} y_i \, x_i^{\nu} \,, \tag{E.5}$$

Similarly,

$$\frac{\partial\varepsilon}{\partial\nu} = \sum_{i=1}^{N} 2 a x_i^{\nu} \ln x_i \left(a x_i^{\nu} - y_i \right),$$

gives

$$a \sum_{i=1}^{N} x_i^{2\nu} \ln x_i = \sum_{i=1}^{N} y_i \, x_i^{\nu} \ln x_i \,, \tag{E.6}$$

Equations (E.5) and (E.6) are a pair of nonlinear equations in the unknowns a and ν. We can eliminate a between the two equations to give the single nonlinear equation for ν:

$$F(\nu) = \frac{\sum_{i=1}^{N} y_i \, x_i^{\nu}}{\sum_{i=1}^{N} x_i^{2\nu}} - \frac{\sum_{i=1}^{N} y_i \, x_i^{\nu} \ln x_i}{\sum_{i=1}^{N} x_i^{2\nu} \ln x_i} = 0 \,, \tag{E.7}$$

This equation has to be solved numerically for ν using some means of solving a transcendental equation. The author prefers the bisection method for simplicity and guaranteed, albeit relatively slow convergence. A program, actually written in C, is appended here, which can do that. Having solved for ν, one can substitute into equation (E.5) or (E.6) to give a.

In the case of rating curves, where one might seek a solution of the form $Q = a\eta^{\nu}$, the solution for ν seems always to be within the range $[1,3]$ (and often is close to 2), and that could be used as the initial interval in which the solution is to be found. Newton's method, a well-known method for solving transcendental equations, does not work, as, in addition to the derivative $dF / d\nu$ of equation (E.7) being very difficult to obtain, the gradient nature of Newton's method can throw the solution right off and fail to converge to the only reasonable solution.

E.2 Computer programs

Table E.1 contains some possibly useful computer programs written in the C language which automate some of the procedures described in this section.

Table E.1: Computer programs for linear interpolation and solution of transcendental equations

```
#define or ||
#define real float
//*********************************************
/* For a value of t, and N+1 values [x[0], .., x[N]], finds the interval where it
lies - i.e. the integer i such that x[i] < t < x[i+1]. It uses the bisection method
such that it finds it in log_2(N) steps */
//*********************************************
int Interval(real t, real *x, int N)
{
int ia, ib, ic;
if (t < x[0] or t > x[N]) {printf("Outside limits"); return -1;}
ia = 0;
ib = N;
while (ib-ia > 1)
   {
   ic = (ia+ib)/2; // This rounds down to the integer
   if (t >= x[ic]) {ia = ic;}
   else {ib = ic;}
   }
return(ia);
}
//*********************************************
/* Given two vectors,  x[0], .., x[N] and y[0], .., y[N] each with N+1 values this
linearly interpolates for a given value of t. */
//*********************************************
real Linear(real t, real x[], real y[], int N)
{
int j;
j = Interval(t, x, N);
return y[j]+(t-x[j])*(y[j+1]-y[j])/(x[j+1]-x[j]);
}

// Example function which returns the value of x^2-3
real F(real x)
{return x*x-3.;}

//*********************************************
/* With a supplied function F(x) which can return a value of F given any x, this
uses the bisection method to solve the equation F(x)=0 to within an interval of
'Accuracy', within range [a,b]. Using Bisection(0., 10., 1.e-5) with the above
function it will find sqrt(3) , correct to 5 places initially knowing that it is
between 0 and 10 */
//*********************************************

real Bisection(real a, real b, real Accuracy)
{
real fa, fb, fm, xm;
fa = F(a);
fb = F(b);
if(fa*fb > 0.)
   {printf("Bounds on initial solution wrong"); exit(0);}
while (b-a > Accuracy)
   {
   xm=0.5*(a+b);
   fm = F(xm);
   if (fm*fa < 0 ) {b = xm;}
   else {a = xm; fa = fm;}
   }
return xm;
}
```

Appendix F. Tables of data

Table F.1. Data scaled from Figure 5 of Australian Standard 3778.2.3

Stage m	Discharge m^3/s	Fall m	Stage m	Discharge m^3/s	Fall m
2.91	697	0.318	6.52	2655	0.300
3.18	628	0.180	6.98	3204	0.318
3.49	783	0.234	7.13	3259	0.288
3.79	983	0.234	7.67	3717	0.288
4.07	1093	0.234	8.08	4331	0.336
4.07	1155	0.243	8.32	4455	0.324
4.78	1576	0.283	8.47	4366	0.297
4.95	1669	0.297	8.56	4583	0.300
5.43	1973	0.300	8.79	4538	0.288
5.62	2249	0.306	8.91	4714	0.288
5.71	2200	0.300	9.20	5180	0.318
5.90	2317	0.288	9.36	5283	0.318
6.24	2569	0.300	9.53	5386	0.306
6.43	2583	0.288			

Table F.2. Data from Figure 8 of AS 3778.2.3

Stage m	Discharge m^3/s	Fall m	Stage m	Discharge m^3/s	Fall m
3.7	2.4	1.59	13.4	23.6	1.473
5.3	4.4	1.707	15.9	25.7	1.038
6.1	5.1	1.737	14.2	23.1	0.905
8.2	8.8	1.488	17.6	33.9	1.420
9.6	9.6	0.525	19.1	45.9	1.683
10.1	13.3	1.305	19.7	51.4	1.794
13.9	19.9	0.831			

Table F.3. Data from Table 8.1 of Herschy (1995)

No.	Stage (base) m	Stage (auxiliary) m	Discharge m^3/s	No.	Stage (base) m	Stage (auxiliary) m	Discharge m^3/s
1	5.907	3.990	1160	9	2.755	2.054	317
2	7.105	4.923	1520	10	2.963	2.347	289
3	5.026	3.429	889	11	2.359	2.155	156
4	7.013	4.788	1490	12	2.286	1.996	145
5	11.558	8.678	2830	13	3.206	2.279	411
6	8.108	6.188	1640	14	2.036	1.978	39.9
7	8.638	5.986	1990	15	2.012	1.951	66
8	3.139	2.331	399				

Table F.4. Data from Table 8.5 of Herschy (1995)

No.	Stage (base) m	Stage (auxiliary) m	Discharge m^3/s	No.	Stage (base) m	Stage (auxiliary) m	Discharge m^3/s
1	5.956	4.386	214	13	5.416	4.514	133
2	5.907	4.715	178	14	1.512	0.686	25.7
3	5.614	4.614	154	15	3.895	2.264	120
4	5.246	4.380	134	16	3.487	2.384	89.8
5	4.865	4.048	119	17	1.859	0.929	36.8
6	3.725	3.131	70.5	18	6.690	4.809	259
7	2.916	2.431	48.7	19	8.001	5.526	524
8	6.559	4.526	217	20	3.158	2.238	78.2
9	7.705	5.242	391	21	3.697	2.438	110
10	6.514	4.944	233	22	5.334	4.188	156
11	8.077	5.595	767	23	1.817	0.113	43
12	3.868	2.243	134	24	1.585	1.039	23.8

Table F.5. Data from Herschy (1995), Table 4.1. Stages in metres, discharges in m^3/s.

Stage	*Q*	Stage	*Q*	Stage	*Q*
0.53	94.6	1.05	193	1.79	387
0.61	104	1.05	189	2.03	440
0.68	120	1.17	202	2.13	469
0.69	113	1.26	235	2.33	540
0.7	124	1.35	240	3.09	930
0.73	125	1.37	246	3.49	1152
0.79	139	1.44	287	3.87	1374
0.91	163	1.55	300	3.93	1452
0.94	168	1.61	306		
0.96	169	1.7	340		

Table F.6. Data from Table 3 of Australian Standard AS 2360.7.2

Stage	*Q*	Stage	*Q*	Stage	*Q*
4.92	1390	6.5	2950	8.6	7350
4.95	1450	6.7	3300	9	8900
5.05	1500	6.9	3410	9.5	10100
5.15	1600	7.1	3800	9.6	12200
5.21	1650	7.2	3810	10.1	14000
5.3	1750	7.3	4800	10.5	14600
5.47	1820	7.5	4500	11.4	22500
5.5	1890	7.6	5100	11.9	28700
5.58	2000	7.7	5300	12.1	31500
5.61	2010	7.8	5220	12.6	36000
5.73	2100	7.9	5400	13.2	45000
5.81	2160	7.9	6100	13.5	52000
5.9	2270	8	6500	13.5	51000
6.1	2500	8.1	6100	13.8	56000
6.25	2750	8.4	6900		

Table F.7. Stage-discharge data from Gwydir River used for various figures

Bundarra		Gravesend		Pallamallawa	
Stage (m)	*Q* (Ml/d)	Stage (m)	*Q* (Ml/d)	Stage (m)	*Q* (Ml/d)
0.85	0	1	0	0.76	0
0.9	0.12	1.01	1	0.77	1
0.92	0.25	1.02	1.37	0.8	2.5
0.96	2.25	1.03	1.8	0.84	7.4
0.98	3.85	1.04	2.3	0.88	17.6
1	6	1.05	2.9	0.9	24.2
1.02	8.6	1.06	3.6	0.92	32.5
1.04	11.3	1.07	4.39	0.94	42.8
1.06	15.6	1.08	5.3	0.96	56
1.08	20.9	1.09	6.3	0.98	70
1.1	27	1.1	7.5	1	87
1.15	46.85	1.12	10.2	1.05	141
1.2	75	1.14	13.7	1.1	216
1.22	89	1.16	17.8	1.15	315
1.24	103	1.18	22.6	1.2	440
1.26	118	1.2	28.5	1.25	580
1.28	134	1.22	35	1.3	760
1.3	152	1.24	43	1.35	930
1.35	205	1.26	52	1.4	1120
1.4	260	1.28	62	1.45	1310
1.45	325	1.3	72.5	1.5	1500
1.5	400	1.32	85	1.6	1860
1.55	490	1.34	98	1.7	2240
1.6	580	1.36	113	1.8	2650
1.65	700	1.38	130	1.9	3080
1.7	830	1.4	150	2	3550
1.75	980	1.45	200	2.5	6350
1.8	1160	1.5	255	3	9500
1.9	1580	1.55	313	3.5	14000
2	2100	1.6	375	4	19000
2.1	2650	1.65	440	4.5	23900
2.3	4000	1.7	510	5.5	35200
2.4	4800	1.75	560	6.5	48000
2.5	5700	1.8	660	7	56500
2.6	6600	1.9	830	8	76000
2.8	8500	2.4	2100	9	97000
2.9	9600	2.9	3950	10	170000
3.3	14300	3.4	6400	10.7	280000
3.8	21410	3.9	9500		
3.9	23070	4.9	17400		
4	24800	5.9	27200		
4.8	39400	6.9	39500		
5.8	62888	7.9	53500		
6.8	88570	8.9	72700		
7.8	119050	9.9	90900		
8.8	157417	10.9	115000		
9.8	199227	11.9	144000		
10.8	245220	12.9	179000		
11.8	296216	13.9	219000		
12.8	362325	14.9	268000		
13	377000	15.9	344000		
		16.9	471000		
		17.38	558000		