Calculating flow over rectangular sharp-edged weirs

Alternative Hydraulics Paper 6

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We demonstrate the widely-known but also widely not-acknowledged fact that traditional sharp-edged weir theory is based on wildly incorrect assumptions and a misleading approach. Dimensional analysis is a more rational way of proceeding, and obtains the same result as the hydraulic analysis much more simply and without assumptions.

Experimental results are then examined to determine the coefficient of discharge which emerges from the analysis. Kindsvater and Carter (1957) presented results in the form of a number of graphical curves, which are not so useful now. We approximate those curves to give a formula which can be used to compute the flow over weirs. When compared with the Swiss SIA formula from 1924, here converted to fully dimensionless form, both sets of results are in good agreement. The formula obtained from Kindsvater and Carter's figures is slightly simpler.

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1 Introduction

The calculation of the flow over sharp-crested weirs is one of the most useful applications of elementary hydraulic theory, both in introducing students to the thought processes of hydraulics and providing one of the most widely-used formulae for practical application. However, the application of the theory is wrong, and its continued use, while often justified in the name of pragmatism, may do more to confuse students than to educate them. The theory is usually rescued by the introduction of a coefficient of discharge which obscures the nature of the approximations. These facts are widely known. For example, Rehbock (1929) stated: "This formula is based on the fallacious assumption that water flows through the entire cross-section", *i.e.* that the top surface has not dropped down at all at the crest, and "... a pressure distribution ... is assumed, which is altogether different from that which exists in the actual jet", and "the fact that ... the nappe rises to a point ... above the weir crest is neglected entirely". Yet many hydraulics textbooks, some of an otherwise scholarly reputation, introduce the theory in apparent disregard of the facts of the problem.

This note shows how the application of the theory is wrong, and suggests that it be no longer presented. Next, the fundamental formulae are simply established using dimensional analysis. The majority of previous work has been for the two-dimensional case where the weir extends the whole width of the (rectangular) channel. The effects of finite weir width were included in a formula published in 1924 by the *Schweizerischer Ingenieur- und Architekten-Verein* (*SIA*), the Swiss Society of Engineers and Architects. Kindsvater and Carter (1957) also studied the problem and presented results as figures, which are not so convenient for calculations. Here we approximate those results and generate mathematical formulae which are easier to implement than the original graphical form. However, the results agree quite well with the earlier Swiss formula, and both describe the experimental results quite satisfactorily. The Swiss formula is here converted to full dimensionless form, and as it is simpler than the results of Kindsvater and Carter, it is to be preferred if accuracy is important.

2 Traditional theory

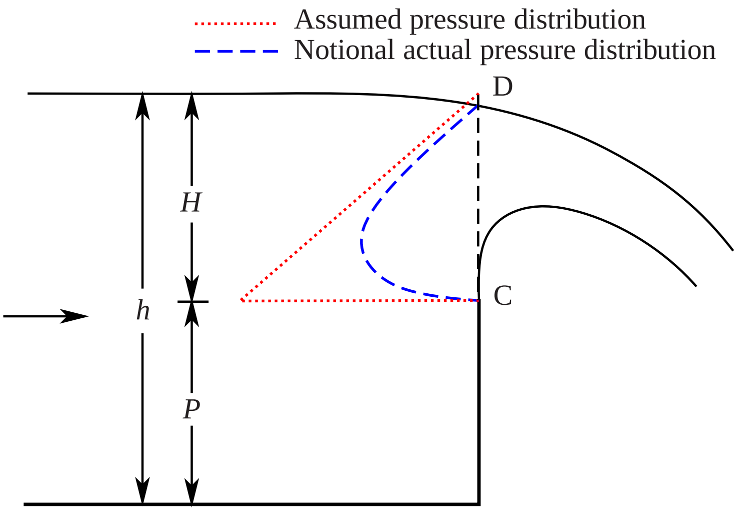

Figure 1: Side view of sharp-crested weir

Consider the flow problem as shown in Figure 1, which in this section we will consider to be two-dimensional only, where a weir extends the whole width of the (rectangular) channel. A uniform flow of discharge q per unit width and depth d encounters the sharp-edged weir, whose crest at C is of height P above the bed, such that the head over the weir is H. The flow springs clear of the weir at C, which is where we place the co-ordinate origin. The notional pressure distribution is shown, increasing from atmospheric (zero) at the crest to a maximum value and then decreasing back to zero at D vertically above the crest on the upper nappe. In producing Figure 1 as if the bed and the surface were horizontal we have made the reasonable assumption that gravity and friction in the flow upstream are in balance, the flow is uniform, and we pretend that both bed and surface are actually horizontal. In a similar sense we assume that the real slope would be roughly balancing friction losses and we make no allowance here for them. Now we apply Bernoulli's theorem along a streamline from a point far upstream at which the elevation is Y to a point y above the crest where the velocity components are (u, v):

$$\frac{1}{2}U^2 + gH = \frac{1}{2}\left(u^2 + v^2\right) + \frac{p}{\rho} + gy, \tag{1}$$

where $U(Y(y))$ is the horizontal velocity far upstream which, as the flow in the channel is a shear flow, is a function of the elevation Y of the streamline there, which in turn depends on y, the elevation at the crest of the particular streamline chosen, g is gravitational acceleration, p is the pressure in the fluid and ρ is the fluid density. The assumptions widely made in the literature, explicitly or implicitly, are:

- The fluid velocity is horizontal over the crest and the vertical velocity component is neglected –

the figure shows instead, how the fluid actually flows vertically upwards before being deflected over the crest, and the assumption is nonsense;

- The pressure above the crest is hydrostatic – it is actually something as shown in Figure 1 by the dashed line.
- The elevation $y_D = 2H/3$ – this has not a lot of justification either.
- The upstream flow is irrotational, such that U is independent of y – ignoring the fact that the flow upstream is a shear flow.

Hence, setting $v = 0$ and $p = 0$ in equation (1) gives an expression for the horizontal velocity above the crest:

$$u = \sqrt{U^2 + 2g(H - y)}. \tag{2}$$

Now, having obtained the velocity distribution, we integrate it from the crest C to the surface at D directly above to obtain the discharge over the weir. If q is the discharge per unit width, then

$$q = \int_0^{2H/3} u\,dy,$$

and on substituting equation (2) we obtain

$$q = \int_0^{2H/3} \sqrt{U^2 + 2gH - 2gy}\,dy = \frac{2}{3}\sqrt{2g}\left(\left(U^2/2g + H\right)^{3/2} - \left(U^2/2g + H/3\right)^{3/2}\right). \tag{3}$$

As $U = q/d$ this is actually a transcendental equation for q which may be solved by direct iteration. Commonly, the approach velocity head is ignored, such that we set $U = 0$ on the right side of equation (3), giving

$$q \approx 0.81 \times \frac{2}{3}\sqrt{2g}H^{3/2}. \tag{4}$$

The factor of 0.81 has been found experimentally to be too high, so usually simply a Coefficient of Discharge C_D is introduced and the expression written

$$q = \frac{2}{3}C_D\sqrt{2g}H^{3/2}. \tag{5}$$

This is the expression which is traditionally used and which has been extensively investigated. The coefficient of discharge in practice covers the fact that the underlying assumptions are poor.

3 Analysis of McNown *et al.* results

McNown *et al.* (1953), amongst other problems, solved the problem of irrotational flow over a high weir, $d \to \infty$, and presented a Figure 13 showing the streamlines of the flow. We used that figure here to give us an idea of the nature of the flow over the weir, even if the irrotational approximation was used. We scaled the intercepts of the streamlines above the weir from their diagram, and obtained the results shown in Table 1, in which ψ/q is the value of the dimensionless stream function. The horizontal velocity is obtained from $u = \partial\psi/\partial y$. We examined the nature of the data points and chose to fit a polynomial in y/H with leading term varying like $(y/H)^{3/2}$, such that after differentiation the polynomial representing the horizontal velocity would vary like $(y/H)^{1/2}$ such that the horizontal velocity is zero at the crest (as it is in practice) but increases very quickly above that. We chose a polynomial with four terms, increasing

ψ/q	y/H
0	0.0000
1/64	0.0200
1/16	0.0580
1/4	0.1902
1/2	0.3750
3/4	0.5978
1	0.8505

Table 1: Position of streamlines above crest on Figure 13 of McNown *et al.*

by half a power each time, in terms of y/H raised to powers of $3/2$, 2, $5/2$ and 3. We used a least-squares procedure to determine the value of the coefficients, and found

$$\frac{\psi}{q} = 6.65\left(\frac{y}{H}\right)^{3/2} - 11.17\left(\frac{y}{H}\right)^{2} + 7.33\left(\frac{y}{H}\right)^{5/2} - 1.67\left(\frac{y}{H}\right)^{3}. \tag{6}$$

The results are shown as a dotted line on Figure 2, and it can be seen that this equation does provide a plausible fit to the seven data points, although at the scale plotted the vertical slope with very high curvature at the origin is not obvious.

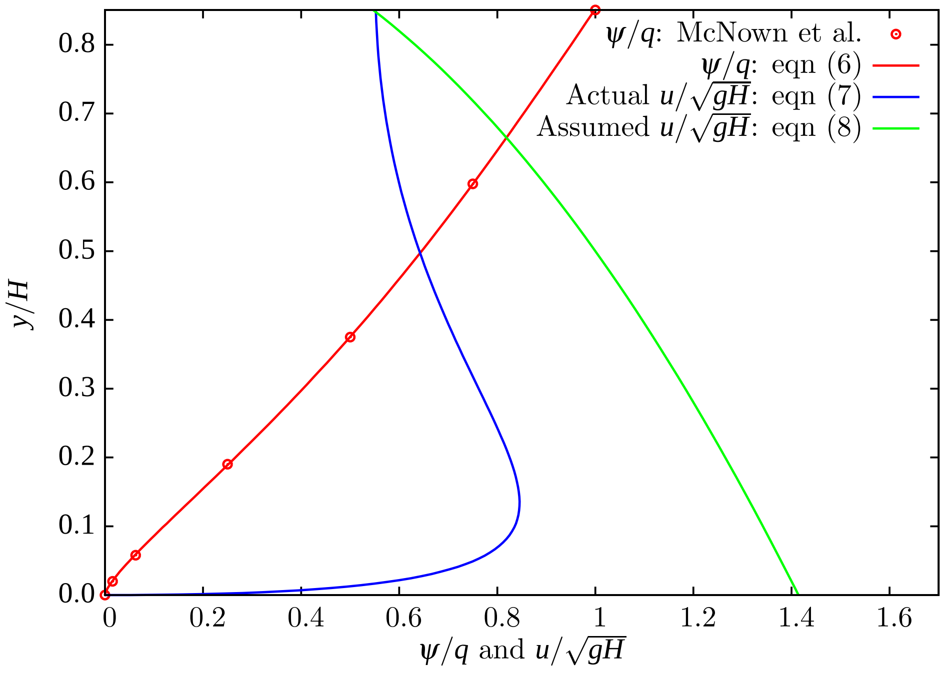

Figure 2: Stream function and velocity profiles above crest

To obtain the velocity we differentiated to give an expression for $\partial\psi/\partial y$ and using the figure quoted by McNown *et al.* that $q/\sqrt{2gH^3} = 0.408$, we converted to non-dimensionalise the velocity in terms of the head:

$$\frac{u}{\sqrt{gH}} = 5.76\left(\frac{y}{H}\right)^{1/2} - 12.89\frac{y}{H} + 10.57\left(\frac{y}{H}\right)^{3/2} - 2.89\left(\frac{y}{H}\right)^{2}. \tag{7}$$

Now to obtain the traditional approximation, we used equation (2) with $U = 0$ as the water is infinitely deep, and non-dimensionalising as above we obtained

$$\frac{u}{\sqrt{gH}} = \sqrt{2\left(1-y/H\right)}. \tag{8}$$

Both these equations for the velocity are plotted on Figure 2 as solid lines, and it can be seen that the conventional approximation used in weir theory, shown by the thin line, bears no relationship to the detailed solution of the (irrotational) flow field. The two do agree at $y = y_D$ ($= 0.851H$ in this example), but that is where the pressure is atmospheric, and the assumption that that is the pressure over the whole crest is at least valid at that point. What is really notable is that at the crest, $y = 0$, the agreement is very poor. In a real fluid the velocity is zero there, because of viscosity and the fluid adhering to the wall of the weir, while in the irrotational example here the horizontal velocity is zero because at the crest the fluid is travelling vertically upwards. Here, the assumption in conventional theory that the horizontal velocity, the basis for discharge calculations, is equal to the magnitude of the fluid velocity, is clearly wrong.

We can gain a little more insight by calculating the volume flux over the weir. Using equation (6) with $y = 0.851H$, the surface elevation found by McNown *et al.*, we obtain $\psi/q = 0.999$, quite acceptably accurate, and using the fact that $q/\sqrt{2gH^3} = 0.408$, we find that the dimensionless flux over the weir is $q/\sqrt{gH^3} = 0.408\sqrt{2} = 0.577$. We use equation (4), which is equivalent to equation (8) integrated, to obtain

$$\frac{q}{\sqrt{gH^3}} = \frac{2}{3}\sqrt{2}\left(1-\left(1-y_D/H\right)^{3/2}\right).$$

If we use the common and very rough approximation that there is no contraction above the weir, $y_D = H$, then we have $q/\sqrt{gH^3} = 0.943$, very much in error, but if we take the conventional approximation, equation (5), it would take a coefficient of discharge of 0.612, very close to the widely-assumed value, to give the required value of 0.577. This numerical example seems to show how the incorrect theory is brought to some measure of accuracy by the coefficient of discharge.

4 Dimensional analysis

The problem studied is an interesting example of one where dimensional analysis provides some insight without recourse to the assumptions of the traditional method. In fact, it also shows how the traditional method gives a result that works. We assume that the flow upstream is sufficiently large and fast that we can neglect viscosity and initially consider the simplified two-dimensional problem and where the channel can be assumed to be very deep. The variables of the problem are q, H, and g. We have three variables and two physical dimensions involved, and so by the Buckingham π theorem we have a single dimensionless quantity, which dimensional analysis gives

$$\pi_1\left(\frac{q}{\sqrt{gH^3}}\right) = 0, \tag{9}$$

where the $\pi_1\left(\ldots\right)$ indicates a functional relationship, which we can write as

$$\frac{q}{\sqrt{gH^3}} = \text{constant}. \tag{10}$$

The dimensional analysis has shown us that q varies like $\sqrt{g}H^{3/2}$ and if the constant is written as $2\sqrt{2}/3 \times C_{\mathrm{D}}$, we have the traditional expression (5) with almost no physical assumptions. Dimensional analysis has shown us that the relationship must be like this, which explains the practical success of the conventional hydraulic approach – it simply had to come up with an expression like this.

In the more general case, for a channel of finite depth, two extra variables, the depth d and crest height P are added. However they are connected by $P = d - H$ so that $P/d = 1 - H/d$. Hence we have four variables and two physical dimensions involved, and so by the Buckingham π theorem we can obtain two dimensionless quantities connected like

$$\pi_2\left(\frac{q}{\sqrt{gH^3}},\frac{H}{P}\right)=0, \tag{11}$$

showing that we can express $q/\sqrt{gH^3}$ as a function of the head over the weir relative to the crest height, H/P, as used by several people, cited below. What this dimensional analysis did not tell us is that it is much better to use $q/\sqrt{gH^3}$ as a variable than, for example $q/\sqrt{gP^3}$. In fact, we were led to the form of equation (11) by the deep-channel approximation (9). Neither did dimensional analysis tell us that it is appropriate to use H/P rather than H/d, where we have followed earlier workers. It is obvious that dimensional analysis has its share of problems too. However, the dimensional analysis has already given more information than that provided by the traditional equation (5). We know that the coefficient of discharge C_{D} depends on H/P. This functional dependence has been obtained experimentally by many workers, however, Rehbock (1929), for example, obtained

$$C_{\mathrm{D}}=0.605+0.08\,\frac{H/d}{1-H/d}=0.605+0.08\,\frac{H}{P}.$$

In the three-dimensional case of a weir of finite width b, with discharge $Q\mathrm{m}^3\mathrm{s}^{-1}$ in a channel of finite width B, two more length scales are involved. We obtain the result

$$\pi_3\left(\frac{Q}{b\sqrt{gH^3}},\frac{H}{P},\frac{b}{B},\frac{B}{d}\right)=0, \tag{12}$$

where we have chosen to represent the first term in terms of Q/b, as suggested by the previous two-dimensional results, when $q=Q/b$. There is no experimental evidence for the effect of the channel aspect-ratio term B/d, but it is possible that it has been overlooked.

Clearly it would be more appropriate to recognise the effects of viscosity ν and to add a Reynolds-number-like term $\sqrt{gH^3}/\nu$ to each of the dimensionless groups (9), (11) and (12). Below, in considering the Swiss Society of Engineers and Architects formulae we convert a term involving H in metres to such a form.

5 Kindsvater and Carter's results

Kindsvater and Carter (1957) analysed a number of experimental results for rectangular sharp-edged weirs in rectangular channels of varying dimensions. They found that all the results they analysed could be presented in terms of a traditional 3/2 power law with certain corrections. They wrote

$$Q=C_e\,(b+k_b)\,(H+k_H)^{3/2}, \tag{13}$$

where Q is the discharge; C_e is a coefficient of dimensions $\mathrm{L}^{1/2}\,\mathrm{T}^{-1}$; $b+k_b$ is the effective width of the weir, where b is the actual width of the weir and k_b is a dimensional quantity representing the effects of viscosity and surface tension; $H+k_H$ is the effective head, where H is the actual head over the weir and k_H is a dimensional quantity similar to k_b.

All their results were presented in dimensional terms, using British units. They found that C_e and k_b could be presented as graphs, depending only on the dimensionless length ratios: H/P, the ratio of head over the weir to height of the weir crest above the channel bottom, and on b/B, the ratio of weir width to channel width. The results in the form of graphs are not so simple to implement in computer programs and require some processing before implementing in the almost international standard of units in the metric system. In this note we use Kindsvater and Carter's results to generate mathematical formulae which are easier to implement in computer programs than in the original form.

We prefer to work with units of the *Système Internationale*, and in dimensionless terms wherever possible. Instead of equation (13) we will write equation (12), neglecting B/d as a parameter, and using traditional terminology based on the hydraulic analysis:

$$Q = \frac{2}{3} C_{\text{D}} \sqrt{2g} b H^{3/2}, \tag{14}$$

where C_{D} is the dimensionless coefficient of discharge, a function of H/P and b/B. It is a strong temptation, following Kindsvater and Carter, to abandon the $2/3$ and the $\sqrt{2}$, both of which are products of the severe assumptions in the hydraulic analysis. We would, unlike Kindsvater and Carter, retain the $\sqrt{g}$ term so that the C_{D} would be dimensionless. However, as we are relating this to previous work we choose to retain the traditional notation. From equations (13) and (14) it can be seen that the dimensionless C_{D} is obtained from Kindsvater and Carter's C_e by

$$C_{\text{D}} = \frac{3}{4} \sqrt{\frac{2}{g}} C_e \left(1 + k_b/b\right) \left(1 + k_H/H\right)^{3/2}. \tag{15}$$

We take the value of g to be that at Atlanta where many of the experiments were performed. We use the expression for the variation of g with latitude, in *SI* units:

$$g \ \left(\text{ms}^{-2}\right) = 9.806 - 0.026 \cos 2\lambda,$$

where λ is the latitude which is $33^{\text{o}}44'$ for Atlanta, giving $g = 9.80\text{ms}^{-2}$. Converting to British units we obtain the numerical version of equation (15):

$$C_{\text{D}} = 0.1871 C_e \left(1 + k_b/b\right) \left(1 + k_H/H\right)^{3/2}.$$

6 Formulae for practical use based on experimental results

Now we examine some of the assumptions and obtain a sequence of approximate formulae so that all of the expressions can be programmed without table look-up procedures.

6.1 Width correction k_b

b/B	k_b (ft)	k_b (m)
0.2	0.008	0.0024
0.5	0.010	0.0031
0.8	0.014	0.0043
1.0	-0.003	-0.0009

Table 2: Values of k_b taken from Figure 8 of Kindsvater and Carter and converted to metres.

Table 2 and Figure 3 show the four numerical values from Figure 8 of Kindsvater and Carter, converted from feet to metres. It can be seen that the numerical magnitudes are small (less than 5mm), of some significance for laboratory experiments, but not very important in practice. Nevertheless for completeness we seek to interpolate these data points to express k_b as a function of the relative width b/B. We tried using low order polynomials, but these were generally unsatisfactory in describing the relatively

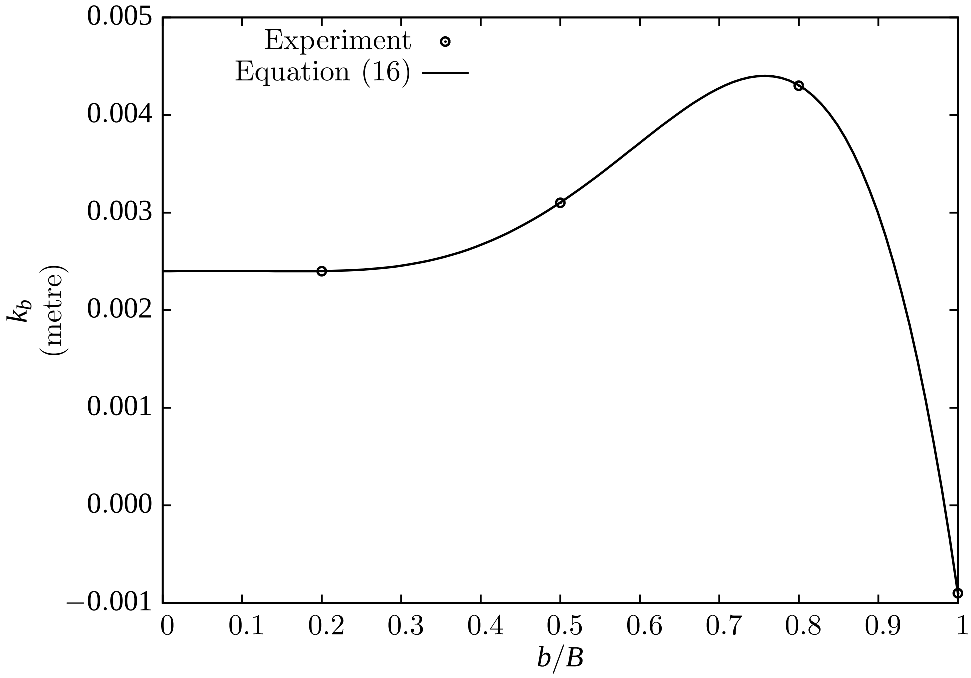

Figure 3: Width correction k_b

slow variation for small b/B and rapid variation as the full width is approached. We found that the most plausible function which mimicked the curve which Kindsvater and Carter drew through their points, was a polynomial of degree 5. Several different limitations on the polynomial were tried (as there are only four data points only four independent coefficients are available). Finally, the best mimicry of the data points, showing no variation in the limit as $b/B \to 0$, as expected for a quantity expressing the effect of channel width, was found to be obtained by requiring a limiting value of 0.0024m at $b/B = 0$, as in the case of Bos (1989, Figure 5.4), as well as a gradient of zero there. This information is sufficient to determine all six coefficients of the fifth-degree polynomial. We used the simple procedure of divided differences for interpolating unequally-spaced data, using the algorithms presented in Fenton (1994 - a more scholarly presentation of the theory is given by Conte and de Boor, 1980) for interpolation using function and derivative information. The procedure yielded the function

$$k_b\ (\mathrm{m}) = 0.0024 + 0.0016 \left(\frac{b}{B}\right)^2 - 0.0210 \left(\frac{b}{B}\right)^3 + 0.0776 \left(\frac{b}{B}\right)^4 - 0.0615 \left(\frac{b}{B}\right)^5. \qquad (16)$$

The coefficients are specified to a higher accuracy than might seem reasonable because their oscillating signs makes the evaluation of the polynomial liable to roundoff errors. It can be seen in Figure 3 that the equation seems to perform a smooth and plausible interpolation of the data points. It should be noted, however, that Kindsvater and Carter stated: ”the authors acknowledge some uncertainty regarding the generality of the curve in (their Figure 8) ... Nevertheless, the adequacy of the effective-width concept was demonstrated by the successful elimination of b as an independent variable ...”.

6.2 Head correction k_H

Kindsvater and Carter found, for all their tests, that a constant value of

$$k_H = 0.001\mathrm{m} \qquad (17)$$

was ”adequate to compensate for fluid-property effects related to the head” but they noted that this might be situation dependent. In any case, it is such a small quantity that in practice this may not be important.

6.3 Formulae for coefficient of discharge C_D

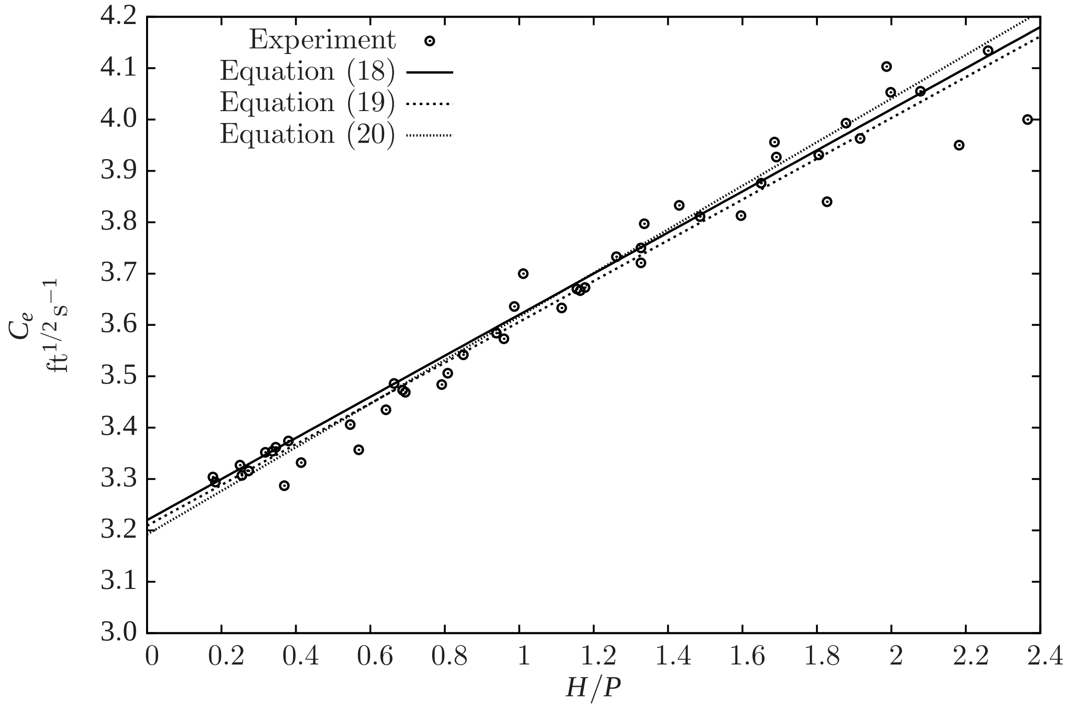

Figure 4: Experimental results with lines of best fit

Kindsvater and Carter presented in their Figure 10 a series of lines on a graph giving the relationship between their C_e and the relative head H/P and relative width b/B. Here we attempt to obtain a single mathematical function approximating all the information in their original figure. The lines on their figure were presented with relatively little justification, so before we go further we examine here the nature of their approximation of the data by considering their Figure 9, one of the cases where they did present the raw data, which is here shown as Figure 4. On our figure we have plotted their dimensional C_e and its variation with relative head, for a constant width of $b/B = 1$, as well as the straight line which they plotted, with no apparent calculation, as representing the experimental points, given by

$$C_e = 3.22 + 0.4\,\frac{H}{P}. \tag{18}$$

We performed a least-squares analysis of the data after scaling it off their figure, and found that it was best approximated by the function

$$C_e = 3.209 + 0.397\,\frac{H}{P}, \tag{19}$$

which clearly is close to their expression, and which is also plotted on Figure 4. If this agreement between their apparently sketched line of best fit and our least-squares analysis were to be repeated throughout all the tests, one would have some confidence in their results, and we will proceed on this basis. We note that several of the results most distant from the curve (including the four points on the far right below the curve) were obtained from a weir of width only 3cm, hardly relevant to practical applications. If we subtract the points from that series, the line of best fit according to least-squares analysis is actually

$$C_e = 3.192 + 0.424\,\frac{H}{P}, \tag{20}$$

still not very different, and which is plotted as a dotted line on Figure 4. We conclude that we have enough faith in the results as plotted on their Figure 10 to proceed with our mathematical approximation.

All variation with H/P on their Figure 10 was linear. We took the lines on that figure and with a scale measured the intercepts at $H/P = 0$, and the gradients of the lines. Converting to the dimensionless

b/B	Equations for C_D
0.2	$0.5895 - 0.0020\,H/P$
0.4	$0.5911 + 0.0060\,H/P$
0.6	$0.5943 + 0.0183\,H/P$
0.8	$0.5962 + 0.0438\,H/P$
0.9	$0.5989 + 0.0638\,H/P$
1.0	$0.6026 + 0.0741\,H/P$

Table 3: Equations for lines on Figure 10 of Kindsvater and Carter converted to dimensionless C_D

coefficient of discharge C_D by using equation (23) we obtain the results presented in Table 3. Similar equations have been presented by Bos (1989, Table 5.2). As these linear equations appear to be adequate in describing the variation with relative head H/P, our problem now is to approximate the variation with b/B of the coefficients in them. We represent the equations as

$$C_D = \alpha\,(b/B) + \beta\,(b/B)\,\frac{H}{P}, \tag{21}$$

and the individual numerical values of α and β shown on the table are plotted on Figure 5 for the α and Figure 6 for the β. Also plotted on each are the recomputed values from equations (19) and (20), to provide some measure of variation which might be expected in a re-analysis of the results.

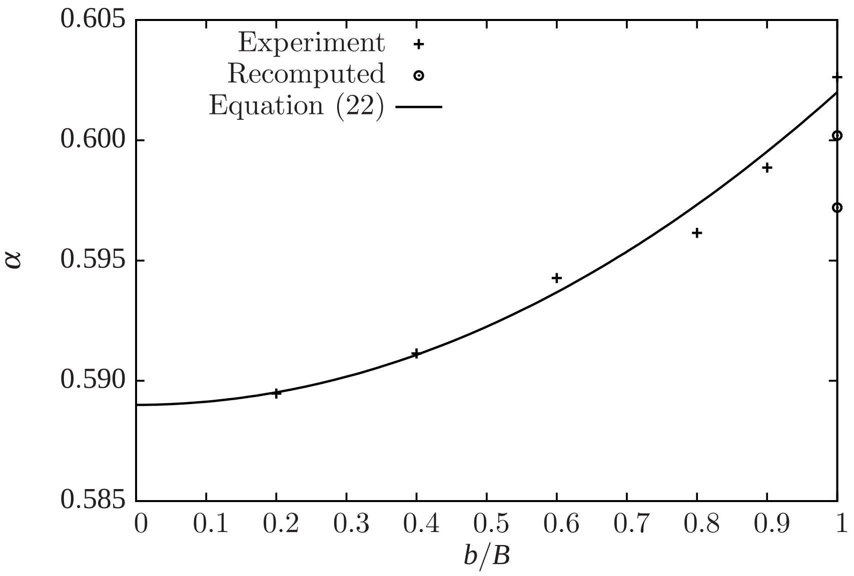

Figure 5: Variation of α with b/B

To approximate the variation of α and β as functions of b/B we considered using linear functions, however these did not seem to represent the data as well as they might, and in both cases we chose an equation of the form $a_0 + a_2\,(b/B)^2$, with the coefficients a_0 and a_2 to be determined by a least-squares

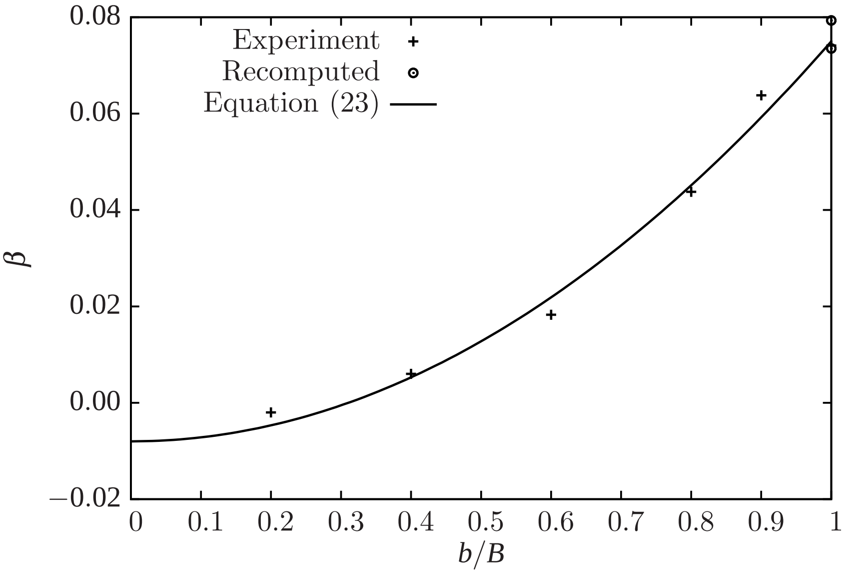

Figure 6: Variation of β with b/B

procedure. The results we obtained are:

$$\alpha = +0.589 + 0.013\left(\frac{b}{B}\right)^2 \tag{22}$$

$$\beta = -0.008 + 0.083\left(\frac{b}{B}\right)^2, \tag{23}$$

which are shown plotted on Figures 5 and 6. It is believed that the choice of quadratic variation with no linear variation like $a_1\,(b/B)$ is appropriate and represents the trend of the points in both cases.

Combining the two expressions in equations (22) and (23) as in equation (21) we have a general expression for the coefficient of discharge and its variation with relative width and head of the weir:

$$C_{\mathrm{D}} = 0.589 - 0.008\,\frac{H}{P} + \left(\frac{b}{B}\right)^2\left(0.013 + 0.083\,\frac{H}{P}\right), \tag{24}$$

which is a least-squares mathematical approximation to the lines on figure 10 of Kindsvater and Carter.

For a weir which extends the width of the channel, $b = B$, with no end contractions, equation (24) becomes $C_{\mathrm{D}} = 0.602 + 0.075\,H/P$. For this full-width case Rehbock obtained $C_{\mathrm{D}} = 0.605 + 0.08H/P$, showing how the results agree.

7 The SIA formula

In 1924 the *Schweizerischer Ingenieur- und Architekten-Verein* (*SIA*), the Swiss Society of Engineers and Architects, published a Standard containing the following expression (which we have here converted from the dimensional expression presented in Kindsvater and Carter and a particular case presented by Rehbock):

$$C_{\mathrm{D}} = \left(0.578 + 0.037\left(\frac{b}{B}\right)^2 + \frac{3.615 - 3\,(b/B)^2}{1000H + 1.6}\right)\left(1 + 0.5\left(\frac{b}{B}\right)^4\left(\frac{H/P}{1 + H/P}\right)^2\right), \tag{25}$$

in which the last term in the first bracket allows for real fluid effects, and the coefficient 1000 is actually a dimensional one of units m^{-1}. Here, we presume that those real fluid effects are due to viscosity, and replace the $1000H$ by the expression in terms of a Reynolds number to the power 2/3, $0.047\left(gH^3/\nu^2\right)^{1/3}$, accurate to 1% for $g = 9.8\mathrm{m}$ and $\nu = 1\times10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$. The full SIA expression is then, in dimensionless terms:

$$C_\mathrm{D} = \left(0.578 + 0.037\left(\frac{b}{B}\right)^2 + \frac{3.615 - 3\,(b/B)^2}{0.047\,(gH^3/\nu^2)^{1/3} + 1.6}\right)\left(1 + \frac{1}{2}\left(\frac{b}{B}\right)^4\left(\frac{H/P}{1+H/P}\right)^2\right). \qquad (26)$$

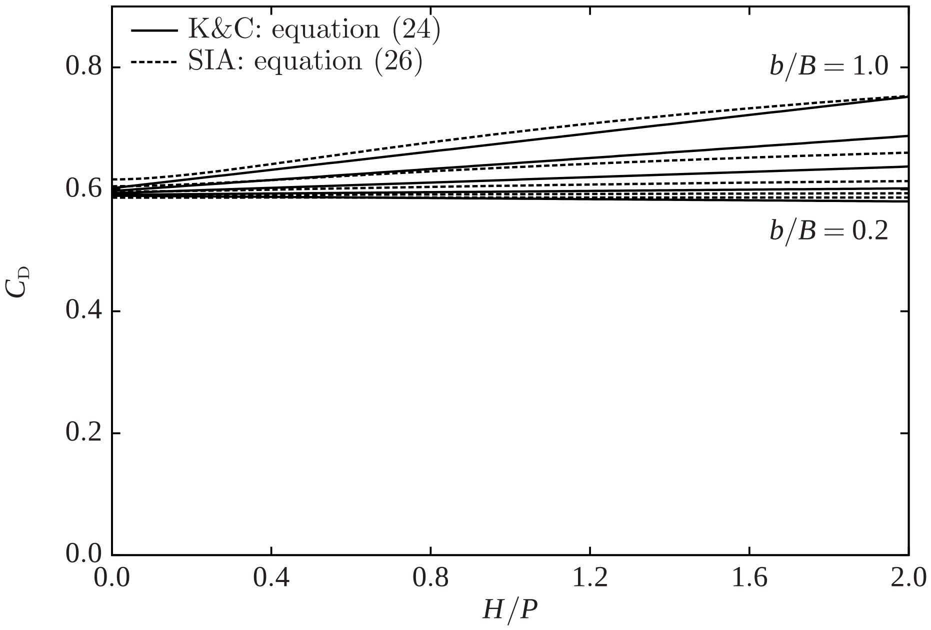

Figure 7: Comparison of formulae for relative width b/B varying from 0.2 to 1 in steps of 0.2

We now compare the predictions of the Kindsvater and Carter and SIA formulae. We ignore the corrections k_b and k_H specified in units of length in the former, and somewhat crudely use $H = 0.5$m in the viscous term of the latter so that the first bracketed term in equation (26) becomes $0.585 + 0.031\,(b/B)^2$. The results are plotted on Figure 7, for a range of widths from $b/B = 0.2$ to 1, and for a range of relative weir heights from $H/P = 0$ to 2, although the stated restriction on equation (25) was that it be limited to $H/P \leq 1$, namely where the weir occupies up to half the channel depth. It can be seen that the agreement between the two expressions is relatively close. Kindsvater and Carter noted that the Swiss formula agreed reasonably well with their results, but that "the complexity of the formula is a deterrent to its use, and the limits of applicability specified by the SIA prevent its being a comprehensive solution for a full, practical variety of notch weirs".

In fact, the evidence was to the contrary, that the Swiss expression was at least an explicit formula, whereas the original Kindsvater and Carter results were presented only as lines on a graph, which have only now in this present work been approximated by equations, culminating in the relatively simple equation (24).

In many practical situations the geometric situation is rather more complicated than that of the experiments, which were for smooth rectangular channels terminated by a smooth planar end containing a single rectangular notch. The variability in results caused by differences of configuration would be rather greater than between the formula which we have obtained and the SIA formula. There is little to choose between the two approaches, but equation (24) is simpler.

8 Conclusions and recommendations

We have shown just how poor the traditional assumptions of the theory of sharp-edged weirs are, and have suggested that dimensional analysis is more rational and insightful than the traditional physical approach. Then we considered the effects of end corrections and noted the experimental results of Kindsvater and Carter that the Francis end correction formula is not accurate. We then obtained convenient mathematical approximation formulae for their results. We also considered the Swiss SIA formula, and the formulae presented here based on Kindsvater and Carter's results. The latter is slightly simpler. Both could be used, to give an idea of the magnitude of uncertainty.

However, one might end on an iconoclastic note, that the variation of C_{D} is so relatively small that, in practical situations where the weir might not be smooth, or the channel not be rectangular, or the weir not be a simple notch – it might be one of two or more as part of a larger structure and there might be piers between them – it might usually be reasonable simply to assume a constant $C_{\mathrm{D}} = 0.64$, say, and to use the expression

$$Q = \frac{2}{3} C_{\mathrm{D}} \sqrt{2g}\, b H^{3/2} \approx 0.6 \sqrt{g} b H^{3/2}.$$

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9 Notation

Latin Symbols

$a_0,\ a_1,\ a_2$	coefficients in polynomial
B	width of channel
b	width of weir
b_e	effective width of weir
C_D	coefficient of discharge
C_e	dimensional coefficient of discharge
d	depth of water in channel
g	gravitational acceleration
H	head over weir
H_e	effective head over weir
k_b	correction to weir width
k_H	correction to head
L	length dimension
P	height of crest above channel bottom
p	pressure
Q	discharge over weir
q	discharge per unit width
T	time dimension
U	velocity of flow upstream
(u,v)	velocity components, horizontally and vertically
Y	elevation of streamline upstream
y	vertical co-ordinate with origin at the crest
y_D	elevation of water surface above the crest

Greek Symbols

α	function of b/B appearing in expression for C_D
β	function of b/B appearing in expression for C_D
λ	latitude
$\pi_1(\ldots)$	function of arguments
ψ	stream function: $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$
ρ	fluid density