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The long wave equations for arbitrary slopes

John D. Fenton

Institute of Hydraulic and Water Resources Engineering
Vienna University of Technology, Karlsplatz 13/222,
1040 Vienna, Austria

URL: *http://johndfenton.com/*
URL: *mailto:JohnDFenton@gmail.com*

Abstract

The long wave equations for a straight slowly-varying channel on an arbitrary slope are derived using the mass and momentum conservation equations in cartesian co-ordinates. The derivation attempts to use a true hydraulic approach, where quantities are modelled as accurately but simply as possible. This document is initially very similar to that of Fenton (2010), however in that document and all other presentations, the long wave equations have been obtained for small slopes. For steep slopes the hydrostatic approximation for pressure is not accurate and neither is the resistance formulation. Here, corrected pressure and resistance formulations for finite slopes are introduced. Evaluating the momentum contribution due to pressure is made rather easier by using the divergence theorem of vector calculus. Then, in evaluating resistance, the Darcy-Weisbach formulation is shown to have several advantages and the Gauckler-Manning approach is criticised. Although use is made of some vector notation, there are few lengthy mathematical operations until it is necessary to relate the derivatives of area to those of surface elevation. A useful result is obtained, however, where the non-prismatic nature of a channel is shown to be simply approximated using the mean downstream slope at a section. The long wave equations are presented in explicit useable form with various combinations of variables, and similarly for steady flow, with different forms of the gradually-varied flow equation. The expressions are valid also for finite slopes. For the special case of steady uniform flow, generalised Chézy-Weisbach and Gauckler-Manning formulae for finite slopes are presented, as well as a quickly-convergent numerical method for finding normal depth in any channel flow.

Revision History	
May 2014	Initial version on Internet site
April 2016	In response to an enquiry from Tom Molls of David Ford Consulting Engineers, I have explained more fully the calculation of wetted perimeter of a section perpendicular to the bed leading to equations (17), and expanded §7 on Chézy-Weisbach and Gauckler-Manning formulae, plus a numerical method for normal depth.
March 2017	Vertical converging/diverging wall term from eqn (39b) now retained in the steady eqns (42)-(45)

This report is: Fenton, J. D. (2014) The long wave equations for arbitrary slopes, Alternative Hydraulics Paper 5.
URL: *http://johndfenton.com/Papers/05-The-long-wave-equations-for-arbitrary-slopes.pdf*

1. Introduction

The one-dimensional long wave equations are widely used in river and canal hydraulics and are the basis of much hydraulic software. They are subject to limitations including both small longitudinal slopes and curvatures. Their origin seems to have been earlier than the usually cited work of Saint-Venant. Dooge (1987, p221) stated that the “first presentation of such a pair of equations would appear to be due either to Dupuit ... or to Kleitz ... and refers to the case of a wide rectangular channel”. Saint-Venant obtained a general mass conservation equation for a uniform waterway of any cross-sectional shape. However his derivation of the momentum equation was just for a rectangular canal of constant width, although he did retain a general resistance term. Dooge further wrote that according to Graeff in 1875 these equations had been previously given in a report by Kleitz in 1858 that was circulated but not published. Saint-Venant’s contribution beyond that of Dupuit and Kleitz has been modest, and it would seem unfair to the earlier workers to continue to use the appellation ”Saint-Venant” for the equations. However, it is interesting that the momentum equation of Dupuit/Kleitz/Saint-Venant, obtained on the basis of a uniform rectangular canal, is in fact valid for the general case of a non-prismatic waterway of arbitrary section.

Boussinesq (1877, p192 – eqn 155), which is described rather more accessibly by Jaeger (1956, p122 *et seq.*), considered steady flow in a wide rectangular channel of varying longitudinal bed topography using mass and momentum conservation. He went to one higher level of approximation, allowing for streamline curvature in determining the pressure, and obtained a third-order (now "Boussinesq") differential equation.

Keulegan (1942) derived both momentum and energy formulations for steady flow, and stated a preference for the momentum approach as it requires only the mechanism of resistance at the boundary and not details of the energy loss processes. Keulegan & Patterson (1943) then considered unsteady motion in a prismatic channel, obtaining the momentum equation, and then for a wide rectangular channel went on to consider the higher Boussinesq approximation with dynamic effects due to streamline curvature.

Two later derivations of the long wave equations went to considerable lengths to make the treatments as general as possible. Strelkoff (1969) considered the full Navier-Stokes equations and made an allowance for turbulence in the form of time averaged Reynolds equations. Yen (1973) also developed a general formulation, including viscosity, turbulence, fluid compressibility and inhomogeneity. In both cases the general forms of the equations were not suitable for practical application, but both writers did go on to present useable equations by making the usual reasonable hydraulic assumptions. Both derivations used a coordinate system based on the bed of the waterway. In many natural streams the bed is poorly known, and it is in general curved in the vertical plane. Such an approach should ideally have included extra terms due to the curvature of the axes, as obtained by Dressler (1978). However, both formulations would have been correct had they simply defined the streamwise axis to be horizontal. Later derivations by Cunge, Holly & Verwey (1980, §2.1) and Lai (1986, pp182-3) did use cartesian coordinates and obtained simpler derivations. Both presented results for different combinations of dependent variables. However in the final presentations, both left some versions with derivatives of combinations of quantities unexpanded and not in a form ready for use. Neither evaluated explicitly the so-called non-prismatic contribution to the derivative of cross-sectional area.

This paper obtains the long wave equations for straight channels of otherwise arbitrary section and topography. It tries to use a true hydraulic approach, where quantities are modelled as accurately but simply as possible. The theory includes flow on steep slopes, overcoming the traditional limitation, following Darvishi, Fenton, and Kouchakzadeh (2014), who considered the problem of flow with curvature in rectangular channels, using a Boussinesq approach. They discovered that previous Boussinesq and long wave theories have all used the hydrostatic approximation for the pressure in the water. In the flow of a real fluid the vertical component of resistance is such that the isobars in the fluid are not horizontal, but are more or less parallel to the free surface. This means that the pressure does not follow a hydrostatic distribution, and further corrections to the equations are necessary. Also, previous work has neglected the effects of the resistance force in the flow being not horizontal, but parallel to the bed. While these effects are usually unimportant, as most streams have a small slope, in principle they should be understood.

Features of the derivation and results

- The equations can be obtained relatively concisely using the integral mass and momentum equations; there are few detailed mathematical manipulations until it is necessary later to relate cross-sectional area and surface elevation.
- The cross-sectional area and discharge arise naturally as the most fundamental dependent variables. The mass conservation equation obtained in terms of them is exact for straight channels.
- The momentum equation requires additional assumptions, listed below.
- Cartesian co-ordinates are simple and require no special attention for axis curvature.
- Use of Gauss' divergence theorem in evaluating the pressure contribution to the momentum equation gives a substantially simpler derivation, as it avoids difficulties of integrals over complicated surfaces.
- The traditional hydrostatic approximation is abandoned, so as to generalize the results to flows on finite slopes. Instead an equivalent approximation for the pressure is made assuming that locally all isobars are parallel to the free surface, which is an isobar.
- An explicit form is given for a term due to a non-prismatic channel, expressed in terms of the local mean downstream slope at a section. Often that is poorly-known, which shows that there is little sense in giving detailed attention to the term and its precise calculation.
- The Darcy-Weisbach resistance formulation fits naturally with the momentum approach, so that the resistance term appears with a clear physical significance in the momentum equation. This is corrected to allow for the effects of finite slope.
- Three pairs of long wave equations are presented, each in terms of discharge plus one measure of cross-section – the actual area (which is found to have an interesting advantage, that approximate simulations can be performed without artificial inclusion of underwater topography), plus surface elevation, plus a depth-like quantity, the elevation relative to an arbitrary axis.
- Three gradually-varied flow equations for steady flow are presented, each in terms of one of the three measures of cross-section.
- In all cases, corresponding equations for small slopes are presented.
- Finally, for the special case of steady uniform flow, generalised Chézy-Weisbach and Gauckler-Manning equations are obtained which are valid for all slopes.

Assumptions

The non-trivial assumptions in the derivation are, roughly in decreasing order of importance or limitation:

1. Resistance to flow is approximated by the Darcy-Weisbach approach. The underlying flow is a turbulent shear flow. Of course the Navier-Stokes equations are not being used.
2. The stream is assumed to be straight, such that effects of curvature of its course are ignored.
3. All surface variation is sufficiently long, but not necessarily of small slope, that the pressure throughout the flow is given by a local uniform flow approximation valid for steep slopes.
4. The water surface across the stream is horizontal.
5. The effects of both non-uniformity of velocity over a section and turbulent fluctuations are approximated by a generalised Boussinesq momentum coefficient multiplying uniform contributions.
6. The fluid density is constant.

2. Mass conservation equation

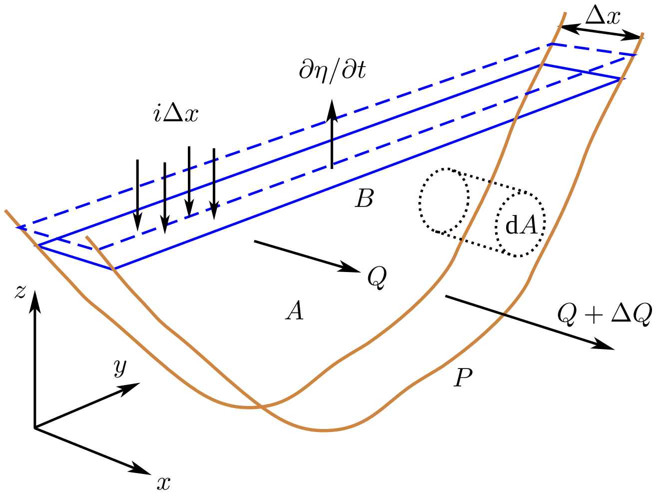

Figure 1. Elemental length of channel showing moving control surface with its enclosed control volume

Consider figure 1, showing an elemental slice of straight channel of length Δx with a control volume formed by four surfaces – two stationary vertical faces transverse to the flow, the stationary stream bed, and the possibly-moving free surface. Cartesian co-ordinates are used, which avoid mathematical problems that bed-oriented co-ordinates introduce. The mass conservation equation for an arbitrarily moving control surface and volume is (*e.g.* White (2009,§3.3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho\,\mathrm{d}\mathcal{V} + \int_{\mathrm{CS}} \rho\,\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0, \tag{1}$$

where the integrals are over CV, the control volume, and CS, its enclosing control surface, ρ is the fluid density, $\mathrm{d}\mathcal{V}$ is an element of volume, t is time, $\mathrm{d}S$ is an element of the control surface, $\mathbf{u}_{\mathrm{r}}$ is the velocity of a fluid particle relative to that of the control surface, $\hat{\mathbf{n}}$ is a unit normal vector directed outwards such that $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}$ is the normal component of fluid velocity relative to the surface. It is this velocity that is responsible for the transport of any quantity across the surface.

The first term in equation (1), is the rate of change of mass inside the elemental control volume. An element of the control volume shown in figure 1 has volume $\mathrm{d}\mathcal{V} = \mathrm{d}A\,\Delta x$, and so, assuming ρ constant and taking constant quantities outside the integral we obtain

$$\rho\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \rho\,\Delta x\frac{\partial A}{\partial t}, \tag{2}$$

where A is the cross-sectional area of the channel flow.

Considering the mass rate of flow crossing the boundary, there is no contribution on the solid stationary bed as $\mathbf{u}_{\mathrm{r}} = \mathbf{u} = \mathbf{0}$ there. On the possibly-moving free surface the fluid particles remain on it, in whatever manner it is moving, such that $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}} = 0$ there, and there is no mass transport across it either. On the stationary vertical faces across the flow, $\mathbf{u}_{\mathrm{r}} = \mathbf{u}$, the actual fluid velocity. On the upstream face $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}} = -u$, where u is the x-component of velocity at a point, and the minus sign is because the velocity is opposite to the outwards normal which is upstream here. The contribution to the integral is then $-\int_A \rho u\,\mathrm{d}A$, which for an incompressible fluid is simply $-\rho Q$, where Q is the volume rate of flow, the discharge. The contribution to the second integral, on the downstream face, is of opposite sign and in general has changed with x. We write it as a Taylor series, giving the combination of the two:

$$-\left.\rho Q\right|_x + \left.\rho Q\right|_{x+\Delta x} = \rho\,\Delta x\frac{\partial Q}{\partial x} + \text{ terms like } (\Delta x)^2\,. \tag{3}$$

Finally, an allowance is made for any fluid entering the control volume from rainfall, seepage, or tributaries, with a volume rate i per unit length and density ρ, assumed to be the same as that already in the channel. Its contribution is $-\rho\Delta x\, i$, negative because it is entering the control volume. Combining this contribution and those of equations (2) and (3) to equation (1), dividing by $\rho\,\Delta x$ and taking the limit as $\Delta x \to 0$ gives

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{4}$$

Unusually in hydraulics, this is almost an exact equation. The only non-trivial assumption that has been is that the channel is straight. No assumption regarding the flow has had to be made. It suggests that the cross-sectional area A and discharge Q are fundamental quantities. Below we will obtain equations also in terms of surface elevation.

3. Momentum conservation equation

The conservation of momentum principle is now applied to the flow in the channel. Consider the x-component of the integral form of the momentum equation for a possibly-moving control volume (White 2009, §3.4):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathrm{CV}} \rho\, u\, \mathrm{d}\mathcal{V} + \int\limits_{\mathrm{CS}} \rho\, u\, \mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\, \mathrm{d}S = F_x, \tag{5}$$

where F_x is the component of force exerted on the fluid in the control volume by both body and surface forces.

3.1 Fluid momentum terms

Here we consider the terms on the left of equation (5).

1. Unsteady term

Again using $\mathrm{d}\mathcal{V} = \mathrm{d}A\,\Delta x$, and taking constant quantities outside the integral we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathrm{CV}} \rho\, u\, \mathrm{d}\mathcal{V} = \rho\,\Delta x\,\frac{\partial}{\partial t}\int\limits_{A} u\, \mathrm{d}A = \rho\,\Delta x\,\frac{\partial Q}{\partial t}, \tag{6}$$

in which no additional approximation has been made, as by definition, the discharge $Q = \int_A u\, \mathrm{d}A$. It is useful that this term has been able to be expressed simply and exactly in terms of Q, further evidence for the utility of that quantity. As the term is linear for constant density, allowing for turbulence by short-term time-averaging gives no additional terms if we ignore turbulent surface fluctuations.

2. Momentum flux term

The second term on the left of equation (5), $\int_{\mathrm{CS}} \rho\, u\, \mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\, \mathrm{d}S$, has no contribution from the material boundaries on top and bottom of the control volume, the surface and the bed, as they are composed of particles that move with and define the control surface such that the velocity of those particles relative to the surface is $\mathbf{u}_{\mathrm{r}} = \mathbf{0}$. On the stationary vertical face upstream, $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}} = -u$, giving the contribution $-\rho\overline{\int_A u^2\, \mathrm{d}A}$, where the overbar has been introduced to denote a short term time averaging operation to allow for turbulence. The downstream face at $x + \Delta x$ has a contribution of a similar nature, but where both A and u have changed with x. The difference between the two contributions is, again using a Taylor series, simply

$$\rho\,\Delta x\,\frac{\partial}{\partial x}\overline{\int\limits_{A} u^2\, \mathrm{d}A} + \text{terms like } (\Delta x)^2 .$$

Evaluating the area integral and the time mean requires a detailed knowledge of the flow distribution and its turbulent nature that is almost always unavailable. Traditionally a Boussinesq momentum coefficient β has been introduced to allow for the non-uniformity of velocity distribution. It should be used also to allow for the effects of turbulence (Fenton 2005). If the velocity at a point is written as $u = \bar{u} + u'$, where $\bar{u}$ is the time mean velocity and u' the fluctuating component, if we ignore fluctuations of the free surface, then performing averaging over a short time interval the integral can be written

$$\overline{\int_A u^2 \, \mathrm{d}A} = \int_A \left(\bar{u}^2 + \overline{u'^2}\right) \, dA$$

and introducing the more general definition of the momentum coefficient

$$\beta = \frac{1}{U^2 A} \int_A \left(\bar{u}^2 + \overline{u'^2}\right) \, dA,$$

where $U = Q/A$, the mean component of x-velocity averaged over the section, the contribution can be written simply

$$\rho \, \Delta x \, \frac{\partial}{\partial x} \overline{\int_A u^2 \, \mathrm{d}A} \approx \rho \, \Delta x \, \frac{\partial}{\partial x} \left(\beta U^2 A\right) = \rho \, \Delta x \, \frac{\partial}{\partial x} \left(\beta \frac{Q^2}{A}\right), \tag{7}$$

The remaining contribution to momentum flux is from inflow. In obtaining the mass conservation equation above, this was lumped together as an inflow i per unit length, such that the mass rate of inflow was $\rho \, i \, \Delta x$, (*i.e.* an outflow of $-\rho \, i \, \Delta x$). If this inflow has a mean streamwise velocity of u_i before it mixes with the water in the channel, the contribution is

$$-\rho \Delta x \, \beta_i \, u_i \, i, \tag{8}$$

where β_i is the Boussinesq momentum coefficient of the inflow. The term is unlikely to be known accurately or to be important in most places, except locally where a significant stream enters.

3.2 Forces – body and surface

1. Gravity

The gravitational force is vertical, such that with our cartesian co-ordinates the horizontal component is zero, and there is no contribution to our x-momentum equation. The manner in which gravity enters is to cause the variation of pressure in the fluid, as we now calculate.

2. Pressure forces, including effects of finite slope

The total pressure force on the fluid around the control surface is $-\int_{\mathrm{CS}} p \, \hat{\mathbf{n}} \, \mathrm{d}S$, where p is the pressure, $\hat{\mathbf{n}}$ is the outward normal and the negative sign shows that the local force due to pressure acts opposite to it, inwards on the fluid in the control volume. In this form the term is difficult to evaluate for non-prismatic waterways, as the pressure and the highly-variable unit normal vector have to be integrated over all the submerged faces of the control surface. Traditional derivations do this using some lengthy calculus, but nevertheless ending with a simple result. It is obtained more easily if the term is evaluated using Gauss' divergence theorem of vector calculus, replacing the integral over the rather complicated control surface by a volume integral (*e.g.* Milne-Thomson 1968, §2.61, equation 3, but with a sign convention for $\hat{\mathbf{n}}$ opposite to ours), so that here

$$-\int_{\mathrm{CS}} p \, \hat{\mathbf{n}} \, \mathrm{d}S = -\int_{\mathrm{CV}} \nabla p \, \mathrm{d}\mathcal{V}, \tag{9}$$

where ∇ is the vector gradient operator such that ∇p is a vector whose components are the pressure gradients in each direction. For the elemental control volume across the channel this can then be simply evaluated in terms of an integral across the section, and taking just the x-component of ∇p, $\partial p/\partial x$, we obtain

$$\text{Pressure contribution to } x\text{-momentum} = -\Delta x \int_{\text{CV}} \frac{\partial p}{\partial x}\, \mathrm{d}A. \tag{10}$$

It is almost obvious that the net pressure force is given by the pressure gradient multiplied by Δx integrated over the section, but without the formality of the mathematics one might be worried about contributions at the bed and free surface.

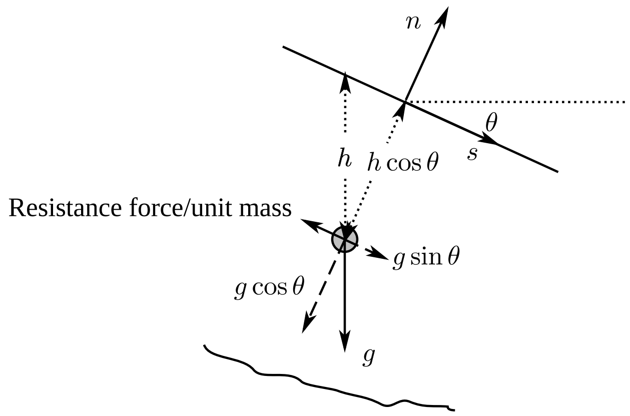

Figure 2. Forces per unit mass acting on a fluid particle in a gradually-varied open channel flow

The usual hydraulic approximation for the pressure p is that the pressure distribution is hydrostatic. That is not so in flow over a finite slope. Consider figure 2 for a slowly varying flow, where surface and bed are not necessarily parallel. The surface is an isobar, and in the upper part of the flow, isobars will be parallel to that. Nearer the bed that may not be quite the case, but the real value is difficult to establish without solving the whole flow problem. We consider variation of pressure on a line perpendicular to the free surface, given by the local normal co-ordinate n as shown. The component of gravity in that direction is $-g\cos\theta$, where θ is the local surface slope. From the Euler equation for equilibrium, the Navier-Stokes equation with viscosity neglected (*e.g.* White 2009, §2.2) we have $\partial p/\partial n = \rho\left(-g\cos\theta - a_n\right)$, where a_n is the acceleration of fluid particles in direction n. This is small and we neglect it, giving $\partial p/\partial n = -\rho g\cos\theta$. Integrating this and requiring that $p = 0$ when $n = 0$, gives $p = -\rho g n\cos\theta$. It follows that at a point vertically h below the surface, such that $n = -h\cos\theta$,

$$\text{Pressure at a depth } h \text{ below the surface:} \quad p \approx \rho g h\cos^2\theta. \tag{11}$$

The expression is a known exact result for uniform flow (Chow 1959, §2-10 and Henderson 1966, eqn 2-2). Even though the result looks like the traditional hydrostatic one, but with a $\cos^2\theta$ modification, it cannot be called a hydro-*static* approximation, as the vertical component of resistance leading to the isobars being tilted actually comes from the flow, and the terminology "static" is no longer available to us. It might be termed the "local uniform flow approximation", corresponding to a uniform flow with the slope of the free surface. It is not exact, but in a spirit of modelling, it is the next best approximation after the hydrostatic one.

Now we write the result in terms of surface elevation η. For a point in the fluid of elevation z, above which the surface elevation is η, such that $h = \eta - z$, and expressing the cosine function in terms of the surface slope $\eta_x = \partial\eta/\partial x$, such that $\cos^2\theta = 1/\left(1+\eta_x^2\right)$, gives the expression for pressure in the fluid in a gradually-varied flow

$$p = \frac{\rho g\left(\eta - z\right)}{1+\eta_x^2}, \tag{12}$$

as used by Darvishi, Fenton & Kouchakzadeh (2014). To substitute into equation (10) we now have to differentiate with respect to x, which will also give contributions from the denominator, with products of first and second derivatives of the surface elevation. However in making a long wave approximation here, we have ignored such curvature terms in obtaining the pressure, and so we continue to do so, giving

$$\frac{\partial p}{\partial x} \approx \frac{\rho g \eta_x}{1+\eta_x^2}. \tag{13}$$

This is constant over a cross-section, so that integration with respect to area A to give the contribution to the momentum equation is trivial and so from equation (10) the contribution to the channel momentum flux is then

$$-\Delta x \int_A \frac{\partial p}{\partial x}\,\mathrm{d}A = -\rho\,\Delta x \frac{gA\eta_x}{1+\eta_x^2}. \tag{14}$$

3. Resistance forces, including effects of finite slope

The forces of the boundary on the flow are incorporated using empirical results from turbulent shear flows. The Darcy-Weisbach formulation here provides insights into the nature of the equations and some convenient quantifications of the effects of resistance. The ASCE Task Force on Friction Factors in Open Channels (1963) recommended its use, but that suggestion has been almost entirely ignored. In the present force determination it is very useful, because it is directly related to stress and force on the boundary.

Consider the expression for the magnitude of the shear force τ on a pipe wall (*e.g.* §6.3 of White 2009)

$$\tau = \frac{\lambda}{8}\rho V^2, \tag{15}$$

where the Weisbach coefficient λ is a dimensionless resistance factor (for which the symbol f is often used, but here we follow the terminology of fundamental researchers in the field in the first half of the twentieth century), and V is the mean velocity in the pipe. Such an expression follows from a dimensional analysis of the problem, suggesting its fundamental nature. The denominator 8 follows from the original introduction of λ in the Darcy-Weisbach formula for head loss in a pipe, with a term $2g$ in the expression for head and a term 4 in the relationship between head loss and τ. The coefficient λ is simply related to Chézy's resistance coefficient C by $\lambda = 8g/C^2$.

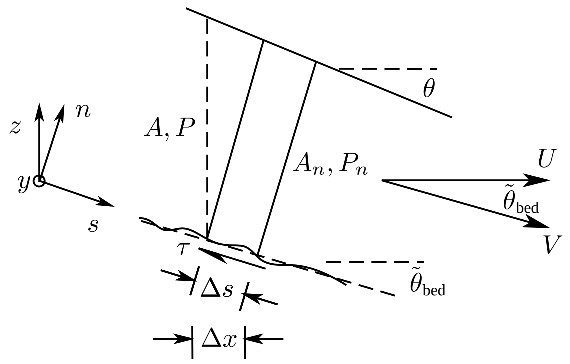

Figure 3. Cross-section of channel with an elemental slice inclined at an angle $\tilde{\theta}_{\text{bed}}$ corresponding to the *mean* downstream bed slope at that section

To calculate the resistance force around the wetted perimeter of the channel, trying to make no approximation as to slope, we consider an elemental volume across the channel shown in figure 3, inclined at an angle to the horizontal of $\tilde{\theta}_{\text{bed}}$, corresponding to the local mean downstream slope $\tilde{S}$ such that

$$\tan\tilde{\theta}_{\text{bed}} = \tilde{S}.$$

Here we have not stated how we define $\tilde{S}$ in terms of bed geometry; the precise definition will be given in equation (31) below.

To evaluate τ we use equation (15) and consider V to be the mean velocity parallel to the local bed, the velocity most analogous to that in a pipe. The mean streamwise velocity is written as $V \approx U/\cos\tilde{\theta}_{\text{bed}}$

in terms of the mean horizontal velocity $U = Q/A$. Equation (15) becomes

$$\tau = \frac{\lambda}{8}\rho\frac{(Q/A)^2}{\cos^2\tilde{\theta}_{\mathrm{bed}}} = \frac{\lambda}{8}\rho\frac{Q^2}{A^2}\left(1+\tan^2\tilde{\theta}_{\mathrm{bed}}\right) = \frac{\lambda}{8}\rho\frac{Q^2}{A^2}\left(1+\tilde{S}^2\right).$$

To calculate the force per unit length we multiply this mean stress by the area $P_n\,\Delta s$ over which it acts, where P_n is the wetted perimeter in the plane of the inclined section, as shown in figure 3. As the force so calculated is parallel to the local bed, we then multiply by $\cos\tilde{\theta}_{\mathrm{bed}} = \Delta x/\Delta s$ to give the x component:

$$\text{Component of resistance in } x \text{ direction} = -\rho\,\Delta x\frac{\lambda P_n}{8}\frac{Q\,|Q|}{A^2}\left(1+\tilde{S}^2\right), \tag{16}$$

where we have replaced Q^2 by $-Q\,|Q|$ such that the direction of the stress is always opposite to the flow direction.

Now we have to obtain the perimeter P_n in the plane of the inclined section from conventional vertical measurements of bed elevation. Consider an element of perimeter $\mathrm{d}P_n$ consisting of a transverse component $\mathrm{d}y$ which is the same in the normal (y, n) and vertical (y, z) planes, but any element with a $\mathrm{d}n$ component has a different projection on the two planes. A simple example is a rectangular section of width W and vertical depth h: the projection of the bottom W onto a normal section is still W while each side has a projection onto the normal of $h\cos\theta$, giving $P_n = W + 2h\cos\theta$, while $P = W + 2h$.

More generally, we can write

$$\mathrm{d}P_n = \sqrt{(\mathrm{d}y)^2 + (\mathrm{d}n)^2} = \sqrt{(\mathrm{d}y)^2 + \cos^2\theta\ (\mathrm{d}Z)^2}, \quad \text{whereas}$$
$$\mathrm{d}P = \sqrt{(\mathrm{d}y)^2 + (\mathrm{d}Z)^2},$$

where Z is the bed elevation. We see that P_n and P are not simply related by a constant factor of $\cos\theta$, and although any of several approximations are possible, in general for simplicity we leave the result, equation (16) written in terms of P_n itself, as we leave other section properties such as area A unevaluated until required for a particular channel. If a value of P_n were required, to evaluate it we can write the result in two forms, either integrating with respect to the transverse co-ordinate y across the channel, or with respect to Z, from the lowest point of the section Z_{min} to the surface η:

$$P_n = \int_B \sqrt{1+\cos^2\theta\,(\mathrm{d}Z/\mathrm{d}y)^2}\,\mathrm{d}y, \quad \text{or} \tag{17a}$$

$$= \int_{Z_{\mathrm{min}}}^{\eta} \sqrt{(\mathrm{d}y/\mathrm{d}Z)^2 + \cos^2\theta}\,\mathrm{d}Z. \tag{17b}$$

In general, the first form, equation (17a) might be more useful, as usually bed elevation Z is specified as a function of transverse co-ordinate y. For the special case of a trapezoidal channel of horizontal bottom width W and batter slopes $\gamma : 1$ (H:V) such that $\mathrm{d}y/\mathrm{d}Z = \gamma$ on the sides, it is convenient to break the integral up into three parts, dealing with the horizontal bottom separately, simply giving W, and to use equation (17b) for the sides to give

$$P_n = W + 2\int_0^h \sqrt{\gamma^2+\cos^2\theta}\,\mathrm{d}Z = W + 2h\sqrt{\gamma^2+\cos^2\theta}. \tag{18}$$

Special cases are:

- A rectangular channel, $\gamma = 0$, $P_n = W + 2h\cos\theta$, as obtained simply above.
- For small slopes, when $\cos\theta \approx 1$, $P_n = W + 2h\sqrt{\gamma^2+1} = P$.

3.3 Collecting all terms in the momentum equation

Now all contributions to the momentum equation (5) are collected, from equations (6), (7), (8) on the left, and (14) and (16), contributions to F_x, on the right. Dividing by $\rho\Delta x$, and bringing all derivatives of dependent quantities to the left and others to the right, gives the momentum equation:

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+\frac{gA}{1+\eta_x^2}\frac{\partial\eta}{\partial x}=-\frac{\lambda P_n}{8}\frac{Q\,|Q|}{A^2}\left(1+\tilde{S}^2\right)+\beta_i i\,u_i. \tag{19}$$

The equation in this form is simple and shows the significance of each term. Previous presentations using the small slope approximation have scarcely gone beyond this stage (for example Cunge et al. 1980, pp16-17; Lai 1986, pages 182-3), and did not expand the inertia term $\partial\left(\beta Q^2/A\right)/\partial x$. That is easily performed, and we obtain

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}-\beta\frac{Q^2}{A^2}\frac{\partial A}{\partial x}+\frac{gA}{1+\eta_x^2}\frac{\partial\eta}{\partial x}=-\frac{\lambda P_n}{8}\frac{Q\,|Q|}{A^2}\left(1+\tilde{S}^2\right)+\beta_i i\,u_i-\frac{Q^2}{A}\beta'(x). \tag{20}$$

In the case of the usual *small slope approximation* we neglect terms in η_x^2 and $\tilde{S}^2$, and set $P_n = P$, giving

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}-\beta\frac{Q^2}{A^2}\frac{\partial A}{\partial x}+gA\frac{\partial\eta}{\partial x}=-\frac{\lambda P}{8}\frac{Q\,|Q|}{A^2}+\beta_i\,u_i\,i-\frac{Q^2}{A}\beta'(x), \tag{21}$$

where $\beta'(x)=\mathrm{d}\beta/\mathrm{d}x$. The discharge has emerged as being fundamental in this integrated momentum formulation, the only time derivative being $\partial Q/\partial t$. However the equation is not yet in useable form, for the derivatives of cross-sectional area $\partial A/\partial x$ and surface elevation $\partial\eta/\partial x$ are not independent. In fact, some of the more important practical steps lie before us, in the relating of these quantities, which requires some effort.

3.4 Discussion of the resistance term

The significance of the resistance term, $-\lambda P/8\times Q\,|Q|\,/A^2$ (written here in its small slope form) is clear, written as a dimensionless coefficient multiplied by wetted perimeter times second power of the mean velocity, giving the force per unit length divided by density.

The resistance coefficient λ has been extensively investigated (ASCE Task Force on Friction Factors in Open Channels 1963). Yen considered the results presented and obtained a convenient formula in terms of the relative roughness and the Reynolds number of the flow (Yen 2002, equation 19):

$$\lambda=\left(-2\,\log_{10}\left(\frac{\epsilon_s}{12}+\frac{1.95}{R^{0.9}}\right)\right)^{-2}, \tag{22}$$

where $\epsilon_s = k_s/\left(A/P\right)$ is the relative roughness, k_s is the equivalent sand-grain diameter and $R = Q/P\nu$ is the channel Reynolds number with ν the kinematic viscosity. Yen stated that his formula was applicable for $R > 30\,000$ and $\epsilon_s < 0.05$.

This explicit formula for computing resistance seems to provide a solution to some of the problems identified by the ASCE Task Force. It treats both smooth and rough boundaries, and for non-vegetated streams at least seems to be superior to that of using values of Manning's coefficient n often obtained roughly by using tables or pictures from books. For streams where vegetation provides an important contribution to resistance, it does not help, but recent research on the effects of vegetation in streams has certainly used the framework of fluid drag, which can be fitted into the Darcy-Weisbach formulation. As λ is dimensionless, it is not necessary to modify any formulae if one uses non-*S.I.* units.

If one wanted to add an allowance for resistance such as that due to vegetation or bed-forms, one advantage of the Weisbach formulation, being directly related to force, is that one can linearly superimpose contributions so that in a more complicated situation, the resistance contributions can be simply combined, including the perimeter over which they act:

$$\lambda P = \sum\nolimits_i \lambda_i P_i. \tag{23}$$

Another simple example where a formula such as this would be useful is a glass-walled laboratory flume with a rough bed, which would cause difficulties for Gauckler-Manning, which is not based on rational mechanics. An idea of the problems which its empiricism causes is given by the different formulae for the compound Manning coefficient n, all found in one recent report on resistance in streams:

$$n = \sum\nolimits_i n_i \quad \text{or} \quad n = \left(\sum\nolimits_i n_i^2\right)^{1/2} \quad \text{or} \quad \frac{1}{n} = \left(\sum\nolimits_i \frac{1}{n_i^2}\right)^{1/2}. \tag{24}$$

The report presented different recommendations in the report as to when each method would be preferred. There was no weighting according to the fraction of perimeter for each contribution.

That notwithstanding, we mention other explicit forms of the resistance term including those of Chézy and Gauckler-Manning. For the different formulations to agree for steady uniform flow (for the traditional small slope approximation):

$$\frac{\lambda}{8} = \frac{g}{C^2} = \frac{gn^2 P^{1/3}}{A^{1/3}} = \frac{g}{k_{\mathrm{St}}^2} \frac{P^{1/3}}{A^{1/3}}, \tag{25}$$

where C is the Chézy coefficient, n the Manning coefficient, and k_{St} the Strickler coefficient. For Gauckler-Manning for small slopes, the resistance term in the momentum equation (21) becomes:

$$-gn^2 \frac{Q\,|Q|\,P^{4/3}}{A^{7/3}}. \tag{26}$$

The indices $4/3$ and $7/3$ seem just a little fussy. In some applications the conveyance of the stream $K = 1/n \times A^{5/3}/P^{2/3}$ has been used so that the resistance term becomes

$$-gA \frac{Q\,|Q|}{K^2}. \tag{27}$$

The term is to do with resistance to fluid motion, and has nothing to do gravity – the presence of g here and in equation (26) is an artefact from the original Gauckler-Manning equation which does not include gravity, even though it is the driving force. Manning's n implicitly contains g, as does Chézy's C.

Possibly because of the non-trivial nature of the Manning form, a more vague way of writing the resistance term that has been used is $-gAS_{\mathrm{f}}$, where S_{f} is called the "friction slope". That is apparently a dangerous procedure, for in some works (*e.g.* Lyn & Altinakar 2002) S_{f} has been assumed to be a constant even where Q and A vary. More importantly, in terms of understanding, this has led to it being mistaken for an energy slope, which it is not. It comes from the shear force on the perimeter. The Weisbach form as it appears in the momentum equation (20) seems simple and clear.

4. Relationships between area and elevation derivatives

The momentum equation contains time and space derivatives of A and η. To be able to express the equations in terms of derivatives of one or the other, we have to relate them. Consider the integral for area

$$A = \int_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}} (\eta - Z)\ \mathrm{d}y, \tag{28}$$

where $z = Z(x, y)$ is the bed elevation, and the right and left waterlines are defined by the intersections between the side surfaces $y = Y_{\mathrm{L/R}}(x, z)$ and the free surface $z = \eta\,(x, t)$ such that the limits are $y = Y_{\mathrm{L/R}}(x, \eta\,(x, t))$, using notation "L/R" in the subscript showing that either or both can be taken.

The derivative of area with respect to time is obtained from Leibniz' theorem for the derivative of an integral, which gives

$$\frac{\partial A}{\partial t}=\int_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}}\frac{\partial \eta}{\partial t}\mathrm{d}y+(\eta-Z_{\mathrm{L}})\ \frac{\partial Y_{\mathrm{L}}}{\partial t}-(\eta-Z_{\mathrm{R}})\ \frac{\partial Y_{\mathrm{R}}}{\partial t},$$

where $(\eta-Z_{\mathrm{L}})$ is the water depth at the left bank and $(\eta-Z_{\mathrm{R}})$ that at the right. Both the last two terms are zero in most situations where the bank is sloping. The only way that they contribute is if the sides of the channel are vertical and are moving with time. This seems unlikely, and so they will be neglected. It has already been assumed that the free surface is level across the channel, so that the integrand $\partial\eta/\partial t$ is independent of y and can be taken outside the integral, giving

$$\frac{\partial A}{\partial t}=B\frac{\partial \eta}{\partial t}, \tag{29}$$

where $B=Y_{\mathrm{L}}-Y_{\mathrm{R}}$ is the surface width.

Now the x derivative of A is considered. Differentiating equation (28) with respect to x and using Leibniz' theorem again:

$$\frac{\partial A}{\partial x}=\underbrace{\int_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}}\frac{\partial \eta}{\partial x}\,\mathrm{d}y}_{\mathrm{I}}-\underbrace{\int_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}}\frac{\partial Z}{\partial x}\,\mathrm{d}y}_{\mathrm{II}}+\underbrace{(\eta-Z_{\mathrm{L}})\ \left.\frac{\partial Y_{\mathrm{L}}}{\partial x}\right|_t-(\eta-Z_{\mathrm{R}})\ \left.\frac{\partial Y_{\mathrm{R}}}{\partial x}\right|_t}_{\mathrm{III}}, \tag{30}$$

where, as A and η are functions of x and t, both $\partial A/\partial x$ and $\partial\eta/\partial x$ imply that t is considered constant. The bed elevation Z is a function of x and y, so $\partial Z/\partial x$ implies y is constant, and it is just a streamwise derivative, the local bed slope. A slightly different notation $\partial/\partial x|_t$ has been necessary for the x-derivatives of $Y_{\mathrm{R}}(x,\eta\,(x,t))$ and $Y_{\mathrm{L}}(x,\eta\,(x,t))$, to show that t is held constant, as here using just $\partial/\partial x$ would imply η constant. Now we evaluate each of the terms in the expression:

Term I: We have assumed that free surface elevation η is constant across the channel, so that the first term becomes simply $(Y_{\mathrm{L}}-Y_{\mathrm{R}})\ \partial\eta/\partial x=B\,\partial\eta/\partial x$.

Term II: The second term is the integral across the channel of the downstream bed slope. We introduce the symbol $\tilde{S}$ for the local mean downstream bed slope evaluated across the section:

$$\tilde{S}=-\frac{1}{B}\int_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}}\frac{\partial Z}{\partial x}\,\mathrm{d}y, \tag{31}$$

defined with a minus sign such that in the usual situation where the bed slopes downwards in the direction of x, so that Z decreases, $\tilde{S}$ will be positive. If the bottom geometry is precisely known, this can be precisely evaluated, however it is much more likely to be only approximately known and a typical bed slope of the stream used. With this definition, the term in equation (30) can be just written $+B\tilde{S}$.

Term III: this term is zero almost everywhere. It is the contribution from the integrand at each limit multiplied by the derivative of the limit. The contributions in equation (30) are:

$$(\eta-Z_{\mathrm{L/R}})\ \left.\frac{\partial Y_{\mathrm{L/R}}}{\partial x}\right|_t=(\eta-Z)_{\mathrm{L/R}}\left(\frac{\partial Y_{\mathrm{L/R}}}{\partial x}+\left.\frac{\partial Y_{\mathrm{L/R}}}{\partial z}\right|_{z=\eta(x,t)}\frac{\partial \eta}{\partial x}\right). \tag{32}$$

The factor $(\eta-Z_{\mathrm{L/R}})$ is the water depth at the banks. We now consider different geometric cases:

1. The usual case for natural streams and most canals, with sloping (*i.e.* not vertical) banks:

   $$\eta-Z_{\mathrm{L/R}}=0,$$

   and so in this common case the contribution of the whole term is zero.

2. If a side is vertical, then $\partial Y_{\mathrm{L/R}}/\partial z\big|_{z=\eta(x,t)} = 0$ so the second term in the brackets in equation (32) is zero and we are left with the contribution $\left(\eta - Z_{\mathrm{L/R}}\right)\,\partial Y_{\mathrm{L/R}}/\partial x$. We have to consider two cases for this.
   a. The usual case for a vertical sided channel such as a flume or race or lock: the walls are parallel to the x-axis, so that they neither converge nor diverge, then $\partial Y_{\mathrm{L/R}}/\partial x = 0$ and so the contribution of the whole term is zero again.
   b. The rare case where the channel walls are both vertical and converging or diverging, such that $\partial Y_{\mathrm{L/R}}/\partial x$ is not zero, in something like a Parshall flume (although one would be careful about applying long wave theory in such a case).

As that last case, with a diverging vertical wall such that $Y_{\mathrm{L/R}}$ is independent of η, is the only possibility for a non-zero contribution, we can replace the partial derivatives by ordinary derivatives, and we write the net contribution of the last two terms in equation (30) as A_x^{V}:

$$A_x^{\mathrm{V}} = (\eta - Z_{\mathrm{L}})\frac{\mathrm{d}Y_{\mathrm{L}}}{\mathrm{d}x} - (\eta - Z_{\mathrm{R}})\frac{\mathrm{d}Y_{\mathrm{R}}}{\mathrm{d}x}, \tag{33}$$

the symbol V used to refer to **V**ertical side walls. For such vertical walls it is highly probable that we are dealing with a man-made structure, so that the bed is transversely horizontal too, such that in that case $Z_{\mathrm{L}} = Z_{\mathrm{R}} = Z$, and using the depth $h = \eta - Z$ we obtain

$$A_x^{\mathrm{V}} = h\left(\frac{\mathrm{d}Y_{\mathrm{L}}}{\mathrm{d}x} - \frac{\mathrm{d}Y_{\mathrm{R}}}{\mathrm{d}x}\right) = h\frac{\mathrm{d}B}{\mathrm{d}x}.$$

Almost everywhere, of course, $A_x^{\mathrm{V}} = 0$.

Collecting contributions from Terms I, II, and III: the relationship between area and elevation derivatives, equation (30), is written

$$\frac{\partial A}{\partial x} = B\frac{\partial \eta}{\partial x} + B\tilde{S} + A_x^{\mathrm{V}}. \tag{34}$$

The quantity $B\tilde{S}$ contains contributions from what has been called the "non-prismatic" term, which in other presentations has usually been written in vague terms like $\partial A/\partial x|_{h=\text{const}}$ and has not been explicitly evaluated. The slope $\tilde{S}$, based on the formal definition, equation (31), allows for the fact that the effective mean slope in a non-prismatic stream is different from that of a prismatic one even if the talweg has the same slope. In any case, the bed geometry is rarely able to be evaluated with any accuracy, and instead in practice, a typical local stream slope might often be used and non-prismatic effects ignored.

5. Three forms of the long wave equations

Presentations elsewhere have given equations in terms of the mean horizontal velocity U in the flow. We do not, believing it to be insufficiently important, as there are very few problems where U might be specified as a boundary condition. In practical problems usually volume flow rate Q is more important. If velocities were required as results, they could be trivially obtained from Q/A.

Here we present three versions of the momentum equation, all in terms of Q, with alternatives for the remaining dependent variable.

5.1 Equations in terms of (A, Q)

We have observed that the mass conservation equation, equation (4), is exact for a straight channel, suggesting that A is a fundamental quantity. This form may not be so important practically, but for some theoretical studies it is useful to use the two integrated quantities A and Q as dependent variables. As well, there is an interesting aspect to the A formulation, such that one can model a channel approximately with relatively little detailed or assumed knowledge of the underwater topography, explained here after the presentation of the equations.

Collecting equations (4) and (20), and using equation (34) to eliminate $\partial\eta/\partial x$ from the latter, gives the long wave equations in terms of A and Q:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=i, \tag{35a}$$

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(\frac{gA/B}{1+\eta_x^2}-\beta\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x} = \frac{gA}{1+\eta_x^2}\left(\tilde{S}+\frac{A_x^{\mathrm{V}}}{B}\right)-\frac{\lambda P_n}{8}\frac{Q\,|Q|}{A^2}\left(1+\tilde{S}^2\right)+\beta_i\,u_i\,i-\frac{Q^2}{A}\beta'(x), \tag{35b}$$

where derivatives of dependent variables have been taken to the left and all others to the right, and in which $\eta_x=1/B\times\partial A/\partial x-\tilde{S}-A_x^{\mathrm{V}}/B$, and of course, almost everywhere the right side is significantly simpler, where A_x^{V}, i, and $\beta'(x)$ are zero. In many situations the dominant terms in the equation are the remaining terms on the right, in terms of bed slope $\tilde{S}$ and resistance coefficient λ. Often neither the details of the underwater topography leading to $\tilde{S}$, nor the value of the resistance coefficient λ are accurately known. This may make one wonder about the necessity of including a value of β not equal to 1, or a value of $\beta'(x)$ at all.

Now we make the usual approximation in equation (35b), ignoring terms in $\tilde{S}^2$ and η_x^2 and taking $P_n=P$, giving the *small slope approximation* to the momentum equation:

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(\frac{gA}{B}-\beta\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x}=gA\left(\tilde{S}+\frac{A_x^{\mathrm{V}}}{B}\right)-\frac{\lambda P}{8}\frac{Q\,|Q|}{A^2}+\beta_i\,u_i\,i-\frac{Q^2}{A}\beta'(x). \tag{36}$$

In practice, in a channel where the geometry is well-known, at each computational value of $x_i,\ i=0,1,\ldots$, one might know the slope $\tilde{S}_i$ and the functional relationships $A_i(\eta)$, $B_i(\eta)$ and $P_i(\eta)$, probably in discrete form, from which one could obtain the corresponding $B_i(A)$ and $P_i(A)$ also in discrete form. However, it is much more likely that the underwater geometry is poorly known, and so the trouble of going to assumed forms of dependence on η is questionable. The formulation of equations (35a) and (35b) in terms of A allows an approximate procedure that is commensurate with the accuracy of knowledge of the whole problem. One could assume approximate values of $\tilde{S}_i$, and approximate, possibly constant, values of B_i and P_i at each computational point i, which for most streams are not going to vary much with flow anyway. Then to calculate the initial values of A at computational points along the stream one could assume notional approximate values of A at the points, with the given initial constant flow Q and perform a simulation with the equations until the incorrect values flow out of the computational domain and values of A became steady. These could then be used as initial values for real simulations with varying input discharge with time at the upstream boundary. Hence, using A one can perform model simulations with relatively little information required or artificially included.

5.2 Equations in terms of (η, Q)

Surface elevation is an important quantity in practice, so we will present a formulation in terms of it. We use equation (29) to eliminate $\partial A/\partial t$ from equation (4) and use equation (34) again, this time to eliminate $\partial A/\partial x$ from the momentum equation (20):

$$\frac{\partial \eta}{\partial t}+\frac{1}{B}\frac{\partial Q}{\partial x}=\frac{i}{B}, \tag{37a}$$

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(\frac{gA}{1+\eta_x^2}-\beta\frac{Q^2B}{A^2}\right)\frac{\partial \eta}{\partial x} = \beta\frac{Q^2}{A^2}\left(B\tilde{S}+A_x^{\mathrm{V}}\right)-\frac{\lambda P_n}{8}\left(1+\tilde{S}^2\right)\frac{Q\,|Q|}{A^2}+\beta_i\,u_i\,i-\frac{Q^2}{A}\beta'(x). \tag{37b}$$

Again, ignoring terms in $\tilde{S}^2$ and η_x^2, taking $P_n = P$ we obtain the *small slope approximation* to the momentum equation:

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial\eta}{\partial x}=\beta\frac{Q^2}{A^2}\left(B\tilde{S}+A_x^{\mathrm{V}}\right)-\frac{\lambda P}{8}\frac{Q\,|Q|}{A^2}+\beta_i\,u_i\,i-\frac{Q^2}{A}\beta'(x). \quad (38)$$

5.3 Equations in terms of a depth-like variable h

Many works present the equations in terms of a quantity h referred to as "depth", which is an ambiguous and uncertain quantity, especially for natural streams. Here we define it to be the surface elevation relative to a reference axis possibly associated with the bottom of the stream, which could be chosen to be the bed of a canal or the thalweg in a river if that were sufficiently well known. It does not have to be a straight line in the vertical plane.

It should be pointed out, however, that using area A might be a better alternative to using h, as it contains some of the advantages of h, such as being constant for uniform flow, and varying relatively little in most streams for non-uniform flow, but it does not require the definition of an axis.

We let the elevation of the reference axis be $Z_0(x)$ and then in general $\eta\,(x,t) = h\,(x,t) + Z_0(x)$ and the derivatives are $\partial\eta/\partial t = \partial h/\partial t$ and $\partial\eta/\partial x = \partial h/\partial x - S_0\,(x)$, where S_0 is the slope of the axis (positive in the usual downward-sloping channel sense) $S_0 = -\partial Z_0/\partial x$. Substituting these into the mass conservation equation (37a) and the momentum conservation equation (37b) gives

$$\frac{\partial h}{\partial t}+\frac{1}{B}\frac{\partial Q}{\partial x}=\frac{i}{B}, \quad (39a)$$

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(\frac{gA}{1+(h_x-S_0)^2}-\beta\frac{Q^2B}{A^2}\right)\frac{\partial h}{\partial x}$$
$$=\frac{gAS_0}{1+(h_x-S_0)^2}+\beta\frac{Q^2}{A^2}\left(B\left(\tilde{S}-S_0\right)+A_x^{\mathrm{V}}\right)-\frac{\lambda P_n}{8}\left(1+\tilde{S}^2\right)\frac{Q\,|Q|}{A^2}, \quad (39b)$$

where for brevity we have not shown the inflow term or the $\partial\beta/\partial x$ term. They are the same as in all the above equations. Now ignoring squares of slope terms, and taking $P_n = P$ gives the *small slope approximation* to the momentum equation:

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial h}{\partial x}=gAS_0+\beta\frac{Q^2}{A^2}\left(B\left(\tilde{S}-S_0\right)+A_x^{\mathrm{V}}\right)-\frac{\lambda P}{8}\frac{Q\,|Q|}{A^2}. \quad (40)$$

6. Generalised steady gradually-varied flow equations

Any of the above formulations can be trivially modified for the case of steady flow to give a pair of ordinary differential equations. In this case any of the forms of the mass conservation equation has the solution $Q = Q\,(x_0) + \int_{x_0}^{x} i\,(x')\,\mathrm{d}x'$. If there is no distributed inflow i, then the solution is $Q = Q\,(x_0) =$ constant. Equation (35b) in terms of area A becomes very long, so we will not present its steady version. The *small slope approximation* is

$$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{B\tilde{S}-\lambda PF^2/8}{1-\beta\mathsf{F}^2}, \quad (41)$$

where $\mathsf{F}^2 = Q^2B/gA^3$ is the square of the Froude number.

In the momentum equation (37b) in terms of η, where we can now use $Q\,|Q| = Q^2$ as the flow is unidirectional, we set $\partial\beta/\partial x = 0$ to give

$$\left(\frac{1}{1+(\mathrm{d}\eta/\mathrm{d}x)^2}\frac{1}{\mathsf{F}^2}-\beta\right)\frac{\mathrm{d}\eta}{\mathrm{d}x}=\beta\left(\tilde{S}+\frac{A_x^{\mathrm{V}}}{B}\right)-\frac{\lambda}{8}\frac{P_n}{B}\left(1+\tilde{S}^2\right). \quad (42)$$

The equation now has the unpleasant property that when multiplied out, it is a cubic equation for $\mathrm{d}\eta/\mathrm{d}x$ that cannot be solved by elementary means. As an approximation we replace the square of the surface slope by that of the bed slope in the troublesome term on the left, probably not a bad approximation where slopes are large and flow supercritical, such as on spillways, we obtain

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=\frac{\beta\left(\tilde{S}+A_x^{\mathrm{V}}/B\right)-\dfrac{\lambda}{8}\dfrac{P_n}{B}\left(1+\tilde{S}^2\right)}{\dfrac{1}{1+\tilde{S}^2}\dfrac{1}{\mathsf{F}^2}-\beta}. \tag{43}$$

Many streams are wide, such that $P_n \approx B$, and almost the only dependence on the dependent variable η in the right side of the equation is in the Froude number term $1/\mathsf{F}^2$ in the denominator. The *small slope approximation* is

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=\frac{\beta\left(\tilde{S}+A_x^{\mathrm{V}}/B\right)-\dfrac{\lambda}{8}\dfrac{P}{B}}{1/\mathsf{F}^2-\beta}. \tag{44}$$

In terms of the depth-like quantity $h=\eta-Z_0$, the equation obtained from equation (39b) becomes rather complicated, and is still a cubic. Instead, we go straight to the *small slope approximation*

$$\frac{\mathrm{d}h}{\mathrm{d}x}=\frac{S_0+\beta\left(\tilde{S}+A_x^{\mathrm{V}}/B-S_0\right)\mathsf{F}^2-\dfrac{\lambda}{8}\dfrac{P}{B}\mathsf{F}^2}{1-\beta\mathsf{F}^2}. \tag{45}$$

Other presentations usually give an equation like (45) for prismatic channels, where $\tilde{S}=S_0$, and where the symbol S_{f} is used for the resistance term. We prefer to keep it explicit – and it is simple anyway. It is interesting that for wide channels, $P \approx B$, all variation with the dependent variable on the right of the differential equations is in $\mathsf{F}^2=Q^2B/gA^3$.

The expressions here are all valid for non-prismatic channels, using the generalised definition of slope $\tilde{S}$, equation (31).

7. Generalised uniform flow equations for a finite slope

7.1 Generalised Chézy-Weisbach and Gauckler-Manning formulae

Further simplification of equation (45) for uniform flow on a constant slope S, such that $\mathrm{d}h/\mathrm{d}x=0$ and $\tilde{S}=S_0=S$ yields the generalised Weisbach and Chézy formulae for steady flow on a finite slope, now with the factor $1/\left(1+S^2\right)$ and with wetted perimeter P_n measured around a planar section normal to the mean local bed slope:

$$Q = \frac{1}{1+S^2}\sqrt{\frac{8g}{\lambda}\frac{A^3}{P_n}S} \tag{46a}$$

$$= \frac{C}{1+S^2}\sqrt{\frac{A^3}{P_n}S}, \tag{46b}$$

where A is the cross-sectional area as measured conventionally over a vertical section (figure 3). Using equation (25) gives the generalised Gauckler-Manning formula

$$Q=\frac{1}{n}\frac{1}{1+S^2}\frac{A^{5/3}}{P_n^{2/3}}\sqrt{S}, \tag{47}$$

in *SI* units. The steep slope corrections will usually only be important on spillways and chutes, but there is a certain cultural value in knowing that the general expressions exist.

7.2 Formulae for trapezoidal sections

As spillways and chutes are likely to be of a simple cross-section, here we present the modified Weisbach and Gauckler-Manning formulae for a trapezoidal section of bottom width W, batter slopes $\gamma : 1$ (H:V), and depth h, substituting $A = h(W + \gamma h)$, and P_n from equation (18) $P_n = W + 2h\sqrt{\gamma^2 + \cos^2\theta}$, where θ is the constant slope angle such that $S = \tan\theta$, and using $\cos^2\theta = 1/\left(1 + S^2\right)$. From equation (46a) we obtain the finite-slope Weisbach formula for a trapezoidal section, giving $Q(h)$:

$$Q = \sqrt{\frac{8g}{\lambda}} \frac{\sqrt{S}}{1 + S^2} \frac{(h(W + \gamma h))^{3/2}}{\left(W + 2h\sqrt{\gamma^2 + 1/(1 + S^2)}\right)^{1/2}}. \tag{48}$$

Similarly, the Gauckler-Manning formula (47) gives

$$Q = \frac{1}{n} \frac{\sqrt{S}}{1 + S^2} \frac{(h(W + \gamma h))^{5/3}}{\left(W + 2h\sqrt{\gamma^2 + 1/(1 + S^2)}\right)^{2/3}}. \tag{49}$$

As n is always specified in units of $\mathrm{m}^{-1/3}\,\mathrm{s}$, if British units are used elsewhere in equation (47), the well-known factor of $0.3048^{-1/3} = 1.486$ must be inserted into the numerator. A similar correction would be necessary for Chézy's C in equation (46b). No such modification is necessary for the Weisbach form, equation (46a), as λ is dimensionless and the equation is dimensionally consistent, whichever units are used. Another more theoretical than a practical advantage of the Weisbach formula is that gravitational acceleration g appears explicitly, and in the unlikely case that it were necessary, one could use the value appropriate to latitude, varying between $9.78\ \mathrm{ms}^{-2}$ at the equator and $9.83\ \mathrm{ms}^{-2}$ at the poles. One has to say, more in keeping with the accuracy with which we know the resistance, using a value of $g = 10\ \mathrm{ms}^{-2}$ would not be inappropriate! In both the Chézy and Gauckler-Manning formulae, g is implicitly contained in the resistance coefficients C and n.

7.3 Numerical solution for uniform depth

For years the author has taught a numerical method to students which gives a simple way of solving for uniform depth, which until now has only been used for the small slope approximation. It is to observe that channel width and wetted perimeter are relatively slowly varying functions of depth h, and so if we divide both sides of equation (46a) by $h^{3/2}$ and re-write, showing quantities which are functions of h:

$$h = \left(Q\left(1 + S^2\right)\sqrt{\frac{\lambda}{8gS}}\right)^{2/3} \frac{P_n^{1/3}(h)}{A(h)/h}, \tag{50}$$

which is in the form of h as a function of h – but which is a slowly varying function, such that if we put in an approximate value of h on the right, evaluate it to give an updated value of h on the left which we again substitute on the right and repeat, the process should quickly converge. Similarly, if we use the Gauckler-Manning form we obtain the computational scheme

$$h = \left(\frac{nQ\left(1 + S^2\right)}{\sqrt{S}}\right)^{3/5} \frac{P_n^{2/5}(h)}{A(h)/h}. \tag{51}$$

An initial rough estimate can be made by assuming that the only contribution to area and perimeter is that above the flat bottom and neglecting finite slope effects

$$Q \approx \frac{1}{n} W h^{5/3} \quad \text{so that} \quad h \approx \left(\frac{nQ}{W}\right)^{3/5}. \tag{52}$$

As an example we take a value of $n = 0.012$ (concrete), $W = 10\,\mathrm{m}$, $\gamma = 0.5$, $S = 0.5$, $h = 0.5\,\mathrm{m}$, not untypical for a chute, equation (49) giving $Q = 145.0\ \mathrm{m^3 s^{-1}}$. Now to work in reverse to solve for normal depth, we take that value of discharge and use equation (52), $h \approx 0.350\,\mathrm{m}$. The first evaluation of equation (51) gives $h = 0.498\,\mathrm{m}$, the second gives the correct value of $h = 0.500\,\mathrm{m}$. The method works for many section shapes, including circular sewers. Here it worked particularly quickly because the walls are steep and γ is small so that width is almost constant.

A very different example is that of a canal with gently-sloping banks and longitudinal slope, $\gamma = 2$, $S = 0.0005$, $n = 0.025$, $W = 10\,\mathrm{m}$, $h = 2\,\mathrm{m}$, for which we obtain $Q = 32.495\ \mathrm{m}^3\mathrm{s}^{-1}$. Using this in the initial estimate, equation (52), $h = 0.222\,\mathrm{m}$ (not very good, ignoring the resistive effects of the wide banks – we might have estimated it better ourselves), and then iterating with equation (51) we obtain the sequence of approximations 2.157, 1.984, 2.002, $2.000\,\mathrm{m}$.

8. Conclusions

The long wave equations for a straight slowly-varying channel have been derived using the integral mass and momentum conservation equations. For flow on finite slopes it has been necessary to correct the traditional assumption of hydrostatic pressure. Pairs of equations have been presented for three combinations of variables: cross-sectional area and discharge (A, Q); surface elevation and discharge (η, Q); and finally using a depth-like quantity h, which is surface elevation relative to an arbitrary axis possibly associated with the bed, also plus discharge, (h, Q). Gradually varied flow equations for steady flow with constant discharge have been presented, which are ordinary differential equations in terms of each of A, η, and h. Finally generalised Chézy-Weisbach and Gauckler-Manning equations for steep slopes have been presented. Of course, there are few open channel problems for which effects of steep slope are important. Those exceptions, for which flow will almost always be super-critical, include flows on spillways and chutes, for which it may be necessary to go to the higher approximation of Boussinesq equations anyway. Nevertheless, the knowledge of the generalised equations and their derivation may be of some interest and possible use.

References

ASCE Task Force on Friction Factors in Open Channels (1963), Friction factors in open channels, *J. Hydraulics Div. ASCE* **89**(HY2), 97–143.

Boussinesq, J. V. (1877), Essai sur la théorie des eaux courantes, *Mémoires présentés par divers savants à l'Académie des Sciences, Paris* **23**, 1–680. **URL:** *http://gallica.bnf.fr/ark:/12148/bpt6k56673076*

Chow, V. T. (1959), *Open-Channel Hydraulics*, McGraw-Hill, New York.

Cunge, J. A., Holly, F. M. & Verwey, A. (1980), *Practical Aspects of Computational River Hydraulics*, Pitman, London.

Darvishi, E., Fenton, J. D. & Kouchakzadeh, S. (2014), Boussinesq-type momentum equations valid for steep slopes. Submitted for publication.

Dooge, J. C. I. (1987), Historical development of concepts in open channel flow, in G. Garbrecht, ed., *Hydraulics and Hydraulic Research: A Historical Review*, Balkema, Rotterdam, pp. 205–230.

Dressler, R. F. (1978), New nonlinear shallow flow equations with curvature, *J. Hydraulic Research* **16**, 205–222.

Fenton, J. D. (2005), On the energy and momentum principles in hydraulics, in *Proc. 31st Congress IAHR, Seoul*, pp. 625–636.

Fenton, J. D. (2010), The long wave equations, Technical report, Alternative Hydraulics Paper 1. **URL:** *http://johndfenton.com/Papers/01-The-long-wave-equations.pdf*

Henderson, F. M. (1966), *Open Channel Flow*, Macmillan, New York.

Jaeger, C. (1956), *Engineering Fluid Mechanics*, Blackie, London.

Keulegan, G. H. (1942), Equation of motion for the steady mean flow of water in open channels, *J. Res. Nat. Bureau Standards* **29**, 97–111.

Keulegan, G. H. & Patterson, G. W. (1943), Effect of turbulence and channel slope on translation waves, *J. Res. Nat. Bureau Standards* **30**, 461–512.

Lai, C. (1986), Numerical modeling of unsteady open-channel flow, in B. Yen, ed., *Advances in Hydroscience*, Vol. 14, Academic.

Lyn, D. A. & Altinakar, M. (2002), St. Venant–Exner equations for near-critical and transcritical flows, *J. Hydraulic Engineering* **128**(6), 579–587.

Milne-Thomson, L. M. (1968), *Theoretical Hydrodynamics*, fifth edn, Macmillan, London.

Strelkoff, T. (1969), One-dimensional equations of open-channel flow, *J. Hyd. Div. ASCE* **95**(HY3), 861–876.

White, F. M. (2009), *Fluid Mechanics*, seventh edn, McGraw-Hill, New York.

Yen, B. C. (1973), Open-channel flow equations revisited, *J. Engng Mech. Div. ASCE* **99**(EM5), 979–1009.

Yen, B. C. (2002), Open channel flow resistance, *J. Hydraulic Engineering* **128**(1), 20–39.