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A note on the role of Area/Perimeter in calculating open channel flow resistance

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Abstract

This note considers the physical significance and role of the quantity A/P, where A is cross-sectional area and P is wetted perimeter, in determining resistance coefficients for flow in open channels. Keulegan's 1938 paper providing the theoretical justification for the use of A/P is found to contain a mathematical step that is fundamentally misleading. However there is a lot of experimental evidence that A/P is indeed important, including the success of the Gauckler-Manning equation and Strickler's experimental validation. To try to understand that success, we consider a simple model of turbulent flow in an open channel, allowing for the depression of the maximum longitudinal velocity below the surface. The hypothesis follows that the fundamental resistance length scale is the mean distance of the region of maximum velocity from the bed, as most points on the bed experience velocity sweeps of that magnitude. Next experimental results for that distance in rectangular channels are considered. A remarkable and probably fortuitous result is found for such channels that A/P is almost exactly equal to the mean distance of the region of maximum velocity from the bed. This provides some understanding as to why A/P has been useful also for more general channels – it is a simpler quantity that merely mimics the magnitude and behaviour of the more fundamental one.

Revision History	
11.09.2011	Initial version
..2011	Addition of section on field measurements

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Introduction

Throughout open channel hydraulics the quantity A/P, where A is the cross-sectional area and P is the wetted perimeter, is widely used as the effective transverse length scale for calculating resistance. In English, it is traditionally called the hydraulic radius, which is a term that we find confusing. In the most important part of resistance calculations, A/P is unambiguously the most important cross-sectional dimension. The component of gravitational force on an element of the stream is proportional to A, while the resistance of the boundary is proportional to P. Equating the two for steady uniform flow, and postulating that boundary resistance is proportional to U^2, Chézy found

$$U = C\sqrt{\frac{A}{P}S},$$

where U is the mean velocity, C is a dimensional coefficient with units of $\mathrm{L}^{1/2}\mathrm{T}^{-1}$, and S is the slope. However, in determining the coefficient C thereafter, it is by no means obvious that the length scale A/P is the best candidate. In the case of the Gauckler-Manning formula it is used empirically as $C = (A/P)^{1/6}/n$, and as found by Strickler and others, it works well.

If instead of Gauckler-Manning, we use the dimensionless Weisbach coefficient λ (the symbol f is often used), as recommended by the ASCE Task Force on Friction Factors in Open Channels (1963) as having a number of advantages, we write

$$\frac{\tau}{\rho} = \frac{\lambda}{8} U^2,$$

τ is the mean stress on the boundary, and ρ is the fluid density. and the Chézy-Weisbach form of the equation for steady uniform flow is

$$U = \sqrt{\frac{8g}{\lambda}\frac{A}{P}S}. \tag{1}$$

The problem is to determine how λ varies with channel and flow parameters, including possible dependence on A/P.

Keulegan's justification for the use of A/P

Keulegan (1938) used a more general fluid mechanics approach to resistance in open channels, based on the work of Prandtl, Kármán and Nikuradse. He obtained formulae for the dimensionless quantity U/u_*, the ratio of the mean flow velocity divided by the shear velocity $u_* = \sqrt{\tau/\rho}$, so that his quantity is $\sqrt{8/\lambda}$, and in this form is a conductance rather than resistance.

What Keulegan did first was to take the logarithmic velocity distribution laws found for circular pipes, both smooth and rough, and apply them to different channel cross-sections, ignoring any effect of cross-section on those velocity distributions, but integrating them from each point on the boundary precisely and mechanically over various geometries to obtain the mean velocity U and hence U/u_*. The geometries included circular pipes (where of course the logarithmic distribution with its finite gradient cannot be exactly right at the centre), between parallel flat plates, semi-circular channels, and then more usefully over trapezoidal channels, where he was forced to consider two cases, where bisectors of the internal angles meet above and below the water surface, and other shapes including semi-circular. There was no specific mathematical attention paid to channel flow phenomena such as the depression of the velocity maximum below the free surface. The whole approach seems unbalanced, with its rigid application of special results to a more general situation, but with precise attention given to the original formulae and the geometry, and not to the actual nature of the flow.

Finally Keulegan obtained the results for channels of general shape:

$$\sqrt{\frac{8}{\lambda}} = \begin{cases} \frac{1}{\kappa}\ln\dfrac{hu_*}{\nu} + \ldots, & \text{for smooth walls} \\ \frac{1}{\kappa}\ln\dfrac{h}{k} + \ldots, & \text{for rough walls} \end{cases}, \tag{2}$$

where κ is the Kármán constant, ν is kinematic viscosity, and k is the boundary roughness size, and importantly, h is the maximum depth of the channel. Terms that are not important for the present discussion have not been shown here.

At this stage, the quantity $R = A/P$ had actually not appeared in the analysis to determine λ. Keulegan may have thought that as it appeared in the Chézy-Weisbach formula (1) along with λ, that it made sense to use it. He re-wrote the expressions in (2), writing $h = R \times h/R$ and separating the logarithmic function into two such functions, as

$$\sqrt{\frac{8}{\lambda}} = \begin{cases} \frac{1}{\kappa}\ln\dfrac{Ru_*}{\nu} + \frac{1}{\kappa}\ln\dfrac{h}{R} + \ldots, & \text{for smooth walls} \\ \frac{1}{\kappa}\ln\dfrac{R}{k} + \frac{1}{\kappa}\ln\dfrac{h}{R} + \ldots, & \text{for rough walls} \end{cases} . \tag{3}$$

Then, for some rectangular and semi-circular channels he calculated the magnitude of the additional term containing $\ln(h/R)$, assigned a constant numerical value to it for *all* channels, and concluded that the resistance parameter $\sqrt{8/\lambda}$ was a function of R as shown in the first term in top and bottom of equation (3)! The procedure seems quite faulty, and has not justified the claim that the transverse length scale to be used in calculations of λ is R.

For fully rough flow, Keulegan's result can be shown to be (ASCE Task Force on Friction Factors in Open Channels 1963, p105)

$$\frac{1}{\sqrt{\lambda}} = -2.03\log_{10}\frac{k_s/R}{12.27}, \tag{4}$$

(complete with four-figure accuracy!) where k_s is the equivalent sand roughness size. Keulegan went on to verify the formula with data from Bazin for rectangular channels, and computed the k_s for several rough surface materials. On p727 he wrote "In this connection it is interesting to note that the equivalent sand roughness of the fine gravel, 1 to $2\,\mathrm{cm}$ in diameter is $0.95\,\mathrm{cm}$, and that of the coarse gravel, 3 to $4\,\mathrm{cm}$ in diameter, is $2.9\,\mathrm{cm}$." The present author has repeated the calculation based on Keulegan's table 3 but using his original equation (2) in terms of maximum depth h instead of the suspect equation (3) in terms of R. We obtained $k_s = 1.3\,\mathrm{cm}$ for the nominally 1 to $2\,\mathrm{cm}$ fine gravel and $k_s = 4.0\,\mathrm{cm}$ for the nominally 3 to $4\,\mathrm{cm}$ coarse gravel, which seem satisfactory. Generally, for the other surfaces the k_s we obtained using h were about $1/3$ greater than Keulegan's values using R.

Model of channel flow

Since Keulegan's work there have been many studies, as itemised in ASCE (1963), obtaining results for λ with an equation such as (4) in terms of $k_s/(A/P)$, some with a viscous term where the Reynolds number was written also with A/P as the length scale. The results were apparently satisfactory, but then the Keulegan result was the dominant paradigm, and going against such paradigms has never been a strong behavioural trait of hydraulic engineers.

Our problem is then, in view of the extremely suspect manner in which Keulegan introduced A/P theoretically, but the apparently satisfactory use of it experimentally, is there an *a priori* physical justification for A/P being the transverse length scale in determining flow resistance? Here, to try to answer that question, we set up a simple model of fully rough flow in a channel of arbitrary section.

Relatively modern ideas about the processes of turbulence, consisting of finite volumes of fluid moving about the flow with some measure of coherence, are described well by Detert (2008, p110). He notes that conditions on the bed are dominated by faster fluid from the outer flow arriving as a sweep event, as anticipated by Sutherland (1967) and discovered by the Stanford school in the same year. Detert notes that this compares with flow over smooth boundaries where "hairpin-like vortices or derivations thereof originate at the bed and then grow and merge with bed distance, leading to the appearance of low and high speed regions that dominate the outer flow".

To examine the effects of such coherent sweeps of faster fluid onto the bed, consider the channel cross-section and flow as shown in figure 1, including isovels, lines on which longitudinal velocity is constant, shown by dashed lines. It is typical of channels that are not narrow that there are secondary circulations such that near the middle of the channel there is a downward component and the region of largest velocity, is actually pushed some way down into the flow. It can be easily imagined intuitively that conditions on the perimeter are indeed determined by larger bodies of fluid coming from faster-flowing regions. Figure 1 also shows with arrows, for a number of points on the boundary, where the fastest-likely fluid for each might come from, and how far it travels. If we were to model the flow at each point

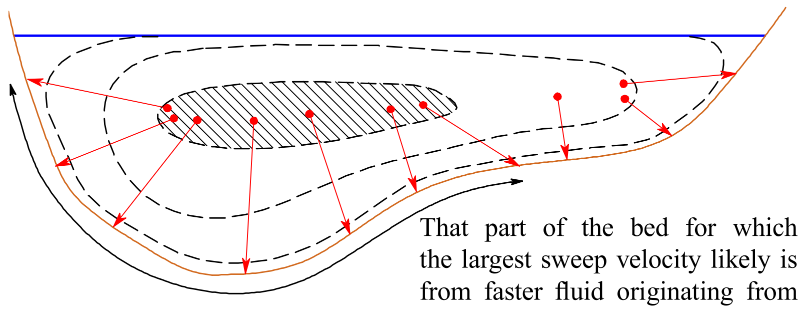

Figure 1. Cross-section of flow showing isovels and, for a number of points on the bed, where the fastest-likely fluid comes from and how far it travels. It will be asserted below that the mean length of these red arrows is the effective length scale for resistance calculations

on the boundary, rather than as Keulegan, integrating a velocity profile right through to the surface, we might say that at each point on the boundary the flow is roughly equivalent to flow in a wide channel (with no depression of the velocity maximum) with a depth equal to the length of the local fluid path shown on the figure. Slower-moving fluid further from the boundary point, above the high-velocity core, is less-likely to arrive at the bed, and in any case might be experienced by the bed as if it came from lower down in the flow. The figure also attempts to show that, even for a channel as irregular as that drawn, over a large fraction of the boundary, fluid with velocity close to the maximum can arrive at the bed. If this model is correct, typical length scales as shown by the red arrows, that we interpret as equivalent depths, are somewhat smaller than the overall mean depth of flow.

This is a highly approximate model, but at least it is in the spirit of modelling, that it is simple and transparent – and so far has not been obscured by mathematical detail. With this simplicity, it is also relatively simple to improve it – or even to disprove it. Now, we attempt to quantify the mean effective length scale of the flow. We suggest that it is some sort of mean value of the hypothetical paths shown in the figure. In reality it will be some weighted mean path length, possibly involving the local largest possible velocity, possibly squared, to give the local resistance stress. Above we have suggested that in fact a substantial part of the bed can experience velocities close to the maximum in the flow, and as a first approximation, we will just obtain the mean of the distances from the region of highest velocity to the bed. To do this properly would require some detailed knowledge of the flow field, and an integration around the boundary. Thus we might introduce $\tilde{R}$ as the spatial mean distance of the velocity maxima from the bed and write

$$\tilde{R} = \frac{1}{P}\int_0^P \sqrt{(Y-y)^2 + (Z-z)^2}\,\mathrm{d}P, \tag{5}$$

integrating around the wetted perimeter with co-ordinates (y, z) and where points on the locus curve of maximum velocity are (Y, Z) which would be those closest to the boundary point, thus $Y = Y(y, z)$ and $Z = Z(y, z)$. The hypothesis of this work is that it is $\tilde{R}$ that is the fundamental channel dimension in determining the resistance.

Comparison with experiment

Now we consider experimental data for the vertical position of the locus of velocity maxima in rectangular channels from Yang, Tan & Lim (2004). They presented the formula

$$\frac{Y}{h} \approx \left(1 + 1.3\exp\left(-\frac{z}{h}\right)\right)^{-1}, \tag{6}$$

where Y is the height above the bed of the velocity maximum; h is the depth; and z is the transverse co-ordinate with origin at the side of the channel such that $0 \leqslant z \leqslant B/2$ where B is the overall breadth of the channel. As an approximation to equation (5) for $\tilde{R}$ we will neglect the contributions from the sides, which are more complicated and for wide channels will be relatively small. Hence we simply integrate horizontally across the bottom of the channel such that $y = 0$ and $Z = z$. Integrating across half the channel:

$$\tilde{Y} \approx \frac{1}{B/2} \int_0^{B/2} Y \, \mathrm{d}z,$$

and substituting equation (6) gives

$$\frac{\tilde{Y}}{h} = \frac{2}{B/h} \ln \left(\frac{\mathrm{e}^{B/h/2} + 1.3}{1 + 1.3} \right), \tag{7}$$

in terms of the aspect ratio B/h.

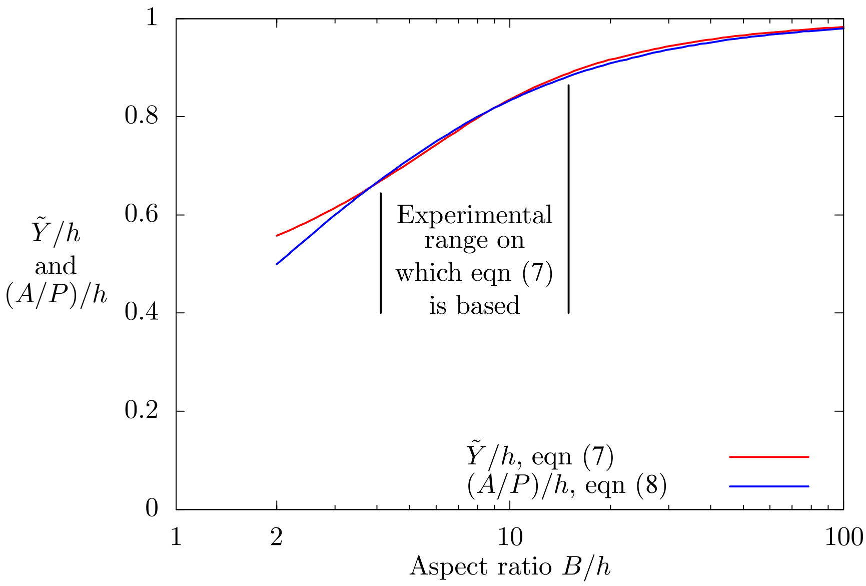

Figure 2. Rectangular channels: the dependence on aspect ratio B/h of the dimensionless mean elevation of the velocity maximum $\tilde{Y}/h$ across the channel and the effective depth $(A/P)/h$

For comparison we calculate the ratio of area to perimeter A/P, giving the dimensionless result also in terms of the aspect ratio B/h:

$$\frac{A/P}{h} = \frac{B/h}{B/h + 2}. \tag{8}$$

Both this and the expression for $\tilde{R}/h$, equation (7) are plotted in figure 2. The two coincide closely over a wide range of aspect ratios, rather more than we expect for such apparently different quantities. It may be that we have discovered something close to exact agreement, but it is extremely unlikely. In fact, we have plotted the comparison far outside the experimental range of channels on which equation (7) was based (figure 4 of Yang et al. 2004), $4.1 \leqslant B/h \leqslant 15$. For small B/h this was done to show how the two functions diverge, as expected. For large B/h it was done to show the strange agreement between the two functions, which was unexpected.

For this case of rectangular channels the agreement between $\tilde{R}$ and the quantity A/P, plus the fact that a large number of experiments have shown that A/P is an important length scale, has provided circumstantial evidence supporting the hypothesis that the most fundamental length scale in determining resistance is $\tilde{R}$, the mean distance of the locus of velocity maxima from the points on the bed. We have been unable to develop a plausible similar hypothesis for the physical significance of A/P in determining the resistance coefficient. We conclude that A/P merely mimics the behaviour of $\tilde{R}$, but that this is fortunate as it is a rather simpler quantity to calculate in practice, usually with no knowledge of the flow field.

For non-rectangular channels we have presented no results. The hypothesis that it is $\tilde{R}$ that is fundamentally important still applies, however, as it was developed for general sections. and as it already appears in the Chézy-Weisbach and Gauckler-Manning equations in expressing the ratio of gravity to resistance forces.

What conclusions might we draw from this? We have suggested that that mean distance of the velocity maximum from the bed, which is rather less than the mean depth for channels that are not very wide, might be the appropriate transverse length scale to use for frictional resistance. We have noted that there are many studies that have successfully used the rather simpler quantity A/P as that scale, including Strickler's verification of the Gauckler-Manning formula. We have shown that for rectangular channels the quantity A/P approximates quite closely the mean elevation of the velocity maximum across the channel. There is nothing contradictory in these results. A tentative conclusion is possible, that the fundamental transverse length scale for determining resistance is the mean distance from the bed of the velocity maxima, but that that quantity, non-trivial to measure or calculate, is modelled sufficiently accurately by the A/P.

Field measurements

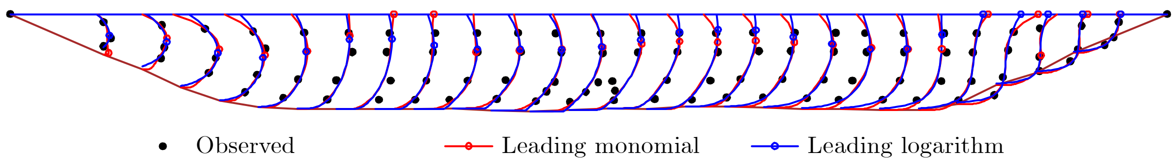

Figure 3. Velocity measurements from a streamgauging exercise and results of calculations: streamwise velocities actually plotted transversely

Any hubris from the probably-fortuitous agreement observed above is easily dispelled by results from a field measurement exercise in which the author participated. A set of hydrographic measurements was made on 8 December 1998 on an irrigation canal above the 95km regulator on the Waranga Western Main Channel near Rochester, Victoria, Australia, by two professional hydrographers, a consulting hydrographer, and the author as consulting engineer. The canal was straight for at least $1\,\mathrm{km}$ upstream, about $30\,\mathrm{m}$ wide, and $2.5\,\mathrm{m}$ deep. The exercise was more careful than usual, and rather more velocity measurements were made to obtain a detailed picture of the velocity field; generally at points below the surface of 90%, 70%, 40%, and 20% of the total depth there. Results are shown in figure 3, where the longitudinal velocity profiles have actually been plotted transversely. The measurements are shown as black disks. It can be seen that there are quite a few stations where the velocity closest to the surface is less than at the next deeper station, indicated by the 15 1s in the following sequence of 26 numbers: 11111100111100110011100000. We have plotted the results from two global approximations to the velocity field, where horizontal variation was represented by a 10th degree polynomial, with each coefficient being a function which was a series in the vertical variable y (here local such that the bed is at $y = 0$):

$$a_n y^{1/7} + b_{1n} y + b_{2n} y^2, \quad \text{or} \quad a_n \ln y + b_{0n} + b_{1n} y + b_{2n} y^2,$$

where the first function with a leading monomial $y^{1/7}$ clearly goes to zero at $y = 0$ while the second function with a leading logarithmic term, varies like $\ln(y/k_0)$ as $y \to 0$, the boundary roughness k_0 being given in terms of the a_n and b_{0n}. It can be seen from the figure that the two global approximations generally agree quite well, but that there is some disagreement in the location of the computed velocity maxima, shown by the red and blue circles. Generally, one can say that over most of the left 80% of the section, the computed approximations believe that the velocity maximum occurs some way down into the flow, and does not seem to be limited to a central region where it could be explained as being due to secondary flow.

From these results we can see that the position of the velocity maximum is a quantity which is extremely sensitive to measurements and details of the local velocity distribution – a small change and it can go from the surface to nearly the middle of the flow. Accordingly, it is possible to suggest that the position of the velocity maximum itself is not such an important quantity in determining conditions at the bed. What is important is the obvious and well-known evidence that with a simple logarithmic or power law velocity distribution, there is quite high velocity fluid relatively close to the bed, and that apparently in channels there is a tendency for the velocity distribution to turn back on itself, such that we might say, based on the profiles in the figure, that in the entire top 70% of the flow, the velocity is within about 10% of the maximum likely. With this observation, we might say that, instead of the height of the velocity maximum above the bed that is a determinant of conditions on the bed, that a scale of roughly 50% of the depth is important, were we comparing with pipe flows.

In the relatively wide channel illustrated, the quantity $A/P \approx A/B$, the mean depth, and it is clear that in this case it is considerably greater than the height of the maxima above the bed.

Maximum depth	$2.52\,\mathrm{m}$
A/B – mean depth	$1.98\,\mathrm{m}$
A/P	$1.91\,\mathrm{m}$
Calculated mean height of maxima above bed – with leading monomial	$1.38\,\mathrm{m}$
Calculated mean height of maxima above bed – with leading logarithm	$1.44\,\mathrm{m}$

Conclusions

1. The theoretical approach of Keulegan (1938) in attempting to show that the length scale A/P is of fundamental importance in determining the resistance coefficient is fundamentally flawed.
2. However there is a large body of experimental evidence that A/P is indeed important and useful.
3. A simple model of turbulent open channel flow has been proposed here that suggests it is the mean distance $\tilde{R}$ of the fastest-flowing fluid from the bed that is actually the fundamental length scale.
4. An approximate calculation of the mean vertical distance $\tilde{R}$ for rectangular channels shows that it is almost exactly equal to A/P.
5. As calculating the mean distance $\tilde{R}$ or $\tilde{Y}$ involves a detailed knowledge of the flow field that is often not available, it is very fortunate that A/P seems to be a good approximation that can continue to be used in formulae such as equation (4).
6. If the worst comes to the worst, the Gauckler-Manning equation can continue to be used. As an additional piece of whimsy, in view of the above claim that $\tilde{R}$ is more fundamental it could be modified so as to use A/P where it is unambiguously correct, but to use the mean distance $\tilde{R}$ of the fastest-flowing fluid from the bed in the form
$$U = \frac{1}{n}\tilde{R}^{1/6}\sqrt{\frac{A}{P}S}.$$
7. However, the author is in agreement with ASCE (1963) that the Chézy-Weisbach form is much to be preferred, and has a document in preparation, a complete revision of Fenton (2010) examining various formulae for the coefficient λ and extending them to a wider range of relative roughnesses and also to cases where the boundary shear exceeds critical such that there is sediment transport.

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