Accuracy of Muskingum-Cunge flood routing

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Abstract

The Muskingum-Cunge approach for computing the propagation of long waves is considered. The corresponding linearised differential equation is obtained and it is shown that the essential approximation required by the method is that diffusion be small. The theoretical propagation behaviour of solutions of the equation is obtained, as well as that of solutions of the linearised advection-diffusion equation. The results show how and when the mathematical diffusion of Muskingum-Cunge methods makes them less accurate for streams with gentle slopes and more rapid flow variation, when they under-predict downstream flood levels.

A subsidiary result is that the numerical solution of the full long wave equations for comparison was done with a simple forwards-time-centred-space scheme. Such a scheme was named the "Unstable Method" by Liggett & Cunge (1975, p111). The author believes their stability analysis to be incorrect (a note is in preparation), and that the method is simple and stable enough to be the method recommended for practical flood routing problems.

Revision History	
08.04.2010	Initial version
04.06.2010	Equations (18) and (19) for N changed so as to be in terms of discharge per unit width q_0 rather than depth h_0.
02.03.2011	Changed definition of V to include $Q(x_0, t)$ in equation (3) as suggested by Fatemeh Soroush

Reference: Fenton, J. D. (2010) Accuracy of Muskingum-Cunge flood routing, Technical report, Alternative Hydraulics Paper 3, http://johndfenton.com/Papers/03-Accuracy-of-Muskingum-Cunge-flood-routing.pdf

1. Introduction

Recently Price (2009) published a paper updating some of his previous work (*e.g.* Price1985), which uses a variant of the Muskingum-Cunge approximation to compute the propagation of flood waves. The author is of the view that the general validity of such Muskingum-Cunge routing has not yet been sufficiently critically examined, although it is known that the method shows significant extra diffusion in some problems. As Cunge wrote in his seminal paper (Cunge 1969, p227): "The attenuation obtained with this equation is not quite the same as the real thing ...".

This note examines the accuracy of the Muskingum-Cunge approximation to determine just when and where that attenuation is not the same. Initially the solutions of two flood propagation problems are presented. In one, for a steep river slope, the approximation is found to be accurate. However, for the same flood and channel, but on a gentle slope, it under-predicts the flood peak, providing some cause for concern.

The paper then goes on to obtain the theoretical propagation behaviour of solutions of, first, the linearised advection-diffusion equation as an approximation to the full equations, and second, the linearised differential equation corresponding to the Muskingum-Cunge approximation. It shows how and when the mathematical diffusion of the latter makes it less accurate for streams with gentle slopes and more rapid flow variation.

2. Two flood propagation problems

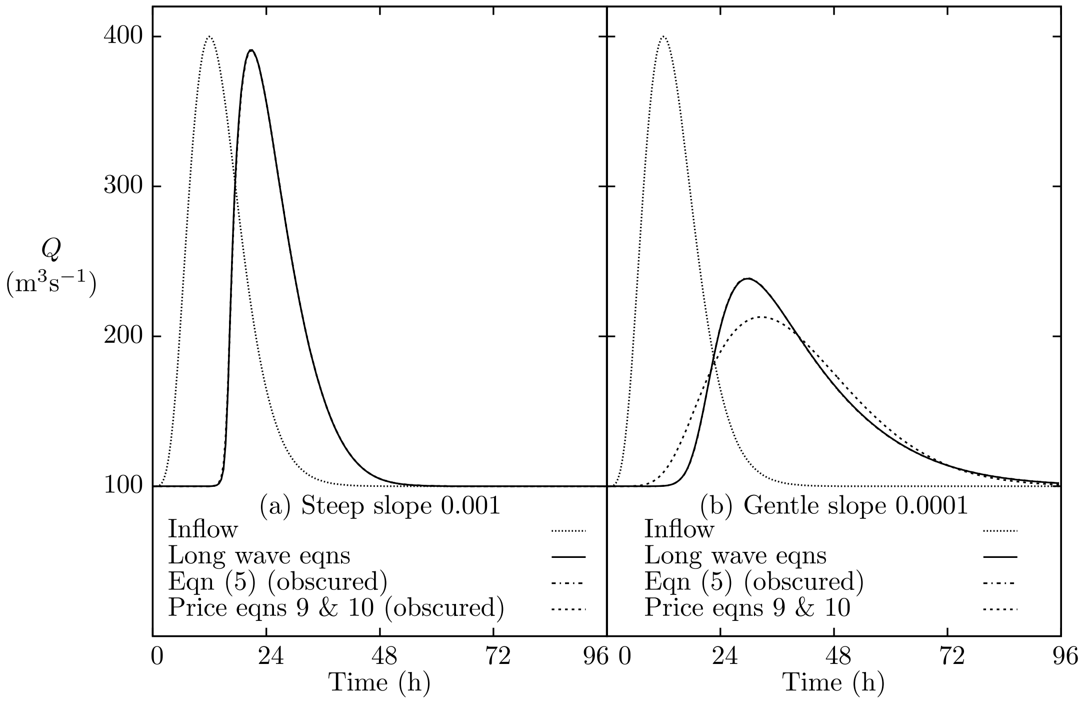

Figure 1. Inflow and computed outflow hydrographs for two model rivers with (a) a steep slope and (b) a gentle slope

Consider the two cases in figure 2 computed for the hypothetical trapezoidal channel in Price (2009), with bottom width $40\,\mathrm{m}$, side slopes H:V 0.5, Manning's $n = 0.035$, and with an inflow hydrograph given by Price's equation (37) with parameters mostly as used in the original paper, with a base flow of $100\,\mathrm{m}^3\mathrm{s}^{-1}$, time to peak $24\,\mathrm{h}$, but with a lower peak flow of $400\,\mathrm{m}^3\mathrm{s}^{-1}$, where no flood-plain extension of the cross-section was considered, and with no distributed inflow. Figure 2 shows the inflow and outflow hydrographs of a $100\,\mathrm{km}$ reach obtained from numerical solutions of the nonlinear Muskingum-Cunge equations (9) and (10) of Price (2009), compared with computational solutions of the long wave equations plus solutions of the volume routing equation (5), a slow-change-slow-flow approximation to the full equations. Part (a) of the figure is for a steep bed slope of 0.001, and the Muskingum-Cunge-Price approach works well. It is clear that diffusion is small. For a gentler slope of 0.0001 part (b) shows that diffusion is more important, and there is significant extra diffusion in the Muskingum-Cunge-Price results. Worse cases could be devised by considering more rapid time variation and to a limited extent, a gentler slope.

3. Equations and computational schemes used

The computational methods used for the simulations were:

1. As already noted, numerical solution of equations (9) and (10) of Price (2009).

2. Explicit finite-difference method for the long wave equations

   Consider the equations with cross-sectional area A and discharge Q as dependent variables:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=i, \tag{1}$$

$$\frac{\partial Q}{\partial t}+\left(\frac{gA}{B}-\beta\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}=gA\left(\tilde{S}-\frac{Q^2}{K^2}\right), \tag{2}$$

   where i is inflow per unit length, B is top width, β is a Boussinesq momentum coefficient, $\tilde{S}$ is mean downstream bed slope at a section (§5, Fenton 2010) and K is the conveyance; lateral inflow momentum has been neglected. If at some point in x and t, the spatial derivatives are approximated by centre-difference expressions such as $\partial A/\partial x \approx (A(x+\delta,t)-A(x-\delta,t))/2\delta$, and the time derivatives by a forward difference approximation such as $\partial A/\partial t \approx (A(x,t+\Delta)-A(x,t))/\Delta$, then the two equations give explicit formulae for each of $A(x,t+\Delta)$ and $Q(x,t+\Delta)$. The method is simply programmed. The only disadvantage is that it has a finite stability criterion on the time step. Liggett & Cunge (1975, p111) called it the "Unstable Method", but the present author believes their stability analysis to be incorrect, and in practice the limit on time step is not a problem. Each case in figure 2 was simply computed using this scheme, admittedly with much smaller steps than the Preissmann scheme or the Muskingum-Cunge scheme, however each problem required less than a second of computer time for solution. This can be compared favourably with implicit methods with their confusing complexity, and with the Muskingum-Cunge method with its potential inaccuracy.

3. Use of upstream volume and the slow-change/slow-flow approximation.

   This approach (Appendix B.2.3, Fenton & Keller 2001) is described here, as it provides a simple method for a single equation in a single variable, which overcomes one of the problems of other methods such as those of Price (1985, p137; 2009), which require complicated and approximate scaling arguments to give an equation in a single variable. As the method requires few significant approximations, it will subsequently be used for a theoretical examination of the Muskingum-Cunge method which is the main purpose of this article.

   The method makes use of the concept of upstream volume $V(x,t)$, the total volume of water contained in a stream between the upstream boundary x_0 and the general point x. The time rate of change of the volume is equal to the rate at which water is flowing into the system at its upstream boundary plus the contribution from lateral inflow i per unit length minus the amount by which water volume is passing the point, thereby becoming no longer "upstream", which is the discharge Q. Hence we have

$$\frac{\partial V}{\partial t}=Q(x_0,t)+\int_{x_0}^{x} i(x')\,\mathrm{d}x'-Q, \tag{3}$$

   where x' is a dummy variable of integration and the inflow integral extends from x_0 the upstream boundary to some arbitrary x. The definition here includes a correction pointed out by Fatemeh Soroush (2011, Personal Communication), such that the inflow at the upstream end $Q(x_0,t)$ has been added, which makes for a more physically-significant and consistent definition. By simple calculus, the spatial derivative of the upstream volume at any section is the local cross-sectional area A, so that

$$\frac{\partial V}{\partial x}=A. \tag{4}$$

   Solving equation (3) for Q and (4) for A and substituting into the volume conservation equation (1), it is found to be identically satisfied, which in retrospect is not surprising. Substituting A and Q in terms of the derivatives of V from equations (3) and (4) into the momentum equation (2) gives a complicated second-order partial differential equation for V, probably too complicated to be of practical use (see equation 24 of Fenton 2010). However the fact that the two equations (1) and (2) are reduced to a single equation in V without approximation may have some theoretical benefits, such as when the equation is linearised. It is much simpler if we consider here the slow-change and slow-flow approximations. The slow change approximation means that the time derivative term is neglected. This can be quantified if necessary, but here we just note that in flood propagation problems it is a good approximation. Also, both fluid inertia terms in β are neglected, which essentially means that we neglect terms of the order of the Froude number $F^2 = Q^2B/gA^3$, hence, limiting to relatively slow flow. In view of these two approximations. equation (2) becomes, to good approximation for many rivers and canals:

$$\frac{1}{B}\frac{\partial A}{\partial x} = \tilde{S} - \frac{Q^2}{K^2},$$

or re-arranged,

$$Q = K\sqrt{\tilde{S} - \frac{1}{B}\frac{\partial A}{\partial x}}.$$

Now substituting A and Q in terms of the derivatives of V, we obtain the approximate governing equation

$$\frac{\partial V}{\partial t} + K(V_x)\sqrt{\tilde{S} - \frac{1}{B(V_x)}\frac{\partial^2 V}{\partial x^2}} = Q(x_0,t) + \int_{x_0}^{x} i(x',t)\,\mathrm{d}x', \tag{5}$$

where $V_x = \partial V/\partial x$ (the area A) has been used in the arguments of breadth B and conveyance K. This is a single equation in a single unknown, the volume V. In previous papers the author has called it the volume routing equation, although it would probably be more correctly described as the Slow-change/slow-flow momentum equation. No linearisation has been necessary. An upstream inflow boundary condition is easily handled, at the upstream point x_0, equation (3) gives $\partial V/\partial t = 0$, and integrating with respect to t gives $V =$constant, which we can set to zero, as there is no volume upstream of this point. If a downstream control exists, then there is a relation between $\partial V/\partial t$ and $\partial V/\partial x$ there.

Numerical solution of equation (5) can be simply carried out using explicit finite difference methods, trying to maintain a numerical approach as simple as the underlying equation. This has meant that stability criterion are finite, but as the computational cost of the method is trivial, these have not been a problem. Once a numerical solution for V has been obtained, numerical differentiation gives A and Q at any point, from equations (3) and (4). Applications include the study of rapid gate movements in a canal, strictly outside the theoretical restriction to slow changes, by Fenton, Oakes & Aughton (1999), while the problem of starting flow in an irrigation ditch over dry soil was simulated in Barlow, Fenton, Nash & Grayson (2006).

Coming to the point of the present work, the results in figure 2 also show that the method gives surprisingly good agreement with the long wave equations, especially as it has assumed Froude number is small while in figure 2(a) the Froude number of the base flow was not so small at $F \approx 0.3$, $F^2 \approx 0.1$.

4. Equations used for theoretical studies

The justification for including the volume routing equation above and demonstrating its accuracy is that it now provides the basis for comparison of the subsequent Muskingum-Cunge methods. We will linearise the volume routing equation (5) by considering small perturbations about a problem with no inflow $i = 0$, a steady uniform flow of area A_0 and discharge $Q_0 = K(A_0)\sqrt{S_0}$ such that we write $V = A_0x + \phi(x,t)$,where ϕ is a relatively small disturbance volume. The advection-diffusion equation is obtained relatively simply:

$$\frac{\partial \phi}{\partial t} + c_{\mathrm{k}}\frac{\partial \phi}{\partial x} = \nu\frac{\partial^2 \phi}{\partial x^2}, \tag{6}$$

where ϕ is a disturbance, whether flow or cross-sectional area, c_{k} is the kinematic wave speed and ν the diffusion coefficient, each given by

$$c_{\mathrm{k}} = K_0'(A_0)\sqrt{S_0} \quad \text{and} \quad \nu = \frac{K_0}{2B_0\sqrt{S_0}} = \frac{Q_0}{2B_0 S_0}, \tag{7}$$

where K_0 is the conveyance of the undisturbed stream on a slope of S_0 such that the underlying discharge $Q_0 = K_0\sqrt{S_0}$, and A_0 and B_0 are the cross-sectional area and width respectively of the undisturbed stream.

On the other hand, considering the seminal work of Cunge (1969), if one takes his equation (10), the finite difference expression of Muskingum routing, together with his equation ($24'$) for a weighting coefficient, and performs an analysis for consistency by writing the point values as Taylor expansions in x and t, linearising, then in the limit of small step sizes one obtains the differential equation that is actually being approximated:

$$\frac{\partial\phi}{\partial t} + c_{\mathrm{k}}\frac{\partial\phi}{\partial x} + \frac{\nu}{c_{\mathrm{k}}}\frac{\partial^2\phi}{\partial x\partial t} = 0. \tag{8}$$

Similarly, if equation (2) of Price (2009) is linearised about a constant uniform flow, this equation is obtained. Its relationship to the advection-diffusion equation (6) can be established by taking that equation and making the additional assumption that diffusion is small, such that in the second derivative on the right one space derivative is eliminated by making the zero-diffusion (kinematic wave) approximation

$$\frac{\partial\phi}{\partial t} + c_{\mathrm{k}}\frac{\partial\phi}{\partial x} \approx 0.$$

It seems that this low-diffusion approximation in the diffusion term is at the heart of such Muskingum-Cunge methods. Here we now compare solutions to the two formulations, equations (6) and (8), showing when they diverge.

5. Solutions of the advection-diffusion equation

Initially, to provide a point of reference, a traditional approach is adopted, where we obtain solutions to equation (6) by assuming periodic variation in x such that $\phi = \exp\left(\mathrm{i}(kx - \omega t)\right)$ where k is real, i$= \sqrt{-1}$, and the nature of ω determines the time behaviour of solutions. Substituting into the advection-diffusion equation (6) gives $\omega = kc_{\mathrm{k}} - \mathrm{i}k^2\nu$ so that the solution can be written

$$\phi = \exp\left(-\nu k^2 t\right)\exp\left(\mathrm{i}k\left(x - c_{\mathrm{k}}t\right)\right),$$

showing that the wave, which is periodic in x, proceeds with a wave speed c_{k} and is damped at a rate $\exp\left(-\nu k^2 t\right)$.

In rivers, however, usually we do not know the wavelength and it is not fixed anyway, unless we are examining the stability of computations on a fixed domain. A more physically-based fundamental solution is to assume that the input to the system is in the form of a time-periodic input, which could be generalised to a Fourier series in time if necessary. Accordingly, the system response is also periodic in time, and we assume the solution

$$\phi = \exp(\mathrm{i}(\mu x - \sigma t)), \tag{9}$$

where σ is a real quantity which is the frequency of the periodic motion, and the nature of μ determines system behaviour. Substituting into the advection-diffusion equation (6) gives $-\mathrm{i}\sigma + \mathrm{i}c_{\mathrm{k}}\mu + \nu\mu^2 = 0$, with solutions

$$\mu = \frac{\mathrm{i}c_{\mathrm{k}}}{2\nu}\left(-1 + \epsilon\sqrt{1 - 4\mathrm{i}N}\right), \tag{10}$$

where the existence of two solutions is shown by $\epsilon = \pm 1$ corresponding to downstream/upstream propagation respectively, and N is a dimensionless diffusion and frequency number:

$$N = \frac{\nu\sigma}{c_{\mathrm{k}}^2}. \tag{11}$$

If we write $\mu = \mu_{\rm r} + {\rm i}\mu_{\rm i}$, the solution (9) can be written

$$\phi = \exp(-\mu_{\rm i} x)\exp\left({\rm i}\mu_{\rm r}\left(x - \frac{\sigma}{\mu_{\rm r}}t\right)\right), \tag{12}$$

so that $\mu_{\rm i}$ is the decay rate in x, the length of the waves is $2\pi/\mu_{\rm r}$, and the propagation velocity is $\sigma/\mu_{\rm r}$.

Extracting these quantities from equation (10) gives the expression for the two propagation velocities (for $\epsilon = \pm 1$) so that, introducing the symbol $c_{\rm AD} = \sigma/\mu_{\rm r}$ for the wave speed from the advection-diffusion equation, and dividing by the kinematic wave speed $c_{\rm k}$ gives a function just of N:

$$\frac{c_{\rm AD}}{c_{\rm k}} = \frac{\sigma}{\mu_{\rm r} c_{\rm k}} = \frac{\epsilon 2\sqrt{2}N}{\sqrt{\sqrt{1+16N^2}-1}} = \epsilon\left(1 + 2N^2 + O\left(N^4\right)\right), \tag{13}$$

where a power series approximation has been added. As this ratio is not unity in general, we obtain the surprising result that in general disturbances periodic in time do not travel at the kinematic wave speed, but at a speed modified by the effects of diffusion.

The imaginary part of μ from equation (10) appears in equation (12) in the term $\exp(-\mu_{\rm i} x)$, giving the downstream decay rate determining how flood peaks diminish in x:

$$\mu_{\rm i}^{\rm AD} = \frac{c_{\rm k}}{\nu}\left(-\tfrac{1}{2} + \epsilon\tfrac{\sqrt{2}}{4}\sqrt{\sqrt{1+16N^2}+1}\right) = \frac{c_{\rm k}}{\nu}\left(\tfrac{1}{2}(\epsilon - 1) + \epsilon N^2 + O\left(N^4\right)\right). \tag{14}$$

For downstream propagation $\epsilon = +1$, the term is positive and as expected, the solution decays with x. For upstream propagation, $\epsilon = -1$, the solution is negative and the water level decays upstream. This solution, determined by changes downstream, is probably not so important.

6. Solutions of the Muskingum-Cunge approximation

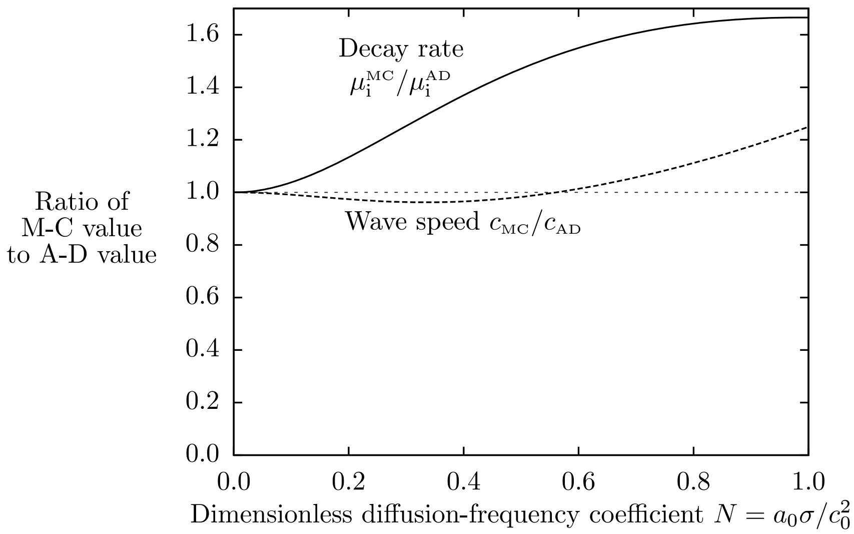

Figure 2. Propagation speeds and downstream decay rates predicted by the Muskingum-Cunge approximation compared with those from the Advection-Diffusion equation

Now the above is repeated, but using equation (8) rather than the advection-diffusion equation (6). A linear equation for μ is obtained, with a single solution, showing propagation in only one direction:

$$\mu = \frac{\sigma c_{\rm k}}{c_{\rm k}^2 - {\rm i}\nu\sigma} = \frac{\sigma}{c_{\rm k}}\left(\frac{1}{1+N^2} + \frac{{\rm i}N}{1+N^2}\right). \tag{15}$$

The real part gives the propagation velocity $c_{\mathrm{MC}} = \sigma/\mu_{\mathrm{r}}$, and again dividing by c_{k}:

$$\frac{c_{\mathrm{MC}}}{c_{\mathrm{k}}} = \frac{\sigma}{c_{\mathrm{k}}\mu_{\mathrm{r}}} = 1 + N^2,$$

which is different from equation (13). The ratio $c_{\mathrm{MC}}/c_{\mathrm{AD}}$ is plotted as a dashed line on figure 2, and for moderate values of N, the ratio is close to unity and the Muskingum-Cunge propagation speed is not far from that of solutions of the advection-diffusion equation.

The other important quantity is the downstream decay rate coefficient μ_{i}, which is given by the imaginary part of equation (15):

$$\mu_{\mathrm{i}}^{\mathrm{MC}} = \frac{\sigma}{c_{\mathrm{k}}}\left(\frac{N}{1+N^2}\right), \tag{16}$$

which is positive and the solution decays downstream as expected. Now we compare this result with the decay rate (14) from the advection-diffusion equation for downstream propagation ($\epsilon = +1$), which gives

$$\frac{\mu_{\mathrm{i}}^{\mathrm{MC}}}{\mu_{\mathrm{i}}^{\mathrm{AD}}} = \frac{N^2/\left(1+N^2\right)}{-\frac{1}{2} + \frac{\sqrt{2}}{4}\sqrt{\sqrt{1+16N^2}+1}}. \tag{17}$$

This is plotted on figure 2 as a solid line. The two solutions now show significant disagreement. Generally the Muskingum-Cunge equation can predict too much decay downstream, by up to 67%, resulting in under-prediction of flood levels, in accordance with the results in figure 2(b).

To obtain the criterion for an error in decay rate of $\leqslant$5%, setting $\mu_{\mathrm{i}}^{\mathrm{MC}}/\mu_{\mathrm{i}}^{\mathrm{AD}} = 1.05$ and solving equation (17) numerically for the corresponding value of N, we find $N \leqslant 0.12$, while for an error of $\leqslant$10%, $N \leqslant 0.17$. Both results can be inferred from figure 2.

The physical significance of these N values can now be explored using Eq. (11). From Eq. (7) for a wide channel $a_0 = q_0/2s_0$, where q_0 is the discharge per unit width Q_0/B_0. For Chézy-Weisbach resistance with uniform flow in a wide channel $c_{\mathrm{k}} = 3U_0/2$ where $U_0 = q_0/h_0$ is the mean velocity and h_0 is the depth, such that $U_0 = \sqrt{(8g/\lambda)\, h_0 S_0}$ where λ is the dimensionless Weisbach resistance coefficient. Solving for h_0 in terms of q_0 gives $h_0 = \left({q_0}^2\lambda/8gS_0\right)^{1/3}$. This gives, with $\sigma = 2\pi/T$, where T is the period of an input wave,

$$N = \frac{q_0}{2S_0}\frac{2\pi}{T}\frac{1}{\left(3q_0/2h_0\right)^2} \approx 0.35\,\frac{q_0^{1/3}\left(\lambda/g\right)^{2/3}}{T\,S_0^{5/3}}. \tag{18}$$

For Gauckler-Manning-Strickler (G-M-S) resistance in metric units, and also for a wide channel, $U_0 = 1/n \times h_0^{2/3}\sqrt{S_0}$, where n is the Manning coefficient, such that $c_0 \approx 5U_0/3$, we obtain

$$N = \frac{q_0}{2S_0}\frac{2\pi}{T}\frac{1}{\left(5q_0/3h_0\right)^2} \approx 1.1\frac{q_0^{1/5}n^{6/5}}{T\,S_0^{8/5}} \tag{19}$$

In both cases it can be seen that, roughly, $N \sim s_0^{-1.6}T^{-1}$, and so for N to be sufficiently small that the Muskingum-Cunge approximation can be used without too much error, small slopes and short period waves might give problems. The criteria are as shown in Table 1.

7. Conclusions

The inherent mathematical diffusion of Muskingum-Cunge methods, with possible under-prediction of downstream water levels, is a potential problem. It is tentatively suggested that the simplest and best way of solving flood propagation problems is to use the full equations with an explicit forwards-time-centred-space scheme which, in the experience of the author, is accurate and surprisingly stable.

Error	N	G-M-S $\dfrac{q_0^{1/5} n^{6/5}}{T\, S_0^{8/5}}$	Chézy-Weisbach $\dfrac{q_0^{1/3}\,(\lambda/g)^{2/3}}{T\, S_0^{5/3}}$
$\leqslant 5\%$	$\leqslant 0.12$	$\lesssim 0.11$	$\lesssim 0.34$
$\leqslant 10\%$	$\leqslant 0.17$	$\lesssim 0.15$	$\lesssim 0.5$

Table 1. Conditions on discharge, resistance, slope, and input period such that the error of the M-C approximation is within certain bounds

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