Alternative Hydraulics Paper 2, 5 April 2010

Calculating resistance to flow in open channels

John D. Fenton

http://johndfenton.com/Alternative-Hydraulics.html
johndfenton@gmail.com

Abstract

The Darcy-Weisbach formulation of flow resistance has advantages over the Gauckler-Manning-Strickler form. It is more fundamental, and research results for it should be able to be used in practice. A formula for the dimensionless resistance coefficient was presented by Yen (1991, equation 30), which bridges between the limits of smooth flow and fully rough flow, to give a general formula for calculating resistance as a function of relative roughness and Reynolds number, applicable for a channel Reynolds number $R > 30\,000$ and relative roughness $\varepsilon < 0.05$. In this paper, the author shows how a closely similar formula was obtained which is in terms of natural logarithms, which are simpler if further operations such as differentiation have to be performed. The formulae are shown to agree with predictions from the Gauckler-Manning-Strickler form, and with a modified form for open channels, have advantages that the equations are simpler and terms are more physically-obvious, as well as using dimensionless coefficients with a greater connectedness to other fluid mechanics studies.

1. Introduction

The ASCE Task Force on Friction Factors in Open Channels (1963) expressed its belief in the general utility of using the Darcy-Weisbach formulation for resistance to flow in open channels, noting that it was more fundamental, and was based on more fundamental research. The Task force noted that the Manning equation could be used for fully rough flow conditions, however it presented a revealing figure for the variation of resistance with Reynolds number, which showed that with Manning's equation there is continual decay of resistance with Reynolds number, even in the limit of large values, so that one could deduce that it is fundamentally flawed.The recommendations of the Task Force have almost entirely been ignored, and the Gauckler-Manning-Strickler formulation continues to dominate, even though, with the exception of the Strickler formula, there are few general research results available, and a picture-book approach seems to prevail.

The reason for the Darcy-Weisbach approach not being adopted may be that, even though the ASCE Task Force presented a number of experimental and analytical results, there was no simple path to follow for general problems.

This document takes the two formulae that the Task Force presented for behaviour in the limits of a smooth boundary and fully rough flow, and obtains a general formula for resistance in open channels which plausibly interpolates between the smooth and rough limits, to give the resistance as a function of relative roughness and Reynolds number. The results are closely similar to the formula given by Yen (1991, equation 30).

2. Experimental results

Consider the expression for the shear force τ on a pipe wall (*e.g.* §6.3 of White 2003)

$$\tau = \frac{\lambda}{8}\rho v^2,$$

where ρ is fluid density, the dimensionless Weisbach coefficient λ is a dimensionless resistance factor, and v is the mean longitudinal velocity in the pipe. The author has used this to obtain the long wave equations for a channel of arbitrary slope in terms of the coefficient λ (Fenton 2010).

The ASCE Task Force on Friction Factors in Open Channels (1963) on p109 presented formulae for the Weisbach friction coefficient in two limits, fully rough and smooth flow. Consider their expressions, where here we use the symbol λ:

$$\text{Fully rough:} \quad \frac{1}{\sqrt{\lambda}} = c \log_{10}\left(\frac{a}{\varepsilon}\right) \tag{1a}$$

$$\text{Smooth:} \quad \frac{1}{\sqrt{\lambda}} = c \log_{10}\left(\frac{R_\text{o}\sqrt{\lambda}}{b}\right) \tag{1b}$$

where a, b, and c are numerical coefficients; $\varepsilon = k_s/\left(A/P\right)$ is the relative roughness, which is the equivalent sand-grain diameter k_s divided by the hydraulic radius A/P, in which A is cross-sectional area and P is wetted perimeter; and where $R_\text{o} = 4Q/P\nu$, where Q is the discharge and ν is the kinematic viscosity, such that R_o is the Reynolds number best suited to a circular pipe. In that case, if U is the mean velocity, and D is the pipe diameter, then $R_\text{o} = 4U\left(\pi D^2/4/(\pi D)\right)/\nu = UD/\nu$. Below, we prefer to use the channel Reynolds number $R = Q/P\nu = UA/P\nu = R_\text{o}/4$, in terms of the discharge per unit of wetted perimeter. Equation (1a) is explicit, giving the fully rough value in terms of relative roughness, however equation (1b) is implicit, as λ appears on both sides of the equation. As presented, neither of them can handle more general situations.

A number of results of investigations were presented by the Task Force, some experimental, others based on analytical integration of logarithmic profiles over typical sections, providing values for the numerical coefficients a, b, and c. It presented the results shown in table 1.

	c	b	a
Bretting	2.0	2.51	14.83
Colebrook	2.0	2.52	14.83
Keulegan, trapezoidal	2.03	3.09	12.27
Keulegan, trapezoidal	2.0	2.27	13.99
Keulegan, wide channel	2.03	3.41	11.09
Keulegan, wide channel	2.0	2.98	12.62
Sayre & Albertson	2.14	7.17	8.89
Thijsse	2.0	2.64	
Thijsse	2.03	3.03	12.20
Tracy & Lester rectangular	2.0	3.86	
Tracy & Lester	2.03	4.37	
Tracy & Lester	2.09	5.58	
Zegzhda	2.0		11.55

Table 1. Results presented by ASCE Task Force on Friction Factors in Open Channels (1963)

3. Explicit formula for smooth channels

Here we seek to unify the results and to provide an explicit formula for λ valid in both smooth and rough limits. Prior to doing this, we obtain an explicit approximation to equation (1b), similar to that of Colebrook (1938, equation 14) for pipes. To do this we consider values of b and c from table 1, and adopt $c = 2.0$ for simplicity and a value of $b = 3.0$, representative of the values in the table. We write the Colebrook-type expression

$$\frac{1}{\sqrt{\lambda}} = c' \log_{10}\left(\frac{R_{\mathrm{o}}}{b'}\right), \tag{2}$$

and solve equation (1b) numerically for λ for five values of $R_{\mathrm{O}} = 10^4$, 10^5, 10^6, 10^7, and 10^8, and use a least-squares method to find the optimal values of $b' = 7.90$ and $c' = 1.79$ in equation (2). Rounding slightly, a quite accurate explicit expression for λ for smooth boundaries is

$$\lambda = \frac{1}{1.8^2}\,\log_{10}^{-2}\left(\frac{R_{\mathrm{o}}}{8.}\right). \tag{3}$$

4. General formula

Now rewriting the two limiting expressions (1a) and (3), using the adopted value of $c = 2.$, the channel Reynolds number $R = Q/P\nu = R_{\mathrm{o}}/4$, and the representative value of $a = 12$ from the experimental and theoretical results of table 1 gives

$$\text{Fully rough: } \lambda = \frac{1}{2.0^2}\log_{10}^{-2}\left(\frac{\varepsilon}{12.}\right) \tag{4a}$$

$$\text{Smooth: } \lambda = \frac{1}{1.8^2}\,\log_{10}^{-2}\left(\frac{2.}{R}\right) = \frac{1}{2.0^2}\log_{10}^{-2}\left(\left(\frac{2.}{R}\right)^{0.9}\right), \tag{4b}$$

where in the last equation we have taken one more step.

Now, to incorporate both limiting behaviours in a single equation, from equations (4) we write the expression which simply combines the limiting forms:

$$\lambda = \frac{1}{2.0^2}\log_{10}^{-2}\left(\frac{\varepsilon}{12.} + \left(\frac{2.}{R}\right)^{0.9}\right). \tag{5}$$

In the limit of a fully rough flow, for large Reynolds number R, this gives equation (4a). In the other limit of a smooth boundary, $\varepsilon = 0$, it gives equation (4b). There have been several presentations of such a simple uniting formula that encompasses such limiting behaviours, as described by Yen (1991, pages 14-15) for pipe flows and for wide channels, due to Churchill, Barr, and Swamee and Jain. Haaland (1983) showed that for pipes a similar equation, but with different exponents of the terms in the argument of the logarithm, gave good results for all intermediate values of roughness and Reynolds number also, agreeing well with the well-known implicit equation of Colebrook and White (equation 4, Colebrook 1938). In fact, the equation developed here is almost exactly equation (30) of Yen (1991):

$$\lambda = \frac{1}{4}\log_{10}^{-2}\left(\frac{\varepsilon}{12} + \frac{1.95}{R^{0.9}}\right), \tag{Yen 1991, eqn 30}$$

with the exception that in our Reynolds number term the numerator $2^{0.9} \approx 1.87$ appears in Yen's expression for wide channels as 1.95. He stated, however, that the value would be smaller for narrower channels, when also the coefficient of 12 in the roughness term would be larger. It is an interesting coincidence that the author did all the calculations leading to (5), requiring certain arbitrary choices of numerical coefficients, before he discovered the existence of the Yen formula, with almost-identical numerical values. The author, however, did not succeed in arriving at the more-careful qualifications of Yen, who stated that it was applicable for $R > 30\,000$ and $\varepsilon < 0.05$, which limitations we also accept.

Now we convert to natural logarithms for simplicity in case further operations such as differentiation have to be carried out, for example in calculating kinematic wave speed, to give the formula here recommended for use in calculating the resistance coefficient:

$$\lambda = \frac{1.33}{\ln^2\left(\frac{\varepsilon}{12.} + \left(\frac{2.}{R}\right)^{0.9}\right)}, \tag{6}$$

in which the term 1.33 is actually $(\ln 10)^2/2.0^2$. In case they are necessary, the derivatives are

$$\frac{\partial\lambda}{\partial\varepsilon} = \frac{-0.222}{\Theta\ln^3\Theta} \quad \text{and} \quad \frac{\partial\lambda}{\partial R} = \frac{1}{R^{1.9}}\frac{4.47}{\Theta\ln^3\Theta}, \tag{7}$$

where $\Theta = \varepsilon/12. + (2./R)^{0.9}$.

We mention here another similar formula that can be deduced with similar limiting behaviours:

$$\lambda = \frac{1}{1.8^2}\log_{10}^{-2}\left(\left(\frac{\varepsilon}{12.}\right)^{10/9} + \frac{2.}{R}\right). \tag{8}$$

but which is not mathematically identical to equation (5). Haaland (1983) adopted this approach for pipes and found that it was within 1-2% of the implicit law of Colebrook and White, which is better known in the form of the curves on the Stanton-Moody diagram (see, for example, White 2003). Colebrook and White's original expression is accurate only to some 3-5%, even if equivalent roughness size were accurately known, and so the level of accuracy of the approximation is reasonable. However, in the case of channels, with different types of roughness and greater physical variability, in addition to the problem of defining equivalent sand roughness, equation (8) would not be expected to give similar high accuracy. In view of this, we choose the alternative where the roughness term is simpler, as in open channels it is more important than the Reynolds number term, so that we will recommend equation (5) or (6). However we mention the other form and in the following section compare its performance to give a feel for the effects of the two different arbitrary approximate expressions.

5. Results

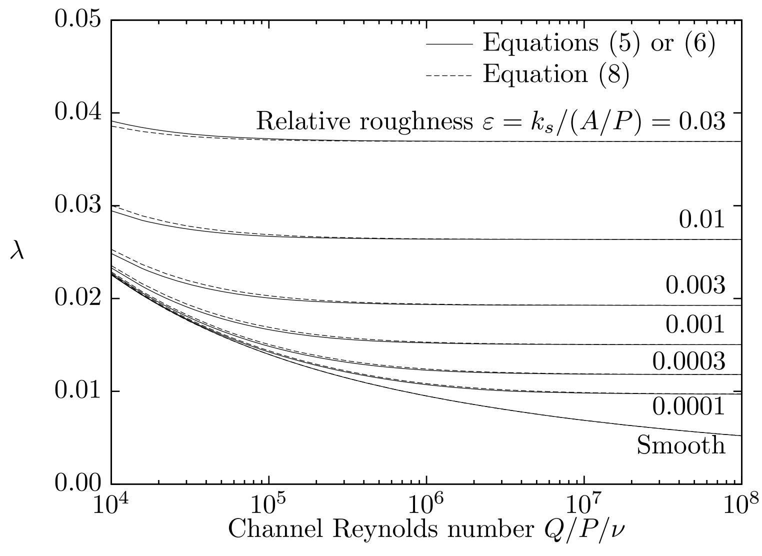

Figure 1. Dependence of λ on relative roughness and channel Reynolds number

Figure 1 shows on a modified Stanton-Moody plot the behaviour of the expressions presented here, in terms of the actual cross-sectional area and wetted perimeter of a channel, and not the equivalent pipe diameter. The abcissa is the channel Reynolds number $R = Q/P/\nu$, and the parameter for the various curves is the relative roughness expressed in terms of the hydraulic radius, $\varepsilon = k_s/(A/P)$. It can be seen that the two families of curves, from equations (6) and (8), show some disagreement. However, as these have been obtained from smooth wall results with $\varepsilon = 0$ and completely rough results $R \to \infty$, and are just a plausible means of incorporating both, it cannot be claimed in this case that one formulation is superior to the other. However, for channel applications, equation (6) is slightly simpler. We do not have general detailed experimental results for verification, however the spirit of the approximation is close to Haaland's approach, which worked very well for pipes.

In most open channel applications the Reynolds number is large enough that its effect is small compared with that of relative roughness, nevertheless Yen's equation and our variant of it here is a way of incorporating it. The result, whether Yen's formula (30) or our slightly modified version of the Yen formula, expression (6), is a formula of some generality which might be of benefit, showing theoretically at least how resistance depends on the hydraulic parameters, and practically giving a means of estimating the resistance.

6. Comparison with Manning's n

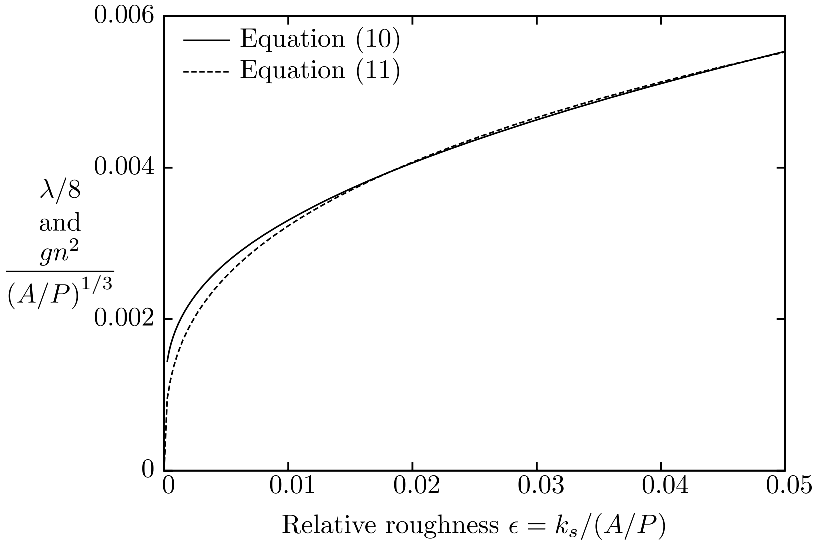

Figure 2. Comparison of two variations of resistance coefficient with relative roughness

Other formulations of the resistance include those of Chézy and Gauckler-Manning-Strickler. For them to agree for a steady uniform flow they are related by

$$\frac{\lambda}{8} = \frac{g}{C^2} = \frac{gn^2}{(A/P)^{1/3}} = \frac{g}{k_{\mathrm{St}}^2\,(A/P)^{1/3}}, \tag{9}$$

where C is the Chézy coefficient, n the Manning coefficient, and k_{St} the Strickler coefficient, the latter two being in *SI* units. We can use equation (6) for λ for fully-rough flow, $R \to \infty$ with this equation to give the formula for $\lambda/8$:

$$\frac{\lambda}{8} = \frac{1.33/8}{\ln^2\left(\frac{\varepsilon}{12.}\right)}. \tag{10}$$

To compare predictions from the Gauckler-Manning-Strickler form we can use Strickler's 1923 result (Jaeger 1956, p30) for the empirical relationship for n in terms of the equivalent sand-grain diameter k_s (in *SI* units):

$$n = \frac{k_s^{1/6}}{21.1} \approx 0.047\, k_s^{1/6}.$$

Henderson (1966, p98) argued that due to bed armouring the coefficient of 0.047 could be reduced by 10-20%. With power of hindsight, here we will take a value of 0.039 (see also p107 of Sturm 2001). We then have for the Gauckler-Manning-Strickler term of equation (9), with $g = 9.8\,\mathrm{ms}^{-2}$

$$\frac{gn^2}{(A/P)^{1/3}} = 0.039^2 g\,\varepsilon^{1/3} \approx 0.015\,\varepsilon^{1/3}. \tag{11}$$

The approximations (10) and (11) are shown plotted on figure 2, and it can be seen that the two agree quite well, with the choice of coefficient we have made here. We interpret this as giving support for choosing the Darcy-Weisbach form, as it is apparently able to describe the variation of the resistance with relative roughness that the widely-used Gauckler-Manning-Strickler form does, but it involves a dimensionless coefficient, is simpler, can be related to more fundamental fluid mechanics results, and it can also include variation with Reynolds number if required.

7. Use of a modified coefficient Λ

In Fenton (2010) it was shown that the long wave equations could also apply to channels on finite slopes. By introducing the symbol Λ:

$$\Lambda = \frac{\lambda}{8}\left(1 + \tilde{S}^2\right), \tag{12}$$

where $\tilde{S}$ is the downstream slope at a section, averaged around the wetted perimeter, the long wave momentum equation (Fenton 2010, eqn 13) becomes, ignoring lateral inflow and assuming unidirectional flow:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\Lambda P\frac{Q^2}{A^2}, \tag{13}$$

where Q is the discharge, t is time, x is the horizontal streamwise co-ordinate, β is a Boussinesq momentum coefficient, and η is surface elevation. The resistance term contains the dimensionless coefficient Λ multiplied by the wetted perimeter around which the resistance acts, multiplied by the square of the mean velocity. This has a relatively simple physically-significant form, which can be compared with use of the Gauckler-Manning-Strickler form, when the two alternative forms of the resistance term are

$$-\Lambda P\frac{Q^2}{A^2} = -\frac{gn^2P^{4/3}Q^2}{A^{7/3}},$$

where the g is there only as an artifact – gravity does not actually play a role in this term. The use of the dimensionless coefficient Λ is clearly simpler.

In the case of steady uniform flow in a prismatic channel of constant slope S_0, the momentum equation (13) reduces to the Chézy-Darcy-Weisbach formula

$$U_0 = \frac{Q_0}{A_0} = \sqrt{\frac{g}{\Lambda}\frac{A_0}{P_0}S_0}, \tag{14}$$

where subscripts 0 denote uniform flow, and where U_0 is the mean flow velocity. The formula is written here in terms of the dimensionless resistance coefficient Λ, rather than Chézy's C which is dimensional, and additionally it allows for the effects of finite slope, as shown in equation (12).

An interesting relationship emerges if equation (14) is re-written to give a formula for the Froude number F_0:

$$F_0^2 = \frac{U_0^2}{gA_0/B_0} = \frac{S_0}{\Lambda}\frac{B_0}{P_0}, \tag{15}$$

where B_0 is the surface width. In wider channels, $P_0 \approx B_0$, giving the relationship

$$F_0^2 \approx \frac{S_0}{\Lambda}, \tag{16}$$

so that the square of the Froude number is approximately equal to the ratio of slope to the dimensionless resistance coefficient Λ. This shows how, now arguing in rough approximate terms, that the Froude number for a given stream is almost independent of flow, if we presume that neither Λ nor the geometry varies much.

The simplicity of all formulae when using Λ is noteworthy.

8. Conclusions

We have considered the Darcy-Weisbach form of resistance for flow in open channels, and presented a formulae by Yen that describes the variation of the dimensionless resistance coefficient λ with both relative roughness and Reynolds number, and have described the derivation of a variant (equation 6) that might be slightly simpler for theoretical studies, as it uses natural logarithms. The formula agrees with the Gauckler-Manning-Strickler form of resistance, but using the Darcy-Weisbach form gives simpler expressions, involves a dimensionless coefficient, and it can be related to more fundamental fluid mechanics results. If a modified form is used, $\Lambda = \lambda/8$, which is actually just g/C^2, where C is the Chézy coefficient, and possibly supplemented by a slope term for the rarely-expected case of very steep channels, the long wave momentum equation has a simple and physically-revealing resistance term, and a simple relationship for the Froude number is obtained in terms of channel slope and that resistance coefficient.

References

ASCE Task Force on Friction Factors in Open Channels (1963), Friction factors in open channels, *J. Hydraulics Div. ASCE* **89**, 97–143.

Barnes, H. H. (1967), Roughness characteristics of natural channels, Water-Supply Paper 1849, U.S. Geological Survey.
URL: *http://il.water.usgs.gov/proj/nvalues/*

Colebrook, C. F. (1938), Turbulent flow in pipes with particular reference to the transition region between the smooth and rough pipe laws, *J. Inst. Civ. Engrs* **11**, 133–156.

Fenton, J. D. (2010), The long wave equations, Technical report, Alternative Hydraulics Paper 1.
URL: *http://johndfenton.com/Papers/01-The-long-wave-equations.pdf*

Haaland, S. E. (1983), Simple and explicit formulas for the friction factor in turbulent pipe flow, *J. Fluids Engng* **105**, 89–90.

Henderson, F. M. (1966), *Open Channel Flow*, Macmillan, New York.

Hicks, D. M. & Mason, P. D. (1991), *Roughness Characteristics of New Zealand Rivers*, DSIR Marine and Freshwater, Wellington.

Jaeger, C. (1956), *Engineering Fluid Mechanics*, Blackie, London.

Sturm, T. W. (2001), *Open Channel Hydraulics*, McGraw-Hill.

White, F. M. (2003), *Fluid Mechanics*, fifth edn, McGraw-Hill, New York.

Yen, B. C. (1991), Hydraulic resistance in open channels, *in* B. C. Yen, ed., ‘Channel Flow Resistance: Centennial of Manning’s Formula’, Water Resource Publications, Highlands Ranch, USA, p. 1–135.