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The long wave equations

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Abstract

The long wave equations for a straight slowly-varying channel of small slope are derived using the mass and momentum conservation equations in cartesian co-ordinates. The derivation attempts to use a true hydraulic approach, where quantities are modelled as accurately but simply as possible. Evaluating the momentum contribution due to pressure is made rather easier by using the divergence theorem of vector calculus. Then, in evaluating resistance, the Darcy-Weisbach formulation is shown to have several advantages and the Gauckler-Manning approach is criticised. Although use is made of some vector notation, there are few lengthy mathematical operations until it is necessary to relate the derivatives of area to those of surface elevation. A useful result is obtained, however, where the non-prismatic nature of a channel is shown to be simply approximated using the mean downstream slope at a section. The long wave equations are presented in explicit useable form with various combinations of variables, and similarly for steady flow, with different forms of the gradually-varied flow equation.

Two appendices are included. The first presents the concept of upstream volume. Using this, the mass conservation equation is satisfied identically, giving the momentum equation as a single equation in terms of a single variable, which is probably more useful for theoretical work.

The second appendix contains a derivation of the equivalent energy conservation equation. It has the same form as the momentum equation, but with more coefficients necessary to express integrals in terms of mean quantities, and energy loss is more difficult to approximate than resistance in the momentum formulation.

Revision History	
28.01.2010	Initial version on Internet site
01.03.2011	Correction of definition of V after Fatemeh Soroush
24.09.2011	Interchanged y to vertical and z to transverse
22.04.2012	Correspondingly corrected the use of Y and Z for the bed
12.05.2014	Re-written in parts, and it has returned to the original y transverse and z vertical, because of a forthcoming paper on meandering rivers. This paper has been somewhat superseded by a paper in this series, very similar to this, but where the equations are derived with no limitation on slope: **URL:** *http://johndfenton.com/Papers/05-The-long-wave-equations-for-arbitrary-slopes.pdf*
13.03.2017	Vertical converging/diverging wall term from eqn (30b) now retained in the steady eqns (31)-(33)

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1. Introduction

The one-dimensional long wave equations are widely used in river and canal hydraulics and are the basis of much hydraulic software. They are subject to limitations including both small longitudinal slopes and curvatures. Their origin seems to have been earlier than the usually cited work of Saint-Venant. Dooge (1987, p221) stated that the "first presentation of such a pair of equations would appear to be due either to Dupuit ... or to Kleitz ... and refers to the case of a wide rectangular channel". Saint-Venant obtained a general mass conservation equation for a uniform waterway of any cross-sectional shape. However his derivation of the momentum equation was just for a rectangular canal of constant width, although he did retain a general resistance term. Dooge further wrote that according to Graeff in 1875 these equations had been previously given in a report by Kleitz in 1858 that was circulated but not published. Saint-Venant's contribution beyond that of Dupuit and Kleitz has been modest, and it would seem unfair to the earlier workers to continue to use the appellation "Saint-Venant" for the equations. However, it is interesting that the momentum equation of Dupuit/Kleitz/Saint-Venant, obtained on the basis of a uniform rectangular canal, is in fact valid for the general case of a non-prismatic waterway of arbitrary section.

Boussinesq (1877, p192 – eqn 155), which is described rather more accessibly by Jaeger (1956, p122 *et seq.*), considered steady flow in a wide rectangular channel of varying longitudinal bed topography using mass and momentum conservation. He went to one higher level of approximation, allowing for streamline curvature in determining the pressure, and obtained a third-order (now "Boussinesq") differential equation.

Keulegan (1942) derived both momentum and energy formulations for steady flow, and stated a preference for the momentum approach as it requires only the mechanism of resistance at the boundary and not details of the energy loss processes. Keulegan & Patterson (1943) then considered unsteady motion in a prismatic channel, obtaining the momentum equation, and then for a wide rectangular channel went on to consider the higher Boussinesq approximation with dynamic effects due to streamline curvature.

Two later derivations of the long wave equations went to considerable lengths to make the treatments as general as possible. Strelkoff (1969) considered the full Navier-Stokes equations and made an allowance for turbulence in the form of time averaged Reynolds equations. Yen (1973) also developed a general formulation, including viscosity, turbulence, fluid compressibility and inhomogeneity. In both cases the general forms of the equations were not suitable for practical application, but both writers did go on to present useable equations by making the usual reasonable hydraulic assumptions. Both derivations used a coordinate system based on the bed of the waterway. In many natural streams the bed is poorly known, and it is in general curved in the vertical plane. Such an approach should ideally have included extra terms due to the curvature of the axes, as obtained by Dressler (1978). However, both formulations would have been correct had they simply defined the streamwise axis to be horizontal. Later derivations by Cunge, Holly & Verwey (1980, §2.1) and Lai (1986, pp182-3) did use cartesian coordinates and obtained simpler derivations. Both presented results for different combinations of dependent variables. However in the final presentations, both left some versions with derivatives of combinations of quantities unexpanded and not in a form ready for use. Neither evaluated explicitly the so-called non-prismatic contribution to the derivative of cross-sectional area.

This paper obtains the long wave equations for straight channels of otherwise arbitrary section and topography. It tries to use a true hydraulic approach, where quantities are modelled as accurately but simply as possible.

Features of the derivation and results

- The equations can be obtained relatively concisely using the integral mass and momentum equations; there are few detailed mathematical manipulations until it is necessary later to relate cross-sectional area and surface elevation.
- The cross-sectional area and discharge arise naturally as the most fundamental dependent variables. The mass conservation equation obtained in terms of them is exact for straight channels.

- The momentum equation requires additional assumptions, listed below.
- Cartesian co-ordinates are simple and require no special attention for axis curvature.
- Use of Gauss' divergence theorem in evaluating the pressure contribution to the momentum equation gives a substantially simpler derivation, as it avoids difficulties of integrals over complicated surfaces.
- An explicit form is given for a term due to a non-prismatic channel, expressed in terms of the local mean downstream slope at a section. Often that is poorly-known, which shows that there is little sense in giving detailed attention to the term and its precise calculation.
- The Darcy-Weisbach resistance formulation fits naturally with the momentum approach, so that the resistance term appears with a clear physical significance in the momentum equation.
- Three pairs of long wave equations are presented, each in terms of discharge plus one measure of cross-section – the actual area (which is found to have an interesting advantage, that approximate simulations can be performed without artificial inclusion of underwater topography), plus surface elevation, plus a depth-like quantity, the elevation relative to an arbitrary axis.
- Three gradually-varied flow equations for steady flow are presented, each in terms of one of the three measures of cross-section.

Assumptions

The non-trivial assumptions in the derivation are, roughly in decreasing order of importance or limitation:

1. Resistance to flow is approximated by the Darcy-Weisbach approach. The underlying flow is a turbulent shear flow. Of course the Navier-Stokes equations are not being used.
2. The stream is assumed to be straight, such that effects of curvature of its course are ignored.
3. All surface variation is sufficiently long and slopes are sufficiently small that the pressure throughout the flow is given by the hydrostatic pressure corresponding to the depth of water above each point.
4. The water surface across the stream is horizontal.
5. The effects of both non-uniformity of velocity over a section and turbulent fluctuations are approximated by a generalised Boussinesq momentum coefficient multiplying uniform contributions.
6. The fluid density is constant.

2. Mass conservation equation

Consider figure 1, showing an elemental slice of straight channel of length Δx with a control volume formed by four surfaces – two stationary vertical faces transverse to the flow, the stationary stream bed, and the possibly-moving free surface. Cartesian co-ordinates are used, which avoid mathematical problems that bed-oriented co-ordinates introduce. The mass conservation equation for an arbitrarily moving control surface and volume is (*e.g.* White 2009,§3.3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho\,\mathrm{d}\mathcal{V} + \int_{\mathrm{CS}} \rho\,\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0, \tag{1}$$

where the integrals are over CV, the control volume, and CS, its enclosing control surface, ρ is the fluid density, $\mathrm{d}\mathcal{V}$ is an element of volume, t is time, $\mathrm{d}S$ is an element of the control surface, $\mathbf{u}_\mathrm{r}$ is the velocity of a fluid particle relative to that of the control surface, $\hat{\mathbf{n}}$ is a unit normal vector directed outwards such that $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}}$ is the normal component of fluid velocity relative to the surface. It is this velocity that is responsible for the transport of any quantity across the surface.

The first term in equation (1), is the rate of change of mass inside the elemental control volume. An

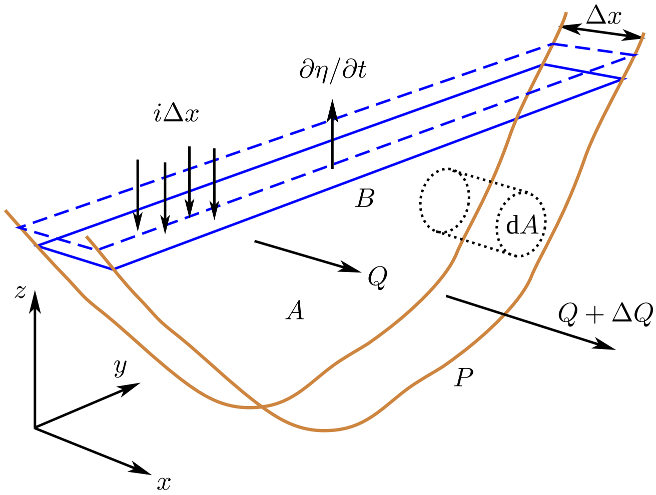

Figure 1. Elemental length of channel showing moving control surface with its enclosed control volume

element of the control volume shown in figure 1 has volume $\mathrm{d}\mathcal{V} = \mathrm{d}A\,\Delta x$, and so, assuming ρ constant and taking constant quantities outside the integral we obtain

$$\rho\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \rho\,\Delta x\frac{\partial A}{\partial t}, \tag{2}$$

where A is the cross-sectional area of the channel flow.

Considering the mass rate of flow crossing the boundary, there is no contribution on the solid stationary bed as $\mathbf{u}_\mathrm{r} = \mathbf{u} = \mathbf{0}$ there. On the possibly-moving free surface the fluid particles remain on it, in whatever manner it is moving, such that $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}} = 0$ there, and there is no mass transport across it either. On the stationary vertical faces across the flow, $\mathbf{u}_\mathrm{r} = \mathbf{u}$, the actual fluid velocity. On the upstream face $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}} = -u$, where u is the x-component of velocity at a point, and the minus sign is because the velocity is opposite to the outwards normal which is upstream here. The contribution to the integral is then $-\int_A \rho u\,\mathrm{d}A$, which for an incompressible fluid is simply $-\rho Q$, where Q is the volume rate of flow, the discharge. The contribution to the second integral, on the downstream face, is of opposite sign and in general has changed with x. We write it as a Taylor series, giving the combination of the two:

$$-\left.\rho Q\right|_x + \left.\rho Q\right|_{x+\Delta x} = \rho\,\Delta x\frac{\partial Q}{\partial x} + \text{ terms like } (\Delta x)^2\,. \tag{3}$$

Finally, an allowance is made for any fluid entering the control volume from rainfall, seepage, or tributaries, with a volume rate i per unit length and density ρ, assumed to be the same as that already in the channel. Its contribution is $-\rho\Delta x\,i$, negative because it is entering the control volume. Combining this contribution and those of equations (2) and (3) to equation (1), dividing by $\rho\,\Delta x$ and taking the limit as $\Delta x \rightarrow 0$ gives

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{4}$$

Unusually in hydraulics, this is almost an exact equation. The only non-trivial assumption that has been is that the channel is straight. No assumption regarding the flow has had to be made. It suggests that the cross-sectional area A and discharge Q are fundamental quantities. Below we will obtain equations also in terms of surface elevation.

3. Momentum conservation equation

The conservation of momentum principle is now applied to the flow in the channel. Consider the x-component of the integral form of the momentum equation for a possibly-moving control volume (White 2009, §3.4):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho\, u\, \mathrm{d}\mathcal{V} + \int_{\mathrm{CS}} \rho\, u\, \mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\, \mathrm{d}S = F_x, \tag{5}$$

where F_x is the component of force exerted on the fluid in the control volume by both body and surface forces.

3.1 Fluid momentum terms

Here we consider the terms on the left of equation (5).

1. Unsteady term

Again using $\mathrm{d}\mathcal{V} = \mathrm{d}A\,\Delta x$, and taking constant quantities outside the integral we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho\, u\, \mathrm{d}\mathcal{V} = \rho\,\Delta x\, \frac{\partial}{\partial t}\int_{A} u\, \mathrm{d}A = \rho\,\Delta x\, \frac{\partial Q}{\partial t}, \tag{6}$$

in which no additional approximation has been made, as by definition, the discharge $Q = \int_A u\, \mathrm{d}A$. It is useful that this term has been able to be expressed simply and exactly in terms of Q, further evidence for the utility of that quantity. As the term is linear for constant density, allowing for turbulence by short-term time-averaging gives no additional terms if we ignore turbulent surface fluctuations.

2. Momentum flux term

The second term on the left of equation (5), $\int_{\mathrm{CS}} \rho\, u\, \mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\, \mathrm{d}S$, has no contribution from the material boundaries on top and bottom of the control volume, the surface and the bed, as they are composed of particles that move with and define the control surface such that the velocity of those particles relative to the surface is $\mathbf{u}_{\mathrm{r}} = \mathbf{0}$. On the stationary vertical face upstream, $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}} = -u$, giving the contribution $-\rho\overline{\int_A u^2\, \mathrm{d}A}$, where the overbar has been introduced to denote a short term time averaging operation to allow for turbulence. The downstream face at $x + \Delta x$ has a contribution of a similar nature, but where both A and u have changed with x. The difference between the two contributions is, again using a Taylor series, simply

$$\rho\,\Delta x\, \frac{\partial}{\partial x}\overline{\int_A u^2\, \mathrm{d}A} + \text{terms like } (\Delta x)^2\,.$$

Evaluating the area integral and the time mean requires a detailed knowledge of the flow distribution and its turbulent nature that is almost always unavailable. Traditionally a Boussinesq momentum coefficient β has been introduced to allow for the non-uniformity of velocity distribution. It should be used also to allow for the effects of turbulence (Fenton 2005). If the velocity at a point is written as $u = \bar{u} + u'$, where $\bar{u}$ is the time mean velocity and u' the fluctuating component, if we ignore fluctuations of the free surface, then performing averaging over a short time interval the integral can be written

$$\overline{\int_A u^2\, \mathrm{d}A} = \int_A \left(\bar{u}^2 + \overline{u'^2}\right)\, dA$$

and introducing the more general definition of the momentum coefficient

$$\beta = \frac{1}{U^2 A}\int_A \left(\bar{u}^2 + \overline{u'^2}\right)\, dA,$$

where $U = Q/A$, the mean component of x-velocity averaged over the section, the contribution can be written simply

$$\rho\,\Delta x\,\frac{\partial}{\partial x}\overline{\int_A u^2\,\mathrm{d}A} \approx \rho\,\Delta x\,\frac{\partial}{\partial x}\left(\beta U^2 A\right) = \rho\,\Delta x\,\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right), \tag{7}$$

The remaining contribution to momentum flux is from inflow. In obtaining the mass conservation equation above, this was lumped together as an inflow i per unit length, such that the mass rate of inflow was $\rho\, i\,\Delta x$, (*i.e.* an outflow of $-\rho\, i\,\Delta x$). If this inflow has a mean streamwise velocity of u_i before it mixes with the water in the channel, the contribution is

$$-\rho\Delta x\,\beta_i\,u_i\,i, \tag{8}$$

where β_i is the Boussinesq momentum coefficient of the inflow. The term is unlikely to be known accurately or to be important in most places, except locally where a significant stream enters.

3.2 Forces – body and surface

1. Gravity

The gravitational force is vertical, such that with our cartesian co-ordinates the horizontal component is zero, and there is no contribution to our x-momentum equation. The manner in which gravity enters is to cause the variation of pressure in the fluid, as we now calculate.

2. Pressure forces

The total pressure force on the fluid around the control surface is $-\int_{\mathrm{CS}} p\,\hat{\mathbf{n}}\,\mathrm{d}S$, where p is the pressure, $\hat{\mathbf{n}}$ is the outward normal and the negative sign shows that the local force due to pressure acts opposite to it, inwards on the fluid in the control volume. In this form the term is difficult to evaluate for non-prismatic waterways, as the pressure and the highly-variable unit normal vector have to be integrated over all the submerged faces of the control surface. Traditional derivations do this using some lengthy calculus, but nevertheless ending with a simple result. It is obtained more easily if the term is evaluated using Gauss' divergence theorem of vector calculus, replacing the integral over the rather complicated control surface by a volume integral (*e.g.* Milne-Thomson 1968, §2.61, equation 3, but with a sign convention for $\hat{\mathbf{n}}$ opposite to ours), so that here

$$-\int_{\mathrm{CS}} p\,\hat{\mathbf{n}}\,\mathrm{d}S = -\int_{\mathrm{CV}} \nabla p\,\mathrm{d}\mathcal{V}, \tag{9}$$

where ∇ is the vector gradient operator such that ∇p is a vector whose components are the pressure gradients in each direction. For the elemental control volume across the channel this can then be simply evaluated in terms of an integral across the section, and taking just the x-component of ∇p, $\partial p/\partial x$, we obtain

$$\text{Pressure contribution to } x\text{-momentum} = -\Delta x\int_{\mathrm{CV}}\frac{\partial p}{\partial x}\,\mathrm{d}A. \tag{10}$$

It is almost obvious that the net pressure force is given by the pressure gradient multiplied by Δx integrated over the section, but without the formality of the mathematics one might be worried about contributions at the bed and free surface.

This expression can be evaluated simply, making the hydrostatic approximation for small slopes, such that the equivalent hydrostatic pressure is written $p = \rho g\,(\eta - z)$ where z is the elevation of a general point, and η is the elevation of the free surface above that point. Differentiating to give the pressure gradient gives $\partial p/\partial x = \rho g\partial\eta/\partial x$, so that the horizontal pressure gradient in the water is due solely to the free surface variation, and so the contribution is

$$-\Delta x \int_A \frac{\partial p}{\partial x}\,\mathrm{d}A \approx -\rho\,\Delta x\, g \int_A \frac{\partial \eta}{\partial x}\,\mathrm{d}A \approx -\rho\,\Delta x\, gA\frac{\partial \eta}{\partial x}, \tag{11}$$

where we have assumed surface elevation η constant across the channel and taken its derivative outside the integral.

3. Resistance forces

The forces of the boundary on the flow are incorporated using empirical results from turbulent shear flows. The Darcy-Weisbach formulation here provides insights into the nature of the equations and some convenient quantifications of the effects of resistance. The ASCE Task Force on Friction Factors in Open Channels (1963) recommended its use, but that suggestion has been almost entirely ignored. We will see that in the present force determination it is very useful, because it is directly related to stress and force on the boundary. We will consider only the case for small slopes, where the square of the slope is small compared with unity, accurate for almost all rivers and canals.

Consider the expression for the shear force τ on a pipe wall (*e.g.* §6.3 of White 2009)

$$\tau = \frac{\lambda}{8}\rho V^2,$$

where the Weisbach coefficient λ is a dimensionless resistance factor (for which the symbol f is often used, but here we follow the terminology of fundamental researchers in the field in the first half of the twentieth century), and V is the mean velocity in the pipe. Such an expression follows from a dimensional analysis of the problem, suggesting its fundamental nature. The denominator 8 follows from the original introduction of λ in the Darcy-Weisbach formula for head loss in a pipe, with a term $2g$ in the expression for head and a term 4 in the relationship between head loss and τ. The coefficient λ is simply related to Chézy's resistance coefficient C by $\lambda/8 = g/C^2$. Here we use it for a channel to calculate the mean stress around the perimeter. We make the small-slope approximation, and so we write $V \approx U = Q/A$, the mean x-component of velocity over the section. To calculate the total force, we write Q^2 as $-Q\,|Q|$ such that the direction of the force is always opposite to the flow direction, and we multiply the stress by $P\,\Delta x$, the elemental area of the channel boundary in figure 1, where P is the wetted perimeter, to give

$$\text{Resistance force} = -\rho\,\Delta x \frac{\lambda P}{8}\frac{Q\,|Q|}{A^2}. \tag{12}$$

3.3 Collecting all terms in the momentum equation

Now all contributions to the momentum equation (5) are collected, from equations (6), (7), (8) on the left, and (11) and (12), contributions to F_x, on the right. Dividing by $\rho\Delta x$, and bringing all derivatives of dependent quantities to the left and others to the right, gives the momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\frac{\lambda P}{8}\frac{Q\,|Q|}{A^2} + \beta_i i\, u_i. \tag{13}$$

The equation in this form is simple and shows the significance of each term. Previous presentations using the small slope approximation have scarcely gone beyond this stage (for example Cunge et al. 1980, pp16-17; Lai 1986, pages 182-3), and did not expand the inertia term $\partial\left(\beta Q^2/A\right)/\partial x$. That is easily performed, and we obtain

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} - \beta\frac{Q^2}{A^2}\frac{\partial A}{\partial x} + gA\frac{\partial \eta}{\partial x} = -\frac{\lambda P}{8}\frac{Q\,|Q|}{A^2} + \beta_i\, u_i\, i - \frac{Q^2}{A}\beta'(x), \tag{14}$$

where $\beta'(x) = \mathrm{d}\beta/\mathrm{d}x$. The discharge has emerged as being fundamental in this integrated momentum formulation, the only time derivative being $\partial Q/\partial t$. However the equation is not yet in useable form, for the derivatives of cross-sectional area $\partial A/\partial x$ and surface elevation $\partial \eta/\partial x$ are not independent. In fact, some of the more important practical steps lie before us, in the relating of these quantities, which requires some effort.

3.4 Discussion of the resistance term

The significance of the resistance term, $-\lambda P/8 \times Q\,|Q|\,/A^2$ is clear, written as a dimensionless coefficient multiplied by wetted perimeter times second power of the mean velocity, giving the force per unit length divided by density.

The resistance coefficient λ has been extensively investigated (ASCE Task Force on Friction Factors in Open Channels 1963). Yen considered the results presented and obtained a convenient formula in terms of the relative roughness and the Reynolds number of the flow (Yen 2002, equation 19):

$$\lambda = \left(-2\,\log_{10}\left(\frac{\epsilon_s}{12} + \frac{1.95}{R^{0.9}}\right)\right)^{-2}, \tag{15}$$

where $\epsilon_s = k_s/\,(A/P)$ is the relative roughness, k_s is the equivalent sand-grain diameter and $R = Q/P\nu$ is the channel Reynolds number with ν the kinematic viscosity. Yen stated that his formula was applicable for $R > 30\,000$ and $\epsilon_s < 0.05$.

This explicit formula for computing resistance seems to provide a solution to some of the problems identified by the ASCE Task Force. It treats both smooth and rough boundaries, and for non-vegetated streams at least seems to be superior to that of using values of Manning's coefficient n often obtained roughly by using tables or pictures from books. For streams where vegetation provides an important contribution to resistance, it does not help, but recent research on the effects of vegetation in streams has certainly used the framework of fluid drag, which can be fitted into the Darcy-Weisbach formulation. As λ is dimensionless, it is not necessary to modify any formulae if one uses non-*S.I.* units.

If one wanted to add an allowance for resistance such as that due to vegetation or bed-forms, one advantage of the Weisbach formulation, being directly related to force, is that one can linearly superimpose contributions so that in a more complicated situation, the resistance contributions can be simply combined, including the perimeter over which they act:

$$\lambda P = \sum\nolimits_i \lambda_i P_i. \tag{16}$$

Another simple example where a formula such as this would be useful is a glass-walled laboratory flume with a rough bed, which would cause difficulties for Gauckler-Manning, which is not based on rational mechanics. An idea of the problems which its empiricism causes is given by the different formulae for the compound Manning coefficient n, all found in one recent report on resistance in streams:

$$n = \sum\nolimits_i n_i \quad \text{or} \quad n = \left(\sum\nolimits_i n_i^2\right)^{1/2} \quad \text{or} \quad \frac{1}{n} = \left(\sum\nolimits_i \frac{1}{n_i^2}\right)^{1/2}. \tag{17}$$

The report presented different recommendations in the report as to when each method would be preferred. There was no weighting according to the fraction of perimeter for each contribution.

That notwithstanding, we mention other explicit forms of the resistance term including those of Chézy and Gauckler-Manning. For the different formulations to agree for steady uniform flow:

$$\frac{\lambda}{8} = \frac{g}{C^2} = \frac{gn^2 P^{1/3}}{A^{1/3}} = \frac{g}{k_{\mathrm{St}}^2}\frac{P^{1/3}}{A^{1/3}}, \tag{18}$$

where C is the Chézy coefficient, n the Manning coefficient, and k_{St} the Strickler coefficient. For Gauckler-Manning, the resistance term in the momentum equation (14) becomes:

$$-gn^2\frac{Q\,|Q|\,P^{4/3}}{A^{7/3}}. \tag{19}$$

The indices $4/3$ and $7/3$ seem just a little fussy. In some applications the conveyance of the stream $K = 1/n \times A^{5/3}/P^{2/3}$ has been used so that the resistance term becomes

$$-gA\frac{Q\,|Q|}{K^2}. \tag{20}$$

The term is to do with resistance to fluid motion, and has nothing to do gravity – the presence of g here and in equation (19) is an artefact from the original Gauckler-Manning equation which does not include gravity, even though it is the driving force. Manning's n implicitly contains g, as does Chézy's C.

Possibly because of the non-trivial nature of the Manning form, a more vague way of writing the resistance term that has been used is $-gAS_\mathrm{f}$, where S_f is called the "friction slope". That is apparently a dangerous procedure, for in some works (*e.g.* Lyn & Altinakar 2002) S_f has been assumed to be a constant even where Q and A vary. More importantly, in terms of understanding, this has led to it being mistaken for an energy slope, which it is not. It comes from the shear force on the perimeter. The Weisbach form as it appears in the momentum equation (14) seems simple and clear.

4. Relationships between area and elevation derivatives

The momentum equation contains time and space derivatives of A and η. To be able to express the equations in terms of derivatives of one or the other, we have to relate them. Consider the integral for area

$$A = \int\limits_{Y_\mathrm{R}}^{Y_\mathrm{L}} (\eta - Z)\ \mathrm{d}y, \tag{21}$$

where $z = Z(x, y)$ is the bed elevation, and the right and left waterlines are defined by the intersections between the side surfaces $y = Y_{\mathrm{L/R}}(x, z)$ and the free surface $z = \eta\,(x, t)$ such that the limits are $y = Y_{\mathrm{L/R}}(x, \eta\,(x, t))$, using notation "L/R" in the subscript showing that either or both can be taken.

The derivative of area with respect to time is obtained from Leibniz' theorem for the derivative of an integral, which gives

$$\frac{\partial A}{\partial t} = \int_{Y_\mathrm{R}}^{Y_\mathrm{L}} \frac{\partial \eta}{\partial t}\mathrm{d}y + (\eta - Z_\mathrm{L})\ \frac{\partial Y_\mathrm{L}}{\partial t} - (\eta - Z_\mathrm{R})\ \frac{\partial Y_\mathrm{R}}{\partial t},$$

where $(\eta - Z_\mathrm{L})$ is the water depth at the left bank and $(\eta - Z_\mathrm{R})$ that at the right. Both the last two terms are zero in most situations where the bank is sloping. The only way that they contribute is if the sides of the channel are vertical and are moving with time. This seems unlikely, and so they will be neglected. It has already been assumed that the free surface is level across the channel, so that the integrand $\partial\eta/\partial t$ is independent of y and can be taken outside the integral, giving

$$\frac{\partial A}{\partial t} = B\frac{\partial \eta}{\partial t}, \tag{22}$$

where $B = Y_\mathrm{L} - Y_\mathrm{R}$ is the surface width.

Now the x derivative of A is considered. Differentiating equation (21) with respect to x and using Leibniz' theorem again:

$$\frac{\partial A}{\partial x} = \underbrace{\int\limits_{Y_\mathrm{R}}^{Y_\mathrm{L}} \frac{\partial \eta}{\partial x}\ \mathrm{d}y}_{\mathrm{I}} - \underbrace{\int\limits_{Y_\mathrm{R}}^{Y_\mathrm{L}} \frac{\partial Z}{\partial x}\ \mathrm{d}y}_{\mathrm{II}} + \underbrace{(\eta - Z_\mathrm{L})\ \left.\frac{\partial Y_\mathrm{L}}{\partial x}\right|_t - (\eta - Z_\mathrm{R})\ \left.\frac{\partial Y_\mathrm{R}}{\partial x}\right|_t}_{\mathrm{III}}, \tag{23}$$

where, as A and η are functions of x and t, both $\partial A/\partial x$ and $\partial\eta/\partial x$ imply that t is considered constant. The bed elevation Z is a function of x and y, so $\partial Z/\partial x$ implies y is constant, and it is just a streamwise derivative, the local bed slope. A slightly different notation $\partial/\partial x|_t$ has been necessary for the x-derivatives of $Y_{\mathrm{R}}(x, \eta(x,t))$ and $Y_{\mathrm{L}}(x, \eta(x,t))$, to show that t is held constant, as here using just $\partial/\partial x$ would imply η constant. Now we evaluate each of the terms in the expression:

Term I: We have assumed that free surface elevation η is constant across the channel, so that the first term becomes simply $(Y_{\mathrm{L}} - Y_{\mathrm{R}})\ \partial\eta/\partial x = B\,\partial\eta/\partial x$.

Term II: The second term is the integral across the channel of the downstream bed slope. We introduce the symbol $\tilde{S}$ for the local mean downstream bed slope evaluated across the section:

$$\tilde{S} = -\frac{1}{B}\int_{Y_{\mathrm{R}}}^{Y_{\mathrm{L}}} \frac{\partial Z}{\partial x}\,\mathrm{d}y, \tag{24}$$

defined with a minus sign such that in the usual situation where the bed slopes downwards in the direction of x, so that Z decreases, $\tilde{S}$ will be positive. If the bottom geometry is precisely known, this can be precisely evaluated, however it is much more likely to be only approximately known and a typical bed slope of the stream used. With this definition, the term in equation (23) can be just written $+B\tilde{S}$.

Term III: this term is zero almost everywhere. It is the contribution from the integrand at each limit multiplied by the derivative of the limit. The contributions in equation (23) are:

$$\left(\eta - Z_{\mathrm{L/R}}\right)\left.\frac{\partial Y_{\mathrm{L/R}}}{\partial x}\right|_t = (\eta - Z)_{\mathrm{L/R}}\left(\frac{\partial Y_{\mathrm{L/R}}}{\partial x} + \left.\frac{\partial Y_{\mathrm{L/R}}}{\partial z}\right|_{z=\eta(x,t)}\frac{\partial \eta}{\partial x}\right). \tag{25}$$

The factor $\left(\eta - Z_{\mathrm{L/R}}\right)$ is the water depth at the banks. We now consider different geometric cases:

1. The usual case for natural streams and most canals, with sloping (*i.e.* not vertical) banks:

   $$\eta - Z_{\mathrm{L/R}} = 0,$$

   and so in this common case the contribution of the whole term is zero.
2. If a side is vertical, then $\left.\partial Y_{\mathrm{L/R}}/\partial z\right|_{z=\eta(x,t)} = 0$ so the second term in the brackets in equation (25) is zero and we are left with the contribution $\left(\eta - Z_{\mathrm{L/R}}\right)\ \partial Y_{\mathrm{L/R}}/\partial x$. We have to consider two cases for this.
   a. The usual case for a vertical sided channel such as a flume or race or lock: the walls are parallel to the x-axis, so that they neither converge nor diverge, then $\partial Y_{\mathrm{L/R}}/\partial x = 0$ and so the contribution of the whole term is zero again.
   b. The rare case where the channel walls are both vertical and converging or diverging, such that $\partial Y_{\mathrm{L/R}}/\partial x$ is not zero, in something like a Parshall flume (although one would be careful about applying long wave theory in such a case).

As that last case, with a diverging vertical wall such that $Y_{\mathrm{L/R}}$ is independent of η, is the only possibility for a non-zero contribution, we can replace the partial derivatives by ordinary derivatives, and we write the net contribution of the last two terms in equation (23) as A_x^{V}:

$$A_x^{\mathrm{V}} = (\eta - Z_{\mathrm{L}})\frac{\mathrm{d}Y_{\mathrm{L}}}{\mathrm{d}x} - (\eta - Z_{\mathrm{R}})\frac{\mathrm{d}Y_{\mathrm{R}}}{\mathrm{d}x}, \tag{26}$$

the symbol V used to refer to **V**ertical side walls. For such vertical walls it is highly probable that we are dealing with a man-made structure, so that the bed is transversely horizontal too, such that in that case $Z_{\mathrm{L}} = Z_{\mathrm{R}} = Z$, and using the depth $h = \eta - Z$ we obtain

$$A_x^{\mathrm{V}} = h\left(\frac{\mathrm{d}Y_{\mathrm{L}}}{\mathrm{d}x} - \frac{\mathrm{d}Y_{\mathrm{R}}}{\mathrm{d}x}\right) = h\frac{\mathrm{d}B}{\mathrm{d}x}.$$

Almost everywhere, of course, $A_x^{\mathrm{V}} = 0$.

Collecting contributions from Terms I, II, and III: the relationship between area and elevation derivatives, equation (23), is written

$$\frac{\partial A}{\partial x} = B\frac{\partial \eta}{\partial x} + B\tilde{S} + A_x^{\mathrm{V}}. \tag{27}$$

The quantity $B\tilde{S}$ contains contributions from what has been called the "non-prismatic" term, which in other presentations has usually been written in vague terms like $\partial A/\partial x|_{h=\mathrm{const}}$ and has not been explicitly evaluated. The slope $\tilde{S}$, based on the formal definition, equation (24), allows for the fact that the effective mean slope in a non-prismatic stream is different from that of a prismatic one even if the talweg has the same slope. In any case, the bed geometry is rarely able to be evaluated with any accuracy, and instead in practice, a typical local stream slope might often be used and non-prismatic effects ignored.

5. Three forms of the long wave equations

Presentations elsewhere have given equations in terms of the mean horizontal velocity U in the flow. We do not, believing it to be insufficiently important, as there are very few problems where U might be specified as a boundary condition. In practical problems usually volume flow rate Q is more important. If velocities were required as results, they could be trivially obtained from Q/A.

Here we present three versions of the momentum equation, all in terms of Q, with alternatives for the remaining dependent variable.

5.1 Equations in terms of (A, Q)

We have observed that the mass conservation equation, equation (4), is exact for a straight channel, suggesting that A is a fundamental quantity. This form may not be so important practically, but for some theoretical studies it is useful to use the two integrated quantities A and Q as dependent variables. As well, there is an interesting aspect to the A formulation, such that one can model a channel approximately with relatively little detailed or assumed knowledge of the underwater topography, explained here after the presentation of the equations.

Collecting equations (4) and (14), and using equation (27) to eliminate $\partial\eta/\partial x$ from the latter, gives the long wave equations in terms of A and Q:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i, \tag{28a}$$

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(\frac{gA}{B} - \beta\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x} = gA\left(\tilde{S} + \frac{A_x^{\mathrm{V}}}{B}\right) - \frac{\lambda P}{8}\frac{Q\,|Q|}{A^2} + \beta_i\, u_i\, i - \frac{Q^2}{A}\beta'(x), \tag{28b}$$

where derivatives of dependent variables have been taken to the left and all others to the right, and of course, almost everywhere the right side is significantly simpler, where A_x^{V}, i, and $\beta'(x)$ are zero. In many situations the dominant terms in the equation are the remaining terms on the right, in terms of bed slope $\tilde{S}$ and resistance coefficient λ. Often neither the details of the underwater topography leading to $\tilde{S}$, nor the value of the resistance coefficient λ are accurately known. This may make one wonder about the necessity of including a value of β not equal to 1, or a value of $\beta'(x)$ at all.

In practice, in a channel where the geometry is well-known, at each computational value of $x_i,\ i = 0, 1, \ldots$, one might know the slope $\tilde{S}_i$ and the functional relationships $A_i(\eta)$, $B_i(\eta)$ and $P_i(\eta)$, probably in discrete form, from which one could obtain the corresponding $B_i(A)$ and $P_i(A)$ also in discrete form. However, it is much more likely that the underwater geometry is poorly known, and so the trouble of going to assumed forms of dependence on η is questionable. The formulation of equations (28a) and (28b) in terms of A allows an approximate procedure that is commensurate with the accuracy of knowledge of the whole problem. One could assume approximate values of $\tilde{S}_i$, and approximate, possibly constant, values of B_i and P_i at each computational point i, which for most streams are not going to vary much with flow anyway. Then to calculate the initial values of A at computational points along the stream one could assume notional approximate values of A at the points, with the given initial constant flow Q and perform a simulation with the equations until the incorrect values flow out of the computational domain and values of A became steady. These could then be used as initial values for real simulations with varying input discharge with time at the upstream boundary. Hence, using A one can perform model simulations with relatively little information required or artificially included.

5.2 Equations in terms of (η, Q)

Surface elevation is an important quantity in practice, so we will present a formulation in terms of it. We use equation (22) to eliminate $\partial A/\partial t$ from equation (4) and use equation (27) again, this time to eliminate $\partial A/\partial x$ from the momentum equation (14):

$$\frac{\partial \eta}{\partial t}+\frac{1}{B}\frac{\partial Q}{\partial x}=\frac{i}{B}, \tag{29a}$$

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial \eta}{\partial x}=\beta\frac{Q^2}{A^2}\left(B\tilde{S}+A_x^{\mathrm{V}}\right)-\frac{\lambda P}{8}\frac{Q\left|Q\right|}{A^2}+\beta_i\,u_i\,i-\frac{Q^2}{A}\beta'(x). \tag{29b}$$

5.3 Equations in terms of a depth-like variable h

Many works present the equations in terms of a quantity h referred to as "depth", which is an ambiguous and uncertain quantity, especially for natural streams. Here we define it to be the surface elevation relative to a reference axis possibly associated with the bottom of the stream, which could be chosen to be the bed of a canal or the thalweg in a river if that were sufficiently well known. It does not have to be a straight line in the vertical plane.

It should be pointed out, however, that using area A might be a better alternative to using h, as it contains some of the advantages of h, such as being constant for uniform flow, and varying relatively little in most streams for non-uniform flow, but it does not require the definition of an axis.

We let the elevation of the reference axis be $Z_0(x)$ and then in general $\eta\left(x,t\right)=h\left(x,t\right)+Z_0(x)$ and the derivatives are $\partial\eta/\partial t=\partial h/\partial t$ and $\partial\eta/\partial x=\partial h/\partial x-S_0\left(x\right)$, where S_0 is the slope of the axis (positive in the usual downward-sloping channel sense) $S_0=-\partial Z_0/\partial x$. Substituting these into the mass conservation equation (29a) and the momentum conservation equation (29b) gives

$$\frac{\partial h}{\partial t}+\frac{1}{B}\frac{\partial Q}{\partial x}=\frac{i}{B}. \tag{30a}$$

$$\begin{aligned}\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial h}{\partial x}\\ =gAS_0+\beta\frac{Q^2}{A^2}\left(B\left(\tilde{S}-S_0\right)+A_x^{\mathrm{V}}\right)-\frac{\lambda P}{8}\frac{Q\left|Q\right|}{A^2}+\beta_i\,u_i\,i-\frac{Q^2}{A}\beta'(x).\end{aligned} \tag{30b}$$

6. Steady gradually-varied flow equations

Any of the above formulations can be trivially modified for the case of steady flow, to give a pair of ordinary differential equations. In this case any of the forms of the mass conservation equation has the solution $Q=Q\left(x_0\right)+\int_{x_0}^{x}i\left(x'\right)\,\mathrm{d}x'$. If there is no distributed inflow i, then the solution is $Q=Q\left(x_0\right)=$ constant. We can now use $Q\left|Q\right|=Q^2$ in the momentum equations as the flow is unidirectional, and consider $\beta'(x)=0$ to give, from equation (28b)

$$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{B\left(\tilde{S}+A_x^{\mathrm{V}}/B\right)-\lambda PF^2/8}{1-\beta\mathsf{F}^2}, \tag{31}$$

where $\mathsf{F}^2 = Q^2B/gA^3$ is the square of the Froude number. In terms of η, equation (29b) gives

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{\beta\left(\tilde{S} + A_x^{\mathrm{V}}/B\right) - \dfrac{\lambda}{8}\dfrac{P}{B}}{1/\mathsf{F}^2 - \beta}. \tag{32}$$

In terms of the depth-like quantity $h = \eta - Z_0$, equation (30b) becomes

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 + \beta\left(\tilde{S} + A_x^{\mathrm{V}}/B - S_0\right)\mathsf{F}^2 - \dfrac{\lambda}{8}\dfrac{P}{B}\mathsf{F}^2}{1 - \beta\mathsf{F}^2}. \tag{33}$$

The expressions here are valid for non-prismatic channels, using the generalised definition of slope $\tilde{S}$ in equation (24). Other presentations usually give an equation like (33) for prismatic channels, such that $\tilde{S} = S_0$, and where the symbol S_{f} is used for the resistance term. We prefer to keep it explicit; it also shows how simple it is. It is interesting that for wide channels, $P \approx B$, all variation with the dependent variable on the right of the differential equations (32) and (33) is in $\mathsf{F}^2 = Q^2B/gA^3$.

7. Conclusions

The long wave equations for a straight slowly-varying channel of small slope have been derived using the integral mass and momentum conservation equations. Pairs of equations have been presented for three combinations of variables: cross-sectional area and discharge (A, Q); surface elevation and discharge (η, Q); and finally using a depth-like quantity h, which is surface elevation relative to an arbitrary axis possibly associated with the bed, also plus discharge, (h, Q). Gradually varied flow equations for steady flow with constant discharge have been presented, which are ordinary differential equations in terms of each of A, η, and h.

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Appendix A. Formulation in terms of upstream volume

A.1 Elimination of mass conservation equation

Here we introduce a quantity V so that we can express two dependent variables A and Q in terms of derivatives of V, which satisfies the mass conservation equation identically. This reduces the number of dependent variables to one, and reduces the number of equations to one, the momentum equation. Although that will be helpful in theoretical studies, for computations the two-equation system is probably still more useful.

Consider the volume of water $V(x,t)$ contained in a stream between the upstream boundary x_0 and the general point x:

$$V(x,t) = \int_{x_0}^{x} A(x',t)\,\mathrm{d}x'.$$

Differentiating with respect to x and using Leibniz' rule it follows that the cross-sectional area is given by the spatial derivative

$$A = \frac{\partial V}{\partial x}. \tag{A-1}$$

Now consider the time rate of change of volume which is increasing upstream due to inflow, and decreasing due to volume leaving by passing the general point. Hence,

$$\frac{\partial V}{\partial t} = I - Q, \tag{A-2}$$

where $I\,(x,t)$ is the net volume rate at which fluid is entering/leaving the channel from inflow. This has contributions from the inflow at the upstream end of the stream $Q(x_{0,}t)$, and distributed inflow i per unit length of channel such that

$$I = Q(x_{0,}t) + \int_{x_0}^{x} i(x',t)\,dx'. \tag{A-3}$$

Including the term $Q(x_0,t)$ on the right here has been suggested by Fatemeh Soroush (2011, Personal Communication), such that V can now more consistently be understood as the total volume in the channel. Fenton, Oakes & Aughton (1999) and Barlow, Fenton, Nash & Grayson (2006) used a different definition. Substituting equation (A-3) into equation (A-2) and solving for Q gives

$$Q = Q(x_0, t) + \int_{x_0}^{x} i\left(x', t\right)\, \mathrm{d}x' - \frac{\partial V}{\partial t}. \tag{A-4}$$

Substituting for Q from this and for A from equation (A-1) into the mass conservation equation (28a) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial x}\left(Q(x_0,t) + \int_{x_0}^{x} i(x', t)\, dx' - \frac{\partial V}{\partial t}\right) = i,$$

and evaluating the left side we obtain i, such that the equation is identically satisfied. This is not so surprising as it is a conservation-of-volume equation for the incompressible fluid.

A.2 Momentum equation in terms of upstream volume V

A single equation in terms of upstream volume V is obtained by substituting the relationships (A-1) and (A-4) into the momentum conservation equation (28b):

$$\begin{aligned}
&\frac{\partial^2 V}{\partial t^2} + 2\beta \frac{Q}{A}\frac{\partial^2 V}{\partial x \partial t} + \left(\beta \frac{Q^2}{A^2} - \frac{gA}{B(A)}\right)\frac{\partial^2 V}{\partial x^2} \\
&\quad - \frac{\lambda P(A)}{8}\left(\frac{Q\left(x_0, t\right) + \int_{x_0}^{x} i(x')\, \mathrm{d}x' - \partial V/\partial t}{\partial V/\partial x}\right)^2 + gA\tilde{S} \\
&\quad\quad = i\left(2\beta \frac{Q}{A} - \beta_i u_i\right) + \frac{dQ\left(x_0, t\right)}{dt} + \int_{x_0}^{x} \frac{\partial i}{\partial t}\, \mathrm{d}x' - \frac{gA}{B} A_x^{\mathrm{V}} + \frac{Q^2}{A}\beta'(x).
\end{aligned} \tag{A-5}$$

where for notational simplicity, single symbols Q and A have been retained in non-derivative terms and in the resistance term the usual Q^2 has been used instead of the formally more general $Q\,|Q|$. The momentum equation has become a second-order partial differential equation in terms of the single variable V. To solve problems numerically, this formulation is probably not of much practical help; as it is a second order equation in time, one would probably use an auxiliary variable $\partial V/\partial t$ and solve it as two first order equations anyway. However, the formulation as a single equation makes theoretical manipulations easier and provides insight into the nature of the equations and their solutions.

Appendix B. Energy conservation equation

Here the energy equation is derived using a similar approach to above, so that comparisons can be made with that based on momentum conservation. It will be found to require more coefficients to approximate integrals, plus there is the difficulty of approximating various forms of energy dissipation which occur throughout the fluid, rather than momentum loss at the boundary.

Consider the energy conservation equation (White 2009, §3.7), where there is no heat added or work done on the fluid in the control volume:

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\, e\, \mathrm{d}\mathcal{V} + \int_{\mathrm{CS}} (p + \rho e)\ \mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\, \mathrm{d}S = 0, \tag{B-1}$$

where e is the internal energy per unit mass of fluid, which in hydraulics is the sum of potential and kinetic energies $e = gz + \left(u^2 + v^2 + w^2\right)/2$. The first term in equation (B-1) becomes

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\, e\, \mathrm{d}\mathcal{V} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\left(gz + \frac{1}{2}\left(u^2 + v^2 + w^2\right)\right)\mathrm{d}\mathcal{V}. \tag{B-2}$$

The contribution of the first term $\rho g z$ using the elemental control volume in figure 1 is

$$\frac{\partial}{\partial t}\int\limits_{\mathrm{CV}} \rho g z \,\mathrm{d}\mathcal{V} = \rho g \Delta x \frac{\partial}{\partial t}\int\limits_{A} z\,\mathrm{d}A.$$

The integral is simply the first moment of area of the cross-section about the transverse y axis. If the surface rises by an amount $\delta\eta$ then the change in the first moment of area is simply $B\,\delta\eta\,(\eta+\delta\eta/2)$, and in the limit $\delta\eta\to 0$ the second part goes to zero, giving the contribution

$$\rho g \Delta x B \eta \frac{\partial \eta}{\partial t}. \tag{B-3}$$

The contribution of the second term in the integrand of equation (B-2) is more difficult to obtain, as we have the problem of integrating the square of the velocity over the section and taking the time mean:

$$\frac{\rho \Delta x}{2}\frac{\partial}{\partial t}\overline{\int\limits_{A}\left(u^2+v^2+w^2\right)\,\mathrm{d}A}.$$

To approximate this we introduce a coefficient α_0 here, defined by the generalised Coriolis coefficient (*e.g.* Fenton 2005):

$$\alpha_n U^{n+2} A = \alpha_n \frac{Q^{n+2}}{A^{n+1}} = \overline{\int\limits_{A}\left(u^2+v^2+w^2\right)\,u^n\,\mathrm{d}A}, \tag{B-4}$$

where α_n expresses the integral of the square of the wave speed weighted with respect to u^n. The contribution of the second term in equation (B-2) is then

$$\frac{\rho\Delta x}{2}\frac{\partial}{\partial t}\left(\alpha_0\frac{Q^2}{A}\right). \tag{B-5}$$

The second term in equation (B-1) gives zero contribution on all the transverse boundaries where $\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}} = 0$. On the upstream face its contribution is, using the hydrostatic pressure contribution $p=\rho g(\eta - z)$:

$$\int\limits_{A}\left(p+\rho e\right)\,\mathbf{u}.\hat{\mathbf{n}}\,\mathrm{d}S = -\rho \int\limits_{A}\overline{\left(g\eta + \frac{1}{2}\left(u^2+v^2+w^2\right)\right)u}\,\mathrm{d}A.$$

As surface elevation η is a constant at any section, it is constant over the integral, and the contribution of the first term in the integrand is

$$-\rho\int\limits_{A} g\eta\, u\,\mathrm{d}A = -\rho g\eta\int\limits_{A} u\,\mathrm{d}A = -\rho g \eta Q,$$

again where fluctuations of the free surface have been ignored. The second term in the integrand again cannot be evaluated exactly, and we use kinetic energy coefficient α_1 as defined in equation (B-4) such that we write

$$\overline{\int\limits_{A}\left(u^2+v^2+w^2\right)\,u\,\mathrm{d}A} = \alpha_1 U^3 A = \alpha_1\frac{Q^3}{A^2}.$$

It is surprising that for many years an integral just in terms of u^3 was used to calculate this contribution. Strelkoff (1969, eqn 21) used the correct definitions in terms of integrals of kinetic energy; Fenton (2005) noted that turbulent contributions should be included.

The total contribution on the upstream face is now

$$-\rho\left(g\eta Q + \alpha_1\frac{Q^3}{2A^2}\right).$$

The contribution on the downstream face will be positive, and composed of a term like this plus its derivative with respect to x, times Δx. The net contribution from upstream and downstream faces is then

$$\rho \Delta x \frac{\partial}{\partial x}\left(g\eta Q+\alpha_1 \frac{Q^3}{2A^2}\right). \tag{B-6}$$

Adding the terms (B-3), (B-5) and (B-6) and dividing by $\rho\Delta x$ we obtain the contribution to the equation thus far as

$$gB\eta\frac{\partial \eta}{\partial t}+\frac{\alpha_0}{2}\frac{\partial}{\partial t}\left(\frac{Q^2}{A}\right)+\frac{\partial}{\partial x}\left(g\eta Q+\alpha_1 \frac{Q^3}{2A^2}\right). \tag{B-7}$$

Now consider energy entering due to the inflow. Let the head of this be H_i, then the net rate of energy flow leaving the control volume is

$$-\rho\,\Delta x\,gH_i i. \tag{B-8}$$

In practical problems it is quite possible that the incoming velocity head will be destroyed by mixing, and just the elevation head η_i should be used here.

Now we have to include energy losses, diffused through the fluid, which are more complicated than the momentum losses which are due just to the action of momentum exchange at the boundary. Here we can use the expression for the resistance force (12) $-\rho\Delta x\,(\lambda/8)\,PQ\,|Q|\,/A^2$ and multiply by the mean velocity at a section Q/A to give the rate of energy dissipation, however the location, mechanisms, and rate of the energy losses are different from the momentum losses so we use a different coefficient as, for example, Yen (1973, after his equation 66):

$$-\frac{\lambda_{\mathrm{e}}}{8}\rho\frac{Q^2}{A^2}P\Delta x\times\frac{Q}{A}, \tag{B-9}$$

where λ_{e} is a dimensionless energy loss coefficient in the same spirit as λ for the stress around the boundary. The momentum and energy coefficients $\lambda=\lambda_{\mathrm{e}}$ would agree in the case of uniform steady flow. Adding contributions (B-7), (B-8) and (B-9) and regrouping:

$$gB\eta\frac{\partial \eta}{\partial t}+\frac{\alpha_0}{2}\frac{\partial}{\partial t}\left(\frac{Q^2}{A}\right)+\frac{\partial}{\partial x}\left(g\eta Q+\alpha_1 \frac{Q^3}{2A^2}\right)=gH_i i-\frac{\lambda_{\mathrm{e}}P}{8}\frac{Q^2\,|Q|}{A^3},$$

which is the equivalent of the unexpanded momentum equation (13), but here the time derivatives are more complicated. Expanding and using equation (28a) to eliminate $\partial A/\partial t$, (29a) for $\partial\eta/\partial t$, (27) for $\partial A/\partial x$, and dividing by Q/A gives the conservation of energy equation

$$\begin{aligned}\alpha_0\frac{\partial Q}{\partial t}+\left(gA-\alpha_1\frac{Q^2B}{A^2}\right)\frac{\partial \eta}{\partial x}+\frac{\alpha_0+3\,\alpha_1}{2}\frac{Q}{A}\frac{\partial Q}{\partial x}=\\ \alpha_1\frac{Q^2B}{A^2}\tilde{S}-\frac{\lambda_{\mathrm{e}}P}{8}\frac{Q\,|Q|}{A^2}+\left(\alpha_0\frac{Q^2}{2A^2}-g\eta+gH_i\right)\frac{iA}{Q}+\alpha_1\frac{Q^2}{A^2}A_x^{\mathrm{V}}-\frac{Q^2}{2A}\frac{\partial\alpha_1}{\partial x}\end{aligned} \tag{B-10}$$

This can be compared with the momentum equation (29b)

$$\frac{\partial Q}{\partial t}+\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial \eta}{\partial x}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}=\beta\frac{Q^2B}{A^2}\tilde{S}-\frac{\lambda P}{8}\frac{Q\,|Q|}{A^2}+\beta_i\,u_i\,i+\beta\frac{Q^2}{A^2}A_x^{\mathrm{V}}-\frac{Q^2}{A}\beta'(x).$$

The structure of the two equations is the same. The differences lie in the coefficients and in the nature of the inflow term on the right. If all α_0, α_1 and β were unity the only difference would be between the inflow terms, and the λ and λ_{e}, which are known to be equal for uniform flow. It is the assertion of this work, however, that the energy dissipation is generally more complicated than momentum loss at the boundary, and the coefficient λ_{e} has to describe more complicated processes than λ from the boundary stresses. The momentum approach is to be preferred.