Institute of Hydraulic and Water Resources Engineering, Vienna University of Technology

River Engineering

Tutorial Sheet 3 – Gradually-varied flow equation

The differential equation governing the water depth in a steady flow in a stream is

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\tilde{S} - Q^2/K^2}{1 - \beta \mathsf{F}^2}, \qquad \text{(Gradually-varied flow equation)}$$

where h is the water depth, $\tilde{S}$ is the bottom slope, Q is the discharge, K is the conveyance, and $\beta\mathsf{F}^2$ (F the Froude number) expresses the fluid inertia effects, which can be ignored for a channel on a sufficiently mild slope.

We have from the G-M-S equation for uniform flow in a *wide* channel of constant slope S:

$$q = k_{\mathrm{St}} h_0^{5/3} \sqrt{S}$$

where q is the discharge per unit width ($\mathrm{m}^2\mathrm{s}^{-1}$) and h_0 is the *normal depth* for the channel.

A wide canal has a bed slope of 1×10^{-4}, $k_{\mathrm{St}} = 50$, and flows uniformly with a depth of $2\,\mathrm{m}$. It is proposed to install a barrage on it which will raise the water level at the barrage to $h = 3\,\mathrm{m}$. You are to investigate how far upstream the enhanced water levels will affect the channel and enable irrigation of the surrounding land.

1. For sufficiently mild slope the term $\beta\mathsf{F}^2 = \beta q^2/gh^3$ will be sufficiently small that it can be ignored. Show that the GVFE for a wide channel can be written

   $$\frac{\mathrm{d}h}{\mathrm{d}x} \approx S\left(1 - \left(\frac{h_0}{h}\right)^{10/3}\right). \qquad \text{(Wide-channel/small-Froude approximation to GVFE)}$$

   Using $\beta \approx 1.05$, check that $\beta\mathsf{F}^2$ is sufficiently small in our case that this is accurate enough.
2. Using a spreadsheet or any other software, solve the Wide-channel/small-Froude approximation to GVFE numerically using Euler's method, starting with $h(0) = 3\,\mathrm{m}$ and stepping upstream backwards in x (*i.e.* Δx is negative) with steps of $5000\,\mathrm{m}$ until the water level is within $1\,\mathrm{cm}$ of the normal depth.
3. How far upstream is it? Are you surprised?
4. Plot the actual surface elevation, which is h plus the local bed elevation.

   Plot all answers to the following questions also.
5. Repeat your calculations with steps of $2500\,\mathrm{m}$. How do your two answers compare – at $x = -20\,\mathrm{km}$ for example?
6. Refine your solution at every $5000\,\mathrm{m}$, using Richardson extrapolation, which in this case can be written for a point X:

   $$h_{\text{Improved}}(X) = 2h_{\Delta/2}(X) - h_{\Delta}(X).$$

7. You could experiment with a different slope, say a steep slope of $S = 1 \times 10^{-3}$, and see how the length of the backwater changes. You will have to use smaller steps! If $\beta\mathsf{F}^2$ is no longer negligible, include it.