Institute of Hydraulic and Water Resources Engineering, Vienna University of Technology

River Engineering

Tutorial Sheet 1 – Resistance laws for steady uniform flow

1. For a trapezoidal cross-section, where the bottom width is W, the depth is h, and the side slopes are (H:V) $\gamma : 1$, show the following properties:

Top width	$B =$	$W + 2\gamma h$
Area	$A =$	$h\,(W + \gamma h)$
Wetted perimeter	$P =$	$W + 2\sqrt{1+\gamma^2}h$

2. Verify that the general relationship $B = dA/dh$ holds true for this type of section.

3. Use the Gauckler-Manning-Strickler formulation and the results for a trapezoidal section to obtain an expression for the steady uniform discharge as a function of depth h:

$$Q = \frac{1}{n}\frac{\left(h\,(W+\gamma h)\right)^{5/3}}{\left(W + 2h\sqrt{1+\gamma^2}\right)^{2/3}}\sqrt{S_0} \tag{1}$$

4. A commonly-encountered problem in river and canal hydraulics is to compute the normal (uniform flow) depth for a given discharge by numerically solving the transcendental equation you obtained in question 3. Here we consider methods for solving the problem:

   a. A traditional method is to plot the relationship and read the solution off. However, this is not suited to computer programs.

   b. Trial and error – substitute numerical values of h in until the required discharge is obtained, also not suited to computer programs.

   c. Solve the equation by a numerical method for solving transcendental equations such as direct iteration, bisection, or Newton's method, or most simply, using Solver in Excel.

   d. The direct iteration scheme in the lectures works well. Examination of equation (1) for $h \ll W$, the usual case, shows that variation with h of two of the terms combined will be slow: $(W + \ldots)^{5/3}/(W + \ldots)^{2/3}$ is almost constant for h relatively small compared with the width W. This suggests extracting out the leading $h^{5/3}$

$$Q = \frac{\sqrt{S_0}}{n}\frac{(A(h)/h)^{5/3}}{P^{2/3}(h)} \times h^{5/3},$$

   giving the direct iteration scheme

$$h = \left(\frac{Qn}{\sqrt{S_0}}\right)^{3/5}\frac{P^{2/5}}{A/h} = \left(\frac{Qn}{\sqrt{S_0}}\right)^{3/5}\frac{\left(W + 2h\sqrt{1+\gamma^2}\right)^{2/5}}{W+\gamma h},$$

   which is then evaluated repeatedly.

   e. Obtain the corresponding iteration scheme for the Weisbach-Chézy formulation.

$$h = \left(\frac{\lambda Q^2}{8gS_0}\right)^{1/3}\frac{P^{1/3}}{A/h} = \left(\frac{\lambda Q^2}{8gS_0}\right)^{1/3}\frac{\left(W + 2h\sqrt{1+\gamma^2}\right)^{1/3}}{W+\gamma h}.$$

   f. Note that such direct iteration does not work for all equations – it did work here because we made sure that our equation was a weakly varying function of h.

5. Consider a canal with $W = 10\,\mathrm{m}$, side slopes of 2:1, a slope of 0.001, Manning $n = 0.04$. For

a flow of $20\,\mathrm{m}^3\,\mathrm{s}^{-1}$, use the direct iteration scheme in question 4 to obtain the normal depth h. (*Ans:* $h = 1.64\,\mathrm{m}$ – if we start with $h = 3$, we then obtain 1.533, 1.646, 1.637, and it has converged to within a millimetre).

6. Consider just when a channel may be considered hydraulically-wide. Calculate the discharge divided by the top width B, using a trapezoidal channel and the Weisbach-Chézy formula

$$\frac{Q}{B} = \frac{A}{B}\sqrt{\frac{8g}{\lambda}\frac{A}{P}S_0},$$

which we will compare with the result for $q = Q/B$ for the hydraulically-wide channel from the lecture notes

$$q = \sqrt{\frac{8g}{\lambda}}h^{3/2}\sqrt{S_0}.$$

From question 1 we substitute values of A, B, and P as functions of h and re-arrange to give, dividing through top and bottom of fractions by W:

$$\frac{Q}{Bh\sqrt{h}} = \sqrt{\frac{8g}{\lambda}S_0}\frac{1+\gamma h/W}{1+2\gamma h/W}\sqrt{\frac{1+\gamma h/W}{1+2\sqrt{1+\gamma^2}h/W}}.$$

We could plot curves for various values of γ and h/W, however it is a little more revealing to obtain a *power series approximation*[1] to quantify it mathematically. Strangely we see that almost everywhere on the right side γ and h/W are combined in the form $\gamma h/W$. If we conduct power series operations assuming that this quantity is small we obtain

$$\frac{Q}{Bh\sqrt{h}} = \sqrt{\frac{8g}{\lambda}S_0}\left(1+\frac{\gamma h}{W}\left(-\sqrt{\frac{1}{\gamma^2}+1}-\frac{1}{2}\right)+\text{terms in }\left(\frac{\gamma h}{W}\right)^2\text{ etc.}\right).$$

Hence we find the expression for the fractional error for discharge in using the wide-channel result as

$$\frac{Q/B-q}{q} \approx -\frac{\gamma h}{W}\left(\sqrt{\frac{1}{\gamma^2}+1}+\frac{1}{2}\right),$$

and as we might write $\gamma h = \Delta B/2$, where ΔB is the effect on top width due to both sloping sides, we obtain

$$\frac{Q/B-q}{q} \approx -\tfrac{1}{2}\frac{\Delta B}{W}\left(\sqrt{\frac{1}{\gamma^2}+1}+\frac{1}{2}\right).$$

For a typical value of $\gamma = 1.5$ we find that the fractional error in calculating discharge from the wide channel value is about $-0.85\,\Delta B/W$ or about 85% of the fractional contribution of sloping sides to the surface width. In many cases that is small. The fact that it is negative feels right, as the sides sloping inwards reduce the carrying capacity from what the channel would have if the sides were vertical.

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[1] **Revision of power series operations**
To make engineering approximations such as that in question 6, it is often helpful to use simple power series approximations, such as use of the *binomial theorem* for small δ:

$$(1+\delta)^\nu \approx 1+\nu\delta+\text{terms in }\delta^2\text{ etc.}$$

For example, what is the effect on area of a square if we increase the length by 10%? We obtain $(1.1)^2 \approx 1+2\times 0.1 = 1.2$, giving 20% (yes, I know you would have got the exact answer 21% very quickly, but that is not the point here). Similarly products of such quantities are easily handled, for example, calculate an approximation to $(1+\alpha\delta)(1+\beta\delta)$. We simply get

$$1+(\alpha+\beta)\,\delta+\text{terms in }\delta^2\text{ etc..}$$

These examples are trivial, but in practice it can be very powerful in giving us insight.

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Tutorial Sheet 2 – Measurement of Discharge

1. In lectures we had a scheme for obtaining the mean velocity in the vertical by measuring the velocity at two points. A simpler scheme is where the velocity is measured at a single point $0.6\,d$ from the surface. Show that the velocity at this point ($0.4\,d$ above the bottom) is a close approximation to the mean velocity for velocity profiles in (a) & (b) by integrating to find the mean velocity, and then finding the value of z for which the velocity is equal to the mean.

   a. The Prandtl-von Kármán law for turbulent flow over a rough bed:

   $$u = \frac{u_*}{\kappa}\log\frac{z}{z_0}.$$

   You will need the result that $\int \log z\,dz = z\log z - z + C$. (Ans: $z/d = e^{-1} = 0.37$, such that the relative depth of measurement should be 0.63).

   b. The simple $1/7$ law, sometimes used as a simpler model for turbulent velocity distributions:

   $$u = U\left(\frac{z}{d}\right)^{1/7},$$

   where U is the surface velocity. (Ans: $z/d = 0.39$, relative depth 0.61)

   c. And, only if you felt like a bit of mathematical fun, do it for the general power law

   $$u = U\left(\frac{z}{d}\right)^{\nu}.$$

   (Ans: $z/d = (1+\nu)^{-1/\nu}$. Plot it and be astonished how little it varies for $0 < \nu < 0.25$. Then take the limit as $\nu \to 0$ and be astonished that it approaches the value $e^{-1} = 0.37$, the same as for the logarithmic law. This is a glorious coming-together of mathematics, for it is Euler's formula for e: $\lim_{x\to 0}(1+x)^{1/x}$!)

2. The Australian water industry uses a non-*SI* unit for flow, namely Megalitre per day (Ml/d).

   a. Verify that a cube $10\,\mathrm{m} \times 10\,\mathrm{m} \times 10\,\mathrm{m}$ contains 1 Megalitre.

   b. It is often said that 1 Megalitre is roughly the size of a $50\,\mathrm{m}$ Olympic swimming pool. Make some estimates of other dimensions and test the truth of that statement.

   c. Show that $1\,\mathrm{m}^3\,\mathrm{s}^{-1} = 86.4$ Ml/d. Often, "$\mathrm{m}^3\,\mathrm{s}^{-1}$" is referred to as "cumec".

3. Hydrographers sometimes use a unit of velocity of km/day for calibrating their propeller meters and presenting their data. This is not as silly as it sounds.

   a. Verify that if velocities in km/day are integrated over cross sectional areas specified in m^2, the result is directly Ml/d. The velocity in km/day gives a practical idea of the distance that the water will travel in a day.

   b. Verify that the velocity in km/day is also roughly the velocity in $\mathrm{cm\,s^{-1}}$, also useful for practical considerations, and show that a velocity of 30 km/day is $34.7\,\mathrm{cm\,s^{-1}}$.

4. Consider the data in Table 1 from a gauging exercise on a large distribution canal in northern Victoria. Calculate the flow in Ml/d using

   a. The Mean-Section Method, and

   b. The Mid-Section Method.

y (m)	Depth (m)	$\bar{u}$ (km/day)
9.5	0.00	0.00
12	1.02	24.31
14	1.64	30.21
15	1.86	28.33
16	2.04	32.76
17	2.26	32.36
18	2.26	33.03
19	2.34	35.72
20	2.44	37.19
21	2.54	38.13
22	2.58	36.79
23	2.58	36.65
24	2.58	39.74
25	2.58	37.33
26	2.52	37.06
27	2.48	37.46
28	2.40	34.51
29	2.42	35.72
30	2.30	29.54
31	2.24	27.66
32	2.08	26.05
33	1.80	25.92
34	1.54	24.31
36	0.78	13.70
38	0.00	0.00

Table 1. Data from an irrigation canal

Channel control – theoretical rating curves using a trapezoidal section

As shown in the lecture notes, if there is no local control such as by a weir, rock ledge, *etc.*, the friction in the channel controls the relationship between stage and discharge, and this relationship is given theoretically by Manning's or Chézy's laws. In practice the roughness varies around the section, and at the moment there is no theory for that, so that the rating curve is determined by measurement. However it is useful for understanding and for some theoretical studies, to model river cross-sections by a trapezoidal cross-section, typically found in canals.

1. For depth varying between 0 and $3\,\mathrm{m}$,
   a. Plot the rating curve obtained from equation (1) on log-log axes,
   b. Also plot it using $\left(\sqrt{Q}, d\right)$ axes.
   c. Criticise both types of plot.

Local control – theoretical rating curves using a weir formula

Sharp-crested rectangular weir

If we now consider a finite width of weir b, and head H, it can be shown using dimensional reasoning that

$$Q = C\sqrt{g}\,b\,H^{3/2},$$

where C is a dimensionless coefficient. There is relatively little variation about a constant value of $C \approx 0.6$.

Sharp-crested triangular weir

Dimensional reasoning and experimental results give

$$Q = 0.44\sqrt{g}\tan\frac{\theta}{2}\,H^{2.5},$$

where θ is the angle between the sides of the triangle.

1. Plot the rating curves using $\left(\sqrt{Q}, d\right)$ axes for both sharp-crested weir formulae over a range of $0 \leqslant H \leqslant 1\,\mathrm{m}$ for $b = 2\,\mathrm{m}$ for the rectangular weir and $\theta = 90^\circ$ for the triangular weir.

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Tutorial Sheet 3 – Rating curves – correction for unsteady effects

1. **Kinematic wave speed for a general section**

   The speed at which the main body of disturbances move in rivers is the *kinematic wave speed*, which we denote by the symbol c. The Kleitz-Seddon law for c is

   $$c = \frac{dQ}{dA} = \frac{1}{B}\frac{dQ}{dd},$$

   where Q can be interpreted as being a function of A or d respectively, given by Manning or Chézy's law.

   a. Show that the value from Manning's law can be written in terms of the mean velocity of flow, Q/A as

   $$c = \frac{Q}{A}\left(\frac{5}{3} - \frac{2}{3}\frac{A}{P}\frac{dP}{dA}\right).$$

   b. It is more convenient to express the derivative in terms of the local water height, and so using the fact that $dA/dd = B$ (interpret that physically) show that

   $$c = \frac{Q}{A} \times \frac{5}{3} \times \left(1 - \frac{2}{5}\frac{A}{PB}\frac{dP}{dd}\right).$$

   The quantity dP/dd is easily shown to be $2\sqrt{1+\gamma^2}$, where γ is the batter slope, so that this might have a value of, for $\gamma = 2$ say, of $2\sqrt{5} \approx 4.5$. This means that the relative contribution due to the perimeter changing is roughly $2 \times A/BP$, which is the twice the mean depth divided by the wetted perimeter, which will be small for typical wide shallow channels. Hence, ignoring this contribution we see that if Manning friction is used, the advection velocity, the velocity with which disturbances are transported, is about $5/3$ times the mean velocity in the waterway, Q/A. If we had used Chézy friction this factor would have been $3/2$.

2. **Correcting for unsteady effects in inferring discharge from stage**

   Consider a stream of slope 0.001, with a trapezoidal section 10m wide at the bottom with batter slopes of 2:1, Manning's friction coefficient of 0.04, carrying a flow of $20\,\mathrm{m}^3\,\mathrm{s}^{-1}$. The expression for correcting for unsteadiness is

   $$Q = Q_\mathrm{r}(\eta)\sqrt{\underbrace{\underbrace{1}_{\text{Rating curve}} + \frac{1}{cS}\frac{d\eta}{dt}}_{\text{Jones formula}} \underbrace{- \frac{\nu}{c^3 S}\frac{d^2\eta}{dt^2}}_{\text{Diffusion term}}} \tag{1}$$

   where $Q_\mathrm{r}(\eta)$ is the rated discharge for the station as a function of stage, c is the kinematic wave speed which can be obtained from expressions above, and the coefficient ν is the diffusion coefficient in advection-diffusion flood routing, given by:

   $$\nu = \frac{K}{2B\sqrt{S}}. \tag{2}$$

   a. Calculate the normal depth for the flow of $20\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

   b. For that depth and flow, calculate the coefficients of the derivative terms in equation (1) and estimate for what values of $d\eta/dt$ and $d^2\eta/dt^2$ the unsteady terms start to become significant, say greater than 5%.

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Tutorial Sheet 4 – Wave propagation and the low-inertia approximation

1. Consider the advection-diffusion equation

$$\frac{\partial d}{\partial t} + c\frac{\partial d}{\partial x} = \nu\,\frac{\partial^2 d}{\partial x^2},$$

where c is the kinematic wave speed, and ν is a diffusion coefficient which are given by

$$c = \frac{\sqrt{S}}{B}\frac{dK}{dd} \quad \text{and} \quad \nu = \frac{K}{2B\sqrt{S}},$$

where $K = 1/n \times A^{5/3}/P^{2/3}$, the conveyance.

a. Show that if we introduce a length scale L which is the length of the reach being considered and a time scale L/c, such that in dimensionless variables $X = x/L$ and $T = tc/L$, the advection-diffusion equation can be written

$$\frac{\partial d}{\partial T} + \frac{\partial d}{\partial X} = \frac{\nu}{cL}\,\frac{\partial^2 d}{\partial X^2},$$

which is to be solved on an interval for $0 \leqslant X \leqslant 1$, so that the only parameter present, which expresses the relative importance of diffusion, is the dimensionless diffusion coefficient ν/cL, which is a Péclet number, or $1/R$ where R is a Reynolds number.

b. Show for a river which is rather wider than it is deep, that this parameter is of the order of the ratio of the mean depth of the river to its fall over the region of interest.

2. Kinematic wave speed

In the last sheet you showed that for a general section the kinematic wave speed can be written

$$c = \frac{Q}{A}\left(\frac{5}{3} - \frac{2}{3}\frac{A}{BP}\frac{dP}{dd}\right).$$

For a trapezoidal channel where W is the bottom width, γ is the batter slope (H:V): $B = W + 2\gamma d$, $A = d\,(W + \gamma d)$, and $P = W + 2d\sqrt{1+\gamma^2}$, show that

$$c = \frac{5}{3}\frac{Q}{A}\left(1 - \frac{4d}{5}\frac{(W+\gamma d)\sqrt{1+\gamma^2}}{(W+2\gamma d)\left(W+2d\sqrt{1+\gamma^2}\right)}\right)$$

3. Consider a trapezoidal channel of bed width $10\,\mathrm{m}$ and batter slopes $2:1$ (H:V) excavated to a slope of 1 in 10,000, and Manning's $n = 0.03$, typical of a large irrigation channel in the Murray Valley. For a depth of $2.5\,\mathrm{m}$,

a. Calculate the discharge Q and the mean velocity in the channel (*Ans:* $Q = 18.3\,\mathrm{m}^3\,\mathrm{s}^{-1}$, $U = 0.49\,\mathrm{m\,s}^{-1}$).

b. Calculate the kinematic wave speed and also calculate it using the wide-channel approximation. Why is the latter not so accurate here? Calculate the "dynamic wave speed", $c_d = \sqrt{gA/B}$. (*Ans:* $c = 0.69\,\mathrm{m\,s}^{-1}$, $0.81\,\mathrm{m\,s}^{-1}$, $c_d = 4.3\,\mathrm{m\,s}^{-1}$).

c. Calculate the estimated time of travel of a disturbance over a distance of $10\,\mathrm{km}$. (*Ans:* $4\,\mathrm{h}$).

d. Estimate the relative importance of diffusion (*Ans:* $3/10\times$ Depth/Drop $= 0.75$, important!).

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Tutorial Sheet 5 – Computational hydraulics – upwinding schemes

1. Consider a river with a cross-sectional area of $200\,\mathrm{m}^2$. The discharge is constant, $100\,\mathrm{m}^3\,\mathrm{s}^{-1}$. At a point $x = 0$ an irrigation drain is discharged from $t = 0$ to $t = 15\,\mathrm{min}$, with a negligible contribution to discharge but with a salt inflow of $10\,\mathrm{kg}\,\mathrm{s}^{-1}$. The river water originally does not contain any salt. You may neglect effects of diffusion, so that you use the advection equation with salt concentration as the dependent variable.

   Find the concentration at (a) $x = 2\,\mathrm{km}$ and $t = 1800\,\mathrm{s}$, and at (b) $x = 1\,\mathrm{km}$ and $t = 2400\,\mathrm{s}$ by using your knowledge of the advective nature of the problem, that is, by drawing a sketch of the (x, t) plane and showing characteristics marking the beginning and end of the salt "plug".

   Note: it would be foolish not to use this approach in such a problem!

2. It is desired to solve the kinematic wave equation

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0,$$

   where c is usually a function of distance co-ordinate x and time t.

   An attempt is to be made to develop a finite difference scheme which uses backwards differences, in order to build in something of the advective nature of solutions of the differential equation. The approximation to the derivative $df(x)/dx$ is

$$\frac{df(x)}{dx} \approx \frac{f(x) - f(x-\delta)}{\delta}.$$

   a. Using this, develop a finite difference scheme for solution of the differential equation, with a "forwards time" and the "backwards space" approximation, using a time step of Δ and a space step δ.

   b. Describe with the aid of diagrams the physical nature of your scheme.

   c. Do you think the scheme is stable, and what, if any limits there are to its stability in terms of the time step Δ and space step δ?

   d. For the interval $x = 0$ to $x = 1$, divided up into ten equal intervals, solve the advection equation for $c = 1$ for three time steps for (i) $\Delta = 0.05$, and (ii) $\Delta = 0.12$, with the *initial conditions* $\phi(x, 0) = 0$ except for the triangular pulse $\phi(0.2, 0) = 0.5$, $\phi(0.3, 0) = 1.0$, $\phi(0.4, 0) = 0.5$, and the *boundary condition* $\phi(0, t) = 0$ for all t.

   e. Do the behaviour of the solutions agree with your stability expectations? ((i) should be stable but should show the numerical diffusivity associated with conventional finite difference schemes; (ii) should be unstable)

3. Consider a $1\,\mathrm{km}$ length of a rectangular concrete canal of slope $S = 0.0001$, Manning's $n = 0.015$, $B = 2\,\mathrm{m}$ and an initial depth of $1\,\mathrm{m}$, with a flow increase such that the flow at the upstream end increases by 25% uniformly over 1min, and then decreases uniformly over another 1 minute back to the original flow.

   a. Calculate the initial flow.

   b. If we use 10 space steps, each of $\delta = 100\,\mathrm{m}$, show that for the initial flow, a time step of $\Delta = 2\,\mathrm{s}$ satisfies both common stability criteria. (That is a very small time step, but it is characteristic of such explicit schemes which make no upwinding allowance.)

   c. Compute the propagation of the disturbance in the canal for 10 minutes, using a centred-time formula for the advection term and a centred-time formula for the diffusion term. For the downstream boundary condition at $x = L = 1000\,\mathrm{m}$, use the two-point backward difference formulae for the first derivative, and the following backwards difference formula for the second derivative:

$$\frac{\partial^2 Q}{\partial x^2}(L,t) = \frac{Q(L,t) + Q(L-2\delta,t) - 2Q(L-\delta,t)}{\delta^2}.$$

It is probably easiest to use a spreadsheet with space horizontal (11 columns for $x = 0, 100, ..., 1000$), and then 300 rows for time, easily copied down, of course.

d. Plot the inflow hydrograph $Q(0,t)$, the triangle wave, plus the outflow hydrograph $Q(L,t)$. Observe the large effect of diffusion even in this short channel.

e. It is interesting to plot several plots of Q against x at several times – you should find that the wave in the channel is rather longer than you might expect, given the outflow hydrograph.

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Tutorial Sheet 6 – Loose-boundary hydraulics

1. The slope of a wide stream is 0.0001. A typical grain size on the bed is $\phi = 1\,\mathrm{mm}$. Use Yalin's approximation to Shields' threshold data to obtain the value of flow depth which will correspond to the initiation of sediment transport. ($\nu = 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $\rho_s/\rho = 2.65$). (*Ans:* $0.6\,\mathrm{m}$).
2. It is intended to study the initiation of sediment transport in that stream with the aid of a physical model where the moveable bed will be formed of light particles with a specific gravity of 1.05. What must be the grain size ϕ_m of the model bed material and the slope S_{0m} if the model flow depth is $d_m = 10\,\mathrm{cm}$? (*Ans:* $3\,\mathrm{mm}$ and 0.00006 – almost impossible to achieve on a laboratory scale).
3. The steady flow in the central part of a wide river has a flow depth of $2\,\mathrm{m}$ and slope 0.00067. The bed is a uniform sand of grain size $2\,\mathrm{mm}$. Determine the specific volumetric transport rate q_{sb} using Bagnold's bed-load formula, but in terms of an as-yet unknown u_b. ($q_{sb} \approx 0.36\,u_b$ L/s per metre width).

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Tutorial Sheet 7 – Computational hydraulics

1. Consider the Taylor series for a function at $x+\delta$ in terms of its derivatives at x:

$$\phi(x+\delta) = \phi(x) + \delta\,\frac{d\phi}{dx}(x) + \frac{\delta^2}{2!}\,\frac{d^2\phi}{dx^2}(x) + \frac{\delta^3}{3!}\,\frac{d^3\phi}{dx^3}(x) + \dots\ .$$

By considering similar series for a function $\phi(x,t)$ at two points, $(x+\delta,t)$ and $(x-\delta,t)$, show that

a. The centred-difference approximation for $\partial\phi/\partial x$ is

$$\frac{\partial\phi}{\partial x}(x,t) = \frac{\phi(x+\delta,t)-\phi(x-\delta,t)}{2\delta} + O(\delta^2),$$

where the Landau symbol $O(\delta^2)$ means ”at least of the order of δ^2”.

b. The centred-difference approximation for $\partial^2\phi/\partial x^2$ is:

$$\frac{\partial^2\phi}{\partial x^2}(x,t) = \frac{\phi(x+\delta,t)-2\,\phi(x,t)+\phi(x-\delta,t)}{\delta^2} + O(\delta^2).$$

2. Verify to your own satisfaction the lecturer’s use of the *shift or advection operator*

$$e^{\delta\,d/dx}\,\phi(x) = \phi(x+\delta),$$

by writing the operator as a power series expansion, treating the exponent as if it were an algebraic quantity, such that $(\delta\,d/dx)^n = \delta^n\,d^n/dx^n$, and comparing with the Taylor series for $\phi(x+\delta)$.

3. Consider the advection equation

$$\frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} = 0.$$

a. Justify to yourself the advective computational procedure adopted by the lecturer, where he argued that this implied $\partial/\partial t = -u\,\partial/\partial x$, and as a general Taylor series solution to a partial differential equation *in the absence of boundary conditions* is

$$\begin{aligned}\phi(x,t+\Delta) &= \phi(x,t) + \Delta\frac{\partial\phi}{\partial t}(x,t) + \frac{\Delta^2}{2!}\,\frac{\partial^2\phi}{\partial x^2}(x,t) + \dots\\ &= e^{\Delta\,\partial/\partial t}\,\phi(x,t)\\ &= e^{-\Delta u\,\partial/\partial x}\,\phi(x,t)\\ &= \phi(x-u\Delta,t).\end{aligned}$$

b. That solution is exact for u constant. Why is it not exact for $u(x,t)$?

c. How might you modify the scheme if $x-u\Delta$ was outside the computational domain?

4. In the case of the *diffusion operator*

$$e^{\nu\Delta\,\partial^2/\partial x^2} = 1 + \nu\Delta\,\frac{\partial^2}{\partial x^2} + \frac{(\nu\Delta)^2}{2!}\,\frac{\partial^4}{\partial x^4} + \dots,$$

see if you can earn yourself First Class Honours, the lecturer’s thanks and a financial consideration from him, by being able to interpret this in similarly simple terms to the shift or advection operator. The answer is probably that of the next question, but there might be a simpler interpretation ...

5. The *diffusion operator*: consider a function which is approximated by a finite Fourier series:

$$\phi(x,t) = \sum_{j=1}^{N} A_j\, e^{ijkx},$$

where $k = 2\pi/L$. Arguing as before, the general solution is the Taylor series solution $\phi(x, t + \Delta) = e^{\Delta\,\partial/\partial t}\,\phi(x,t)$, and if we have the diffusion equation $\partial\phi/\partial t = \nu\partial^2\phi/\partial x^2$ then we can replace time differentiation by two space derivatives so that this can be written

$$\begin{aligned}
\phi(x, t+\Delta) &= e^{\nu\Delta\,\partial^2/\partial x^2}\,\phi(x,t), \\
&= e^{\nu\Delta\,\partial^2/\partial x^2} \sum_{j=1}^{N} A_j\, e^{ijkx} \\
&= \sum_{j=1}^{N} A_j\, e^{\nu\Delta\,\partial^2/\partial x^2} e^{ijkx} \\
&= \sum_{j=1}^{N} A_j\, e^{-\nu\Delta\, j^2k^2} e^{ijkx},
\end{aligned}$$

which is an exact solution of the diffusion equation. That is, after a time step Δ, the jth component of the Fourier series is reduced in magnitude by a factor $e^{-\nu\Delta\, j^2k^2}$, which is very much smaller for high frequency components, large j.

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Tutorial Sheet 8 – Water quality

It is proposed to discharge wastewater into a previously unpolluted river having a minimum flow of 4 m^3s^{-1}. The wastewater flow will not exceed 1.5 m^3s^{-1}. It has a BOD_u of up to 31.5 g m^{-3}. The river is about $10\,\mathrm{m}$ wide and $1.5\,\mathrm{m}$ deep. The initial concentration of dissolved oxygen is $C_0 = 7.5$ g m^{-3}.

For this problem you may assume that the discharge is steady.

The initial dissolved oxygen concentration in the river is 7.5 g m^{-3} after mixing.

Based on laboratory studies the k_1 value for deoxygenation at the temperature of the river and effluent is 0.26 d^{-1} and the value of k_2 for reaeration is 0.41 d^{-1}. At $25°$C, the temperature of river and effluent, the saturation concentration of oxygen is $C_s = 8.2$ g m^{-3}.

1. Describe the nature of the Streeter-Phelps oxygen sag equation.
2. Determine the initial BOD_u of the mixture of pure river water and wastewater. (*Ans*: 8.6 g m^{-3})
3. Calculate the velocity in the river in $\mathrm{km\,d^{-1}}$.(*Ans*: $32\ \mathrm{km\,d^{-1}}$)
4. Substitute numerical values of the parameters to give an explicit formula for $C_{O_2}(x)$. ($C_{O_2}(x) = 8.2 + 14.19\,e^{-0.01294\,x} - 14.89\,e^{-0.008207\,x}$, where x is in km).
5. Plot a graph of oxygen concentration for $400\,\mathrm{km}$.
6. Either analytically or graphically calculate the critical distance and minimum value of the dissolved oxygen concentration (*Ans:* about 5.5 g m^{-3} at $86\,\mathrm{km}$).
7. Comment critically on your results overall.

Institute of Hydraulic and Water Resources Engineering, Vienna University of Technology

River Engineering

Tutorial Sheet 9 – Dimensional analysis & discharge past structures

1. Consider a rectangular notch weir of finite width b, with head over the weir H, in a rectangular channel of finite width B, and with a finite apron height P.
   a. Use dimensional analysis to obtain the expression for the flow Q

$$\frac{Q}{b\sqrt{gH^3}} = f(b/B, H/P, b/P)$$

   b. How sure are you that they are the most appropriate grouping of terms?

2. See if you can use dimensional analysis to establish that the flow past a parabolic notch weir is of the form

$$Q = C\sqrt{g}H^2,$$

   where the width at any elevation $b(z) \propto \sqrt{z}$. (Conventional weir theory obtains $Q = \frac{\pi}{8}C_d\gamma\sqrt{2g}H^2$ where $b(z) = \gamma\sqrt{z}$).

3. Using energy principles, obtain the expression for the discharge under a sluice gate:

$$\frac{q}{\sqrt{2gh_u^3}} = \frac{h_d/h_u}{\sqrt{1 + h_d/h_u}},$$

   where the upstream depth is h_u and h_d is the downstream depth. We relate this to the gate opening D, such that $h_d \approx 0.61\,D$.

4. Repeat, but this time use Coriolis coefficients α_1 and α_2 in front of the kinetic energy terms to allow for the fact that the velocity distribution over each section is not constant. How sensitive is the expression for discharge to these coefficients, which might be as large as 1.25?