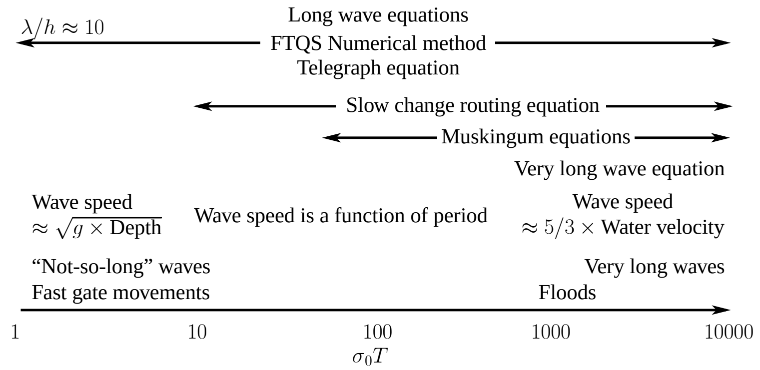
Summary of theories, names, and ranges of application



7.11 Muskingum methods

The advection-diffusion equation, re-written with the symbol D_0 for the coefficient of diffusion, $D_0 = Q/2B_0S$, is

$$\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial x} - D_0\frac{\partial^2 v}{\partial x^2} = 0. \qquad \text{(Advection-diffusion equation)}$$

It is a good simple approximate model of flood propagation – not as good as the fully nonlinear *slow change routing equation*. Both have, however, a finite stability criterion – strangely, the numerical simulation with the second derivative diffusion term makes the computation less stable! However, numerical solution is not a problem – one simply takes smaller steps until it works. In the last 40 years, however, there have been a large number of papers published using *Muskingum methods*, named after a river in the USA where such a method was first applied. They mimic the advection-diffusion equation, are supposed to be simple and plausibly seem so, and have been widely used. People have obtained the methods, sometimes from a simple reservoir routing approach, sometimes from the long wave equations, using long, complicated and arbitrary methods. The problem is to obtain a single finite difference equation in a single variable. (Using upstream volume V solves that problem rather better!). A typical Muskingum scheme is written, where c_m^n is the very long wave speed $dQ_{\mathrm{r}}/dA|_m^n$, and D_m^n is the coefficient of diffusivity $D_m^n = Q_m^n/2B_m^nS$, and so on:

$$\left(-\delta - \Delta c_m^n + 2D_m^n/c_m^n\right) Q_m^n + \left(-\delta + \Delta c_{m+1}^n - 2D_{m+1}^n/c_{m+1}^n\right) Q_{m+1}^n$$
$$+ \left(\delta - \Delta c_m^{n+1} - 2D_m^{n+1}/c_m^{n+1}\right) Q_m^{n+1} + \left(\delta + \Delta c_{m+1}^{n+1} + 2D_{m+1}^{n+1}/c_{m+1}^{n+1}\right) Q_{m+1}^{n+1} = 0\ .$$

This can be re-arranged to give an explicit expression for Q_{m+1}^{n+1}, the top right point shown in the blue computational module in Figure 7.4 in terms of the known two values of the discharge at time n, Q_m^n and Q_{m+1}^n and the known value at the previous space point Q_m^{n+1}.

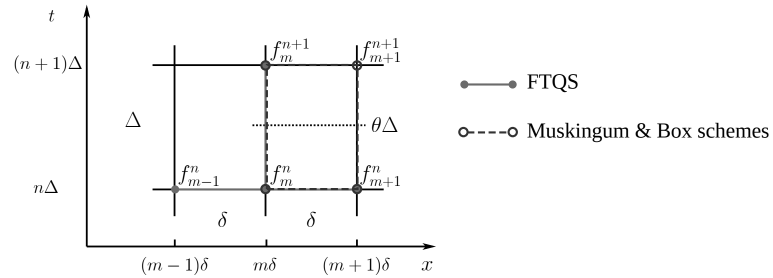


Figure 7.4: Computational stencils

However, if one performs a *Consistency Analysis*, and writes the point values Q_m^n *etc* as bi-dimensional Taylor series, one finds that the differential equation that the Muskingum formula

actually satisfies is

$$\frac{\partial Q}{\partial t} + c_0\frac{\partial Q}{\partial x} + \frac{D_0}{c_0}\frac{\partial^2 Q}{\partial x \partial t} = 0,$$

and *not* the desired Advection-diffusion equation

$$\frac{\partial Q}{\partial t} + c_0\frac{\partial Q}{\partial x} - D_0\frac{\partial^2 Q}{\partial x^2} = 0.$$

One can use the first two terms in the Muskingum equation to write $\partial Q/\partial t \approx -c_0 \partial Q/\partial x$ and substitute this into the mixed derivative $\partial^2 Q/\partial x\partial t$ to give the advection-diffusion equation, but that is accurate only for small diffusion.

Muskingum methods work surprisingly well for small-diffusion problems, but in general, they solve the wrong equation, are numerically diffusive, and are to be avoided.

7.12 The method of characteristics

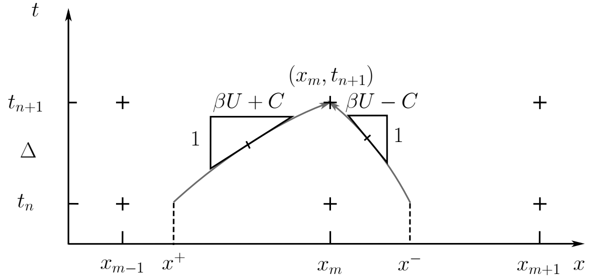


Figure 7.5: (x, t) axes, showing computational module with characteristics

This method is described in many books. The lecturer believes that it is something of an accident of history, and that the deductions that emerge from it are misleading and have caused several important misunderstandings about the nature of wave propagation in open channels.

Each of the pairs of long wave equations (7.17) and (7.18), which are partial differential equations, can be expressed as four ordinary differential equations. Two of the differential equations are for paths for $x(t)$, a path known as a *characteristic*:

$$\frac{dx}{dt} = \beta U \pm C, \tag{7.32}$$

where $U = Q/A$ is the mean fluid velocity in the waterway at that section and the velocity C is

$$C = \sqrt{\frac{gA}{B} + U^2\left(\beta^2 - \beta\right)},$$

often incorrectly described as the “long wave speed”. It is, as equation (7.32) shows, the speed of the characteristics relative to the flowing water. The two contributions $\pm C$ correspond to downstream and upstream propagation of information. Two characteristics that meet at a point are shown on Figure 7.5. The “downstream” or “+” characteristic has a velocity at any point of $\beta U + C$. In the usual case where U is positive, both parts are positive and the term is large. As shown on the diagram, the “upstream” or “-” characteristic has a velocity $\beta U - C$, which is usually negative and smaller in magnitude than the other. Not surprisingly, upstream-propagating disturbances travel more slowly. The characteristics are curved, as all quantities determining them are not constant, but functions of the variable A, B, and Q.

The other two differential equations for η and Q can be established from the long wave equations:

$$B\left(-\beta\frac{Q}{A} \pm C\right)\frac{d\eta}{dt} + \frac{dQ}{dt} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Lambda P\frac{Q\,|Q|}{A^2}, \tag{7.33}$$

On each of the two characteristics given by the two alternatives of equation (7.32), each of these two equations holds, taking the corresponding plus or minus signs in each case. To advance the solution numerically means that the four differential equations (7.32) and (7.33) have to be solved over time, usually using a finite time step Δ. Figure 7.5 shows the nature of the process on a plot of x against t.

The usual computational problem is, for a time $t_{n+1} = t_n + \Delta$, and for each of the discrete points x_m, to determine the values of x^+ and x^- at which the characteristics cross the previous time level t_n. From the information about η and Q at each of the computational points at that previous time level, the corresponding values of η^+, η^-, Q^+, and Q^- are calculated and then used as initial values in the two differential equations (7.33) which are then solved numerically to give the updated values $\eta\left(x_m, t_{n+1}\right)$ and $Q\left(x_m, t_{n+1}\right)$, and so on for all the points at t_{n+1}.

In textbooks and research papers, characteristics seem wrongly to be believed to have an almost supernatural property that the partial differential equations do not. An advantage of characteristics has been believed to be that numerical schemes are relatively stable. The lecturer is not convinced that they are any more stable then finite difference approximations to the original partial differential equations, but this remains to be proved conclusively.

In fact, the use of characteristics has led to a widespread misconception in hydraulics where C is understood to be the speed of propagation of all waves. It is not – it is the speed of *characteristics*. If surface elevation were constant on a characteristic there would be some justification in using the term ”wave speed” for the quantity C, as disturbances travelling at that speed could be observed. However as equation (7.33) holds, in general neither η (surface elevation – the quantity that we see), nor Q, is constant on the characteristics and one does not have observable disturbances, something that we would call a wave, travelling at C relative to the water. While C may be the speed of propagation of information in the waterway relative to the water, it cannot properly be termed the wave speed as it would usually be understood. In this course we have already examined at length the real nature of the propagation speed of waves.

7.13 Implicit methods – the Preissmann Box scheme

The most popular commercial numerical method for solving the long wave equations in time are Implicit Box (Preissmann) models, where the derivatives are replaced by finite-difference equivalents based on the rectangular blue module in Figure 7.4 on page 109:

$$\begin{aligned}
\frac{\partial f}{\partial x}(m,n) &\approx \frac{1}{\delta}\left[\theta\left(f_{m+1}^{n+1} - f_m^{n+1}\right) + (1-\theta)\left(f_{m+1}^{n} - f_m^{n}\right)\right], \\
\frac{\partial f}{\partial t}(m,n) &\approx \frac{1}{2\Delta}\left[\left(f_{m+1}^{n+1} - f_{m+1}^{n}\right) + \left(f_m^{n+1} - f_m^{n}\right)\right], \\
\bar{f}(m,n) &\approx \frac{1}{2}\left[\theta\left(f_{m+1}^{n+1} + f_m^{n+1}\right) + (1-\theta)\left(f_{m+1}^{n} + f_m^{n}\right)\right],
\end{aligned}$$

where θ is a coefficient that determines how much weight is attached to values at time $n+1$ (unknown, shown red) and how much to those at n (known, shown blue). Now, in the long wave equations (7.17) or (7.18) we use these expressions for all derivatives and also the averaged quantities $\bar{f}$ for those that occur algebraically. Considering all the modules at a certain time level, we have a set of $2M$ simultaneous complicated nonlinear algebraic equations in the values of Q and η at all points along the channel. The method is very complicated, but it is robust and stable, and large time steps can be taken. It is neutrally stable if $\theta = \frac{1}{2}$. In practice, one uses a larger value, such as $\theta = 0.6$, and the scheme is stable because it is computationally-diffusive. Several well-known commercial programs are available. For human purposes, it is simpler and better to use an explicit finite difference FTQS scheme.

7.14 Results

Evolution of flood wave – large diffusion case

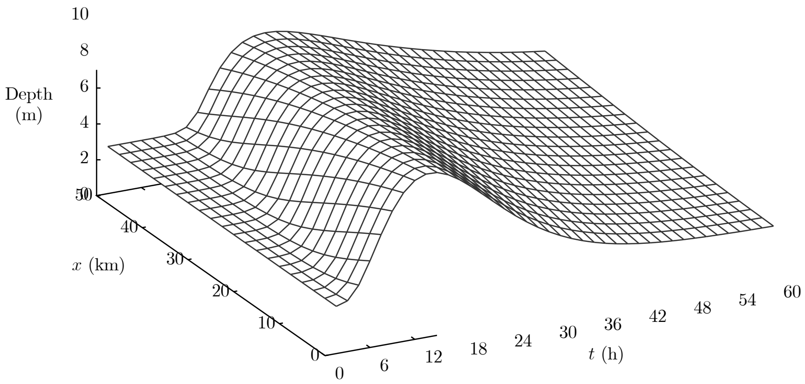
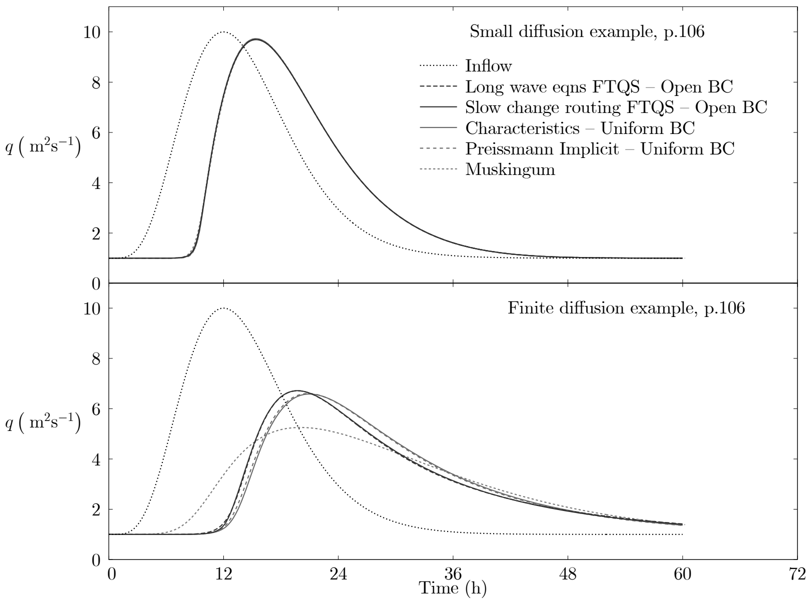


Figure 7.6:

Comparison of different methods



Conclusions

Small diffusion case:

- All methods performed well. Muskingum was accurate; the incorrect uniform flow downstream boundary condition did not matter (all motion is like simple advection – there is little diffusion for downstream effects to be felt upstream).

Finite diffusion case:

- Muskingum showed far too much diffusion when that was important. Such methods have been known to be problematical for small slopes, but nobody has called them out.
- The uniform flow downstream boundary condition: the Preissmann Implicit Box scheme and the Method of Characteristics agreed quite well with each other, but there were finite differences with those of the open downstream boundary condition.
- Both the FTQS finite difference schemes (solving the long wave equations and the slow change routing equation) agreed closely with each other using the more correct open boundary condition. They are the simplest methods and the best. One has to take relatively small time steps ($30\,\mathrm{s}$ used in the examples, compared with $900\,\mathrm{s}$ for the implicit and Muskingum methods), but their simplicity means that computational time is short.
- One can devise an example with a more rapidly-rising flood wave where the slow change routing equation no longer agrees so well. It is simple, and is in terms of a single variable, so that we used it to show the nature of approximations. *In general, however, solving the long wave equations themselves using our explicit FTQS scheme is the best of all.*