7.8 Nature of the equations and solutions

The Telegraph equation as a model for long wave propagation

We recall the long wave equations in terms of area, equations (7.17):

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=i,$$
$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+\frac{gA}{B}\frac{\partial A}{\partial x}=gA\tilde{S}\left(1-\frac{Q^2}{Q_{\mathrm{r}}^2(A)}\right),$$

where we have now written the resistance term in terms of the function of area $Q_{\mathrm{r}}(A)$ which is the Rating Curve relationship, or that given by, say, the Gauckler-Manning-Strickler formula. For steady uniform flow such that all derivatives on the left are zero, the equation becomes $Q=Q_{\mathrm{r}}(A)$, which we require. We also recall the relationships (7.3) for the upstream volume:

$$\frac{\partial V}{\partial x}=A \quad \text{and} \quad \frac{\partial V}{\partial t}=\int^{x} i\left(x'\right)\,\mathrm{d}x'-Q. \tag{7.21}$$

Substituting these into the mass conservation equation, as we did in §7.1.1, we find that it is identically satisfied (we expect volume to satisfy a volume conservation equation). The momentum conservation equation becomes

$$\frac{\partial^2 V}{\partial t^2}+2\beta\frac{Q}{A}\frac{\partial^2 V}{\partial x\partial t}+\left(\beta\frac{Q^2}{A^2}-\frac{gA}{B(A)}\right)\frac{\partial^2 V}{\partial x^2}+gA\tilde{S}\left(1-\left(-\frac{\partial V/\partial t}{Q_{\mathrm{r}}(A)}\right)^2\right)=0$$

where symbols Q and A have been retained in coefficients of second derivatives.

- The momentum equation has become a second-order partial differential equation in terms of the single variable V.
- And it is unusable in this ugly form. It is more useful in theoretical works and where approximations can be made, as we now do.
- We linearise the equation by considering relatively small disturbances about a uniform flow with area A_0 and discharge Q_0. Substituting the series

$$V = A_0 x - Q_0 t + \varepsilon v,\; A = A_0 + \varepsilon v_x,\; Q = Q_0 - \varepsilon v_t,\; \text{and } Q_\text{r}(A) = Q_0 + Q_0'\,\varepsilon v_x,$$

where εv is a small quantity, a deviation of upstream volume from that of uniform flow, $v_t = \partial v/\partial t$, $v_x = \partial v/\partial x$, and $Q_0' = \mathrm{d}Q_\text{r}/\mathrm{d}A|_0$.

Performing power series operations, the second order derivatives can simply be written down with constant coefficients. The gravity and resistance terms become, making use of the binomial theorem to first order, $(1+\varepsilon a)^n = 1 + \varepsilon n a + \ldots$:

$$\begin{aligned}
g\left(A_0 + \varepsilon v_x\right) S_0 \times \left(1 - \left(\frac{Q_0 - \varepsilon v_t}{Q_0 + Q_0'\,\varepsilon v_x}\right)^2\right) &= g A_0 S_0 \times \left(1 - \left(\frac{1 - \varepsilon v_t/Q_0}{1 + Q_0'/Q_0\,\varepsilon v_x}\right)^2\right) \\
&= g A_0 S_0 \times \left(1 - \left(1 - 2\varepsilon v_t/Q_0\right)\left(1 - 2Q_0'/Q_0\,\varepsilon v_x\right)\right) \\
&= g A_0 S_0 \times \left(1 - \left(1 - 2\varepsilon v_t/Q_0 - 2Q_0'/Q_0\,\varepsilon v_x\right)\right) \\
&= \varepsilon \frac{2 g A_0 S_0}{Q_0}\left(\frac{\partial v}{\partial t} + Q_0' \frac{\partial v}{\partial x}\right).
\end{aligned}$$

We obtain the linearised momentum equation as the *Telegraph equation*:

$$\sigma_0 \left(\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} \right) + \frac{\partial^2 v}{\partial t^2} + 2\beta U_0 \frac{\partial^2 v}{\partial x \partial t} - (C_0^2 - \beta^2 \, U_0^2) \frac{\partial^2 v}{\partial x^2} = 0. \tag{7.22}$$

- σ_0 – **resistance parameter / inverse time scale**: this is actually an important channel parameter, determining the nature of wave behaviour and computational solution properties

  $$\sigma_0 = \frac{2gA_0S_0}{Q_0} = 2\frac{gS_0}{U_0} = \frac{\partial}{\partial Q} \left(gAS \frac{Q^2}{Q_\mathrm{r}^2} \right) \bigg|_0$$

  It is the derivative with respect to Q of the resistance term in the momentum equation. We could argue by a rough electrical analogy that the resistance term in the momentum equation is equivalent to potential difference or voltage, while discharge Q is equivalent to current. As the derivative of voltage with respect to current gives electrical resistance, σ_0 can be thought of as a *resistance parameter* in our nonlinear case.

- c_0 – **wave speed**: This will be shown to be the speed of very long period waves, which means for us the propagation speed of flood waves:

  $$c_0 = \frac{\mathrm{d}Q_\mathrm{r}}{\mathrm{d}A} \bigg|_0 .$$

  For the Gauckler-Manning-Strickler equation, $Q_\mathrm{r} = k_\mathrm{St} A^{5/3} / P^{2/3} \sqrt{S}$, for a wide stream, ignoring change of P with A, this gives $c_0 \approx \frac{5}{3} U_0$, so that a good estimate of the speed of propagation of a flood wave is to multiply the stream velocity by $5/3$. This velocity is important, and will be studied more practically later.

- $U_0 = Q_0/A_0$ – **mean fluid velocity**: used for simplicity.
- C_0 – **the speed of not-so-long waves**:

$$C_0 = \sqrt{gA_0/B_0 + \left(\beta^2 - \beta\right) U_0^2},$$

  In most textbooks this is written, not unreasonably, implicitly with $\beta = 1$ such that $C_0 = \sqrt{gA_0/B_0}$, which is usually said to be the "celerity" or "long wave speed" or "dynamic wave speed". Below it will be shown that it is actually the speed of waves only in the limit of shorter waves, but still long enough that the hydrostatic approximation holds. We call these "not-so-long" waves. They occur when waves are due to rapid gate movements. This velocity is less-important than is generally believed.

We now obtain some simple solutions to the Telegraph equation in two limits.

Very long waves – the longest flood waves

- For disturbances with a long period, such that $\partial^2/\partial t^2 \ll \sigma_0 \partial/\partial t$, "very long waves", the last three terms in the equation can be neglected, and it becomes the advection equation

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} = 0, \qquad \text{(Very long wave equation)}$$

$$\text{Solution: } v = f_1\left(x - c_0 t\right), \qquad (7.23)$$

where $f_1\left(.\right)$ is an arbitrary function given by the upstream conditions. To show this consider a moving variable $X = x - c_0 t$, and $v = f_1(X)$. By the *chain rule* for partial differentiation,

$$\frac{\partial v}{\partial t} = \frac{\partial f_1(X)}{\partial t} = \frac{df_1(X)}{dX}\frac{\partial X}{\partial t} = -c_0 \frac{df_1(X)}{dX}, \quad \text{and}$$

$$\frac{\partial v}{\partial x} = \frac{\partial f_1(X)}{\partial x} = \frac{df_1(X)}{dX}\frac{\partial X}{\partial x} = 1 \times \frac{df_1(X)}{dX},$$

and the equation is satisfied for *any* $f_1(X)$, whatever the upstream conditions determine.

- This solution is a wave propagating downstream at speed c_0 with no change or diffusion.
- The equation has been known as the "kinematic wave equation" and c_0 the "kinematic wave speed", because the approximation has previously been believed to be such that dynamic terms of order F^2 in the momentum equation have been neglected.
- Here we have shown that the only approximation has been that the wave period is long. No approximation has been made by neglecting dynamical terms. A better name is the *Very Long Wave Equation*, VLWE.

Not-so-long waves – in the shorter limit of waves from the long wave equations

- In the other limit, for disturbances which are shorter, such that $\partial^2/\partial t^2 \gg \sigma_0 \partial/\partial t$, for which we use the term "not-so-long" waves, the Telegraph equation becomes

$$\frac{\partial^2 v}{\partial t^2} + 2\beta U_0 \frac{\partial^2 v}{\partial x \partial t} - (C_0^2 - \beta^2 U_0^2) \frac{\partial^2 v}{\partial x^2} = 0,$$

which is a second-order wave equation with solutions

$$v = f_{21}\left(x - (\beta U_0 + C_0)\, t\right) + f_{22}\left(x - (\beta U_0 - C_0)\, t\right)$$

where $f_{21}(.)$ and $f_{22}(.)$ are arbitrary functions determined by boundary conditions both upstream and downstream.

- In this case the solutions are waves propagating upstream and downstream at velocities of $\beta U_0 \pm C_0$, such that in the usual terminology C_0 is the "long wave speed", and the waves travel relative to an advection velocity βU_0, where the presence of β is slightly surprising.
- We have shown here that C_0 is the speed of waves that are actually not so long, apparently paradoxically – they are long enough that the pressure distribution in the fluid is still hydrostatic, but they are short in terms of time scales given by the resistance characteristics.

Intermediate period waves

- In the general case, solutions of the long wave equations show wave propagation characteristics, velocity and rate of decay, that depend on the period of the waves, so that the waves are actually
  - diffusive – different period components decay at different rates, and
  - dispersive – different components travel at different speeds
- One can obtain solutions for the propagation behaviour in terms of wave period, but the operations are complicated, and they are not included here.
- The widespread belief, printed in all textbooks, is wrong, that all waves obeying the long wave equations travel at a speed $C \approx \sqrt{gA/B} \approx \sqrt{g \times \text{Depth}}$. The behaviour is very much more complicated. There is no such thing as "**the** long wave speed".

Solving the long wave equations numerically overcomes all such problems, but it is nice to know what physical processes are at work.

7.9 Slow change routing equation

The Telegraph equation (equation 7.22 on page 93) is

$$\sigma_0\left(\frac{\partial v}{\partial t}+c_0\frac{\partial v}{\partial x}\right)+\frac{\partial^2 v}{\partial t^2}+2\beta U_0\frac{\partial^2 v}{\partial x\partial t}+\beta^2 U_0^2\frac{\partial^2 v}{\partial x^2}-C_0^2\frac{\partial^2 v}{\partial x^2}=0.$$

Previously for very long waves, we included just the first σ_0 terms. We now make a rather better approximation, including the last term, ignoring just the terms shown light blue. The corresponding full momentum conservation equation is

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+\frac{gA}{B}\frac{\partial A}{\partial x}=gA\tilde{S}\left(1-\frac{Q^2}{Q_r^2}\right).$$

It can be shown that neglecting those first two terms of the momentum equation, the time derivative and the fluid momentum term, one is making, surprisingly, not a low-Froude number "kinematic" approximation as has been believed, but a *Very Long Wave* or *Slow Change* approximation. The criterion is in terms of the dimensionless time scale $\sigma_0 T$ where $\sigma_0 = 2gS/U_0$, in which U_0 is the mean flow velocity, and T is the time scale.For the approximation to be accurate, waves should be sufficiently long that $\sigma_0 T \gtrsim 20$, where here T is wave period. The equation can be re-arranged to give the momentum equation in the form of a simple expression for Q for the period limitation shown:

$$Q(A)=\underbrace{Q_r(A)}_{\text{Steady, uniform}}\times\sqrt{1-\frac{1}{B(A)S}\frac{\partial A}{\partial x}},\quad \text{for}\quad gST/U_0\gtrsim 10\,. \tag{7.24}$$

It is interesting that just using the rating expression $Q = Q_\mathrm{r}(A)$ is already an approximation to the momentum equation.

This approximate simple momentum equation is surprisingly accurate, and can be used for long wave propagation problems in streams. It is possible to substitute it into the mass conservation equation to give a single partial differential equation in area $A\,(x,t)$. However, that is complicated, and inflow boundary conditions are often expressed in terms of a discharge hydrograph, when the area A formulation is not so convenient.

A more general approach in terms of *Upstream Volume* can be developed. Substituting for $A = \partial V/\partial x$ and $Q = -\partial V/\partial t$ gives the single equation in the single dependent variable V:

$$\frac{\partial V}{\partial t} + Q_\mathrm{r}\left(\partial V/\partial x\right)\sqrt{1 - \frac{1}{S\,B(\partial V/\partial x)}\frac{\partial^2 V}{\partial x^2}} = 0\ , \qquad (7.25)$$

where both breadth B and Q_r have been written as functions of area $A = \partial V/\partial x$. We call this the *Slow change routing equation*. It is a single equation in a single unknown. The only approximation relative to the long wave equations has been that the variation with time is slow, such that $gST/U \gtrsim 10$. Boundary conditions involving discharge Q or stage η can be incorporated using equations (7.21) and the geometrical relationship between A and surface elevation η at a point.

The equation can be used for simulations, for which it is necessary to use the quadratic approximation to the second derivative, similar to those for the first derivative in equation (7.19) on page 85:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_m = \frac{f_{m-1} - 2f_m + f_{m+1}}{\delta^2},$$

and as the second derivative of a quadratic function is constant, this is the value used also at $m-1$ and $m+1$ at boundaries if necessary. For most flood routing problems the slow change routing equation gives results as good as the long wave equations. For short waves it is less accurate. The only approximation relative to the long wave equations has been that the variation with time is slow, such as for flood waves.

7.10 Advection-diffusion model equation

In fact, the *slow change routing equation* is more use to us here for purposes of understanding – to show us the *nature* of solutions and how waves in rivers behave.

It is a fully-nonlinear (no assumption of smallness) advection-diffusion equation, which is made clearer if we write $Q_{\mathrm{r}} = U_{\mathrm{r}} A$, where U_{r} is the mean flow velocity as given by Gauckler-Manning-Strickler *etc.*:

$$\frac{\partial V}{\partial t} + U_{\mathrm{r}}\left(\partial V/\partial x\right)\frac{\partial V}{\partial x}\sqrt{1 - \frac{1}{S\,B(\partial V/\partial x)}\frac{\partial^2 V}{\partial x^2}} = 0\ , \qquad (7.26)$$

so that in this form with little approximation, the advective term with the single derivative $\partial V/\partial x$ is multiplied by a term containing a diffusive correction with $\partial^2 V/\partial x^2$.

Advection term

To the lowest level of approximation, when waves are so long that diffusion plays no role, equation (7.26) becomes the nonlinear advection equation

$$\frac{\partial V}{\partial t} + U_{\mathrm{r}}(V_x)\frac{\partial V}{\partial x} = 0 \ . \tag{7.27}$$

This generally nonlinear equation has some interesting properties. Superficially it seems that volume V is advected at a velocity of U_{r}, the mean velocity of flow in the stream, as one might expect. However nonlinearity of the equation causes us a surprise. We consider a general unsteady flow superimposed on a steady uniform flow, of area A_0 and discharge $Q_0 = U_0 A_0$, such that, as previously to obtain the Telegraph equation, we write $V = A_0 x - U_0 A_0 t + v\,(x,t)$, where $v\,(x,t)$ is the unsteady or non-uniform contribution. Substituting into equation (7.27)

$$-U_0 A_0 + \frac{\partial v}{\partial t} + U_{\mathrm{r}}\left(A_0 + \frac{\partial v}{\partial x}\right) \times \left(A_0 + \frac{\partial v}{\partial x}\right) = 0 \ ,$$

and now assuming that the unsteady/non-uniform terms $\partial v/\partial t$ and $\partial v/\partial x$ are small compared with the underlying flow, and taking a Taylor expansion of U_{r} about A_0, we obtain the advection equation

$$\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} + \left(U_0 + A_0\left.\frac{dU_{\mathrm{r}}}{dA}\right|_0\right)\frac{\partial v}{\partial x} = 0. \tag{7.28}$$

We have obtained, rather more simply this time, the very long wave equation (Very long wave equation). It is now clearer that, even if the underlying flow is carried at a velocity U_0, any variation in the flow is advected at the very long wave speed c_0 given by the Kleitz-Seddon formula:

$$c_0(A) = U_0 + A_0 \left.\frac{dU_\mathrm{r}}{dA}\right|_0 = \left.\frac{dQ_\mathrm{r}}{dA}\right|_0 . \qquad (7.29)$$

This nonlinear mathematical artefact is not so obvious, physically! Evaluating c_0 for some cases, we find

$$c_0 = \begin{cases} \mathrm{d}Q_\mathrm{r}/\mathrm{d}A|_0 \,, & \text{General expression} \\ \frac{3}{2}U_0\left(1 - \frac{1}{3}A_0 P_0'/P_0\right), & \text{Chézy-Weisbach} \\ \frac{5}{3}U_0\left(1 - \frac{2}{5}A_0 P_0'/P_0\right), & \text{Gauckler-Manning-Strickler} \end{cases} ,$$

where $P_0' = \mathrm{d}P/\mathrm{d}A|_0$.

Example 6 Estimate the effect of side resistance on flood wave speed for a river of bottom width $20\ \mathrm{m}$, side slopes 2:1 (H:V) and a depth of $2\ \mathrm{m}$.

Using our relationships $B = W + 2mh$, $A = h\,(W + mh)$, and $P = W + 2\sqrt{1+m^2}h$ we have $\mathrm{d}P/\mathrm{d}h = 2\sqrt{1+m^2}$ and $\mathrm{d}A/\mathrm{d}h = B$

$$-\frac{2}{5}\frac{A_0}{P_0}\left.\frac{\mathrm{d}P}{\mathrm{d}A}\right|_0 = -\frac{2}{5}\frac{A_0}{P_0}\left.\frac{\mathrm{d}P/\mathrm{d}h}{\mathrm{d}A/dh}\right|_0 = -\frac{2}{5}\frac{A_0}{P_0}\left.\frac{\mathrm{d}P/\mathrm{d}h}{B}\right|_0 = -\frac{2}{5}\frac{h\,(W+mh)}{W+2\sqrt{1+m^2}h}\frac{2\sqrt{1+m^2}}{W+2mh}$$

$$= -\frac{2}{5}\frac{2\,(10+2\times 2)}{10+2\times 2\sqrt{1+2^2}}\frac{2\sqrt{1+2^2}}{10+2\times 2\times 2} = -15\%$$

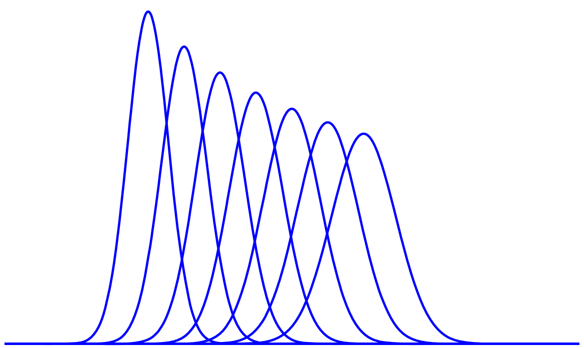
Diffusion term

Now we consider the effects of the "diffusion term", with the second derivative.

In physics, the process of diffusion occurs because of a continuous process of random particle movements, where any irregularities in concentration ϕ of a substance are smoothed out. In a stationary medium, the governing diffusion equation is

$$\frac{\partial \phi}{\partial t} = \kappa \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right), \qquad \text{(Diffusion equation)}$$

where κ is the coefficient of diffusion. The significance of the equation is that any regions of "curvature" (where there is not a linear variation of ϕ) will be smoothed out. For example, near a local maximum in concentration, even in just one dimension, $\partial^2\phi/\partial x^2$ is negative, and this means that $\partial\phi/\partial t$ there is negative, and the concentration is reduced. The reverse applies near a minimum. Quantities that show diffusive behaviour are temperature, electric charge, and pollution concentration.



Linearising the slow change routing equation, now including the second derivative term, gives the *Advection-diffusion equation*:

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} - \frac{Q_0}{2B_0 S} \frac{\partial^2 v}{\partial x^2} = 0, \qquad \text{(Advection-diffusion equation)}$$

and we see that the diffusion coefficient is given by $Q_0/2B_0S$, where B_0 is the width of the undisturbed stream. Typical solutions of the equation are shown in the figure. This is a good simple approximate model of flood propagation. It tells us most importantly, how fast the flood wave moves.

As a first estimate, we might assume that diffusion is unimportant, and that the flood peak might be the same downstream as it was upstream (as implied by the very long wave advection equation). For practical problems, however, the problem then arises as how to estimate the importance of diffusion. We can go back to the momentum equation (7.24) and consider the term which leads to diffusion

$$\left(1 - \frac{1}{B(A)S}\frac{\partial A}{\partial x}\right)^{1/2}.$$

This is not a particularly helpful criterion, as we do not know what the x-derivative is until we solve a problem. However, we can use our knowledge that, *to first approximation*, the flood wave propagates unchanged, satisfying the very long wave equation

$$\frac{\partial A}{\partial t} + c_0 \frac{\partial A}{\partial x} = 0.$$

The term containing diffusion, using a power series expansion to first order becomes

$$\begin{aligned} 1 - \tfrac{1}{2}\frac{1}{B(A)S}\frac{\partial A}{\partial x} &= 1 + \tfrac{1}{2}\frac{1}{c_0 B(A)S}\frac{\partial A}{\partial t} \\ &= 1 + \tfrac{1}{2}\frac{1}{c_0 S}\frac{\partial \eta}{\partial t}, \end{aligned}$$

where η is the surface elevation, such that $\partial A/\partial t = B\partial\eta/\partial t$. We have the result

$$\text{Relative importance of diffusion} = \frac{1}{2c_0S}\frac{\partial\eta}{\partial t}. \tag{7.30}$$

This has an interesting simple physical significance: we can consider it to be

$$\text{Relative importance of diffusion} = \frac{1}{2}\frac{\text{Vertical water surface velocity} \,/\, \text{Horizontal water surface velocity}}{\text{Vertical fall of bed} \,/\, \text{Horizontal distance}}.$$

At the beginning of any problem one does not know $\partial\eta/\partial t$. However, it is not difficult to estimate it to give the relative importance of diffusion. Historical records should give us an approximate idea of the maximum (**P**eak) flood level η_{P} to be expected, as well as the time at which it occurs, or which value is given for a particular event. In this case we have

$$\text{Relative importance of diffusion} = \frac{1}{2c_0S}\frac{\eta_{\text{P}} - \eta_0}{t_{\text{P}} - t_0}, \tag{7.31}$$

and this is enough for an estimate. Our expressions might also be useful to give us a general idea of when diffusion is important. As c_0 is proportional to U_0 and that is proportional to $\sqrt{S}$, whether Gauckler-Manning-Strickler or Chézy-Weisbach resistance formula are used, we obtain the important result that the relative importance of diffusion is proportional to $S^{-3/2}$, which is a very strong result showing that diffusion is more important for small slopes.

Two examples, showing smaller and greater diffusion effects

The figure shows results from two computations, for a $50\,\mathrm{km}$ length of river, first with a very smooth boundary and steeper slope so that diffusion is not so large, the second for a rougher boundary and smaller slope.