6. Reservoir routing

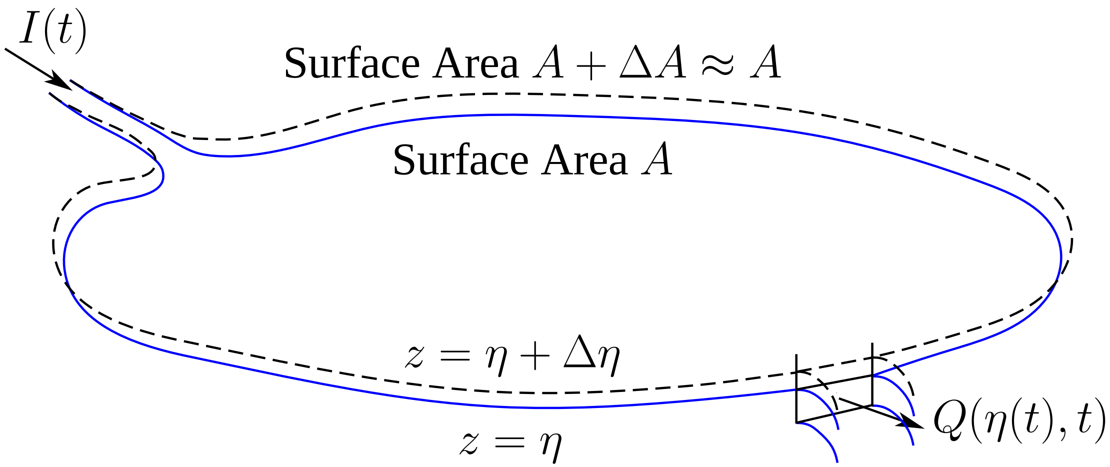

Figure 6.1: Reservoir or tank, showing surface level varying with inflow, determining the rate of outflow

Consider the problem shown in figure 6.1, where a generally unsteady inflow rate $I(t)$ enters a reservoir or a storage tank, and we have to calculate what the outflow rate $Q(t)$ is, as a function of time t. The action of the reservoir is usually to store water, and to release it more slowly, so that the outflow is delayed and the maximum value is less than the maximum inflow. Some reservoirs, notably in urban areas, are installed just for this purpose, and are called *detention* reservoirs or storages. The procedure of solving the problem is also called *Level-pool Routing*.

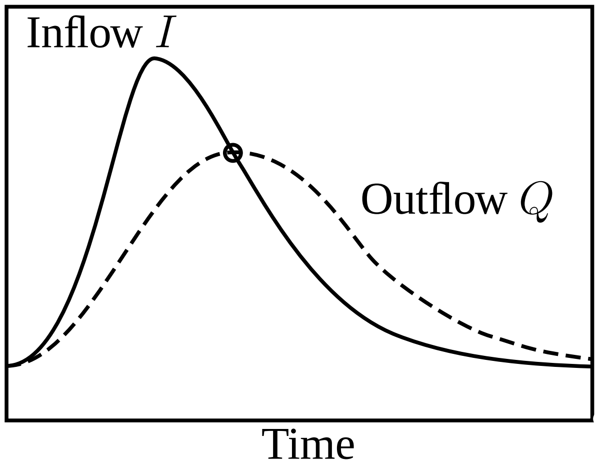

Figure 6.2:

The process is shown in figure 6.2. When a flood comes down the river, inflow increases, the water level rises in the reservoir until at the point O when the outflow over the spillway now balances the inflow. At this point, outflow and surface elevation in the reservoir have a maximum. After this, the inflow might reduce quickly, but it still takes some time for the extra volume of water to leave the reservoir.

Detention reservoir in a public park in Melbourne, Australia

It is simple and obvious to write down the relationship stating that the rate of surface rise $\mathrm{d}\eta/\mathrm{d}t$ is equal to the net rate of volume increase divided by surface area:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{I(t) - Q(\eta, t)}{A(\eta)}\ , \tag{6.1}$$

where η is the free surface elevation, and $A(\eta)$ is the surface area, possibly given from planimetric information from contour maps, and $Q(\eta, t)$ is the volume rate of outflow, which is usually a simple function of the surface elevation η, from a weir or gate formula, usually involving terms like $(\eta - z_{\text{outlet}})^{1/2}$ and/or $(\eta - z_{\text{crest}})^{3/2}$, where z_{outlet} is the elevation of the pipe or tailrace outlet to atmosphere and z_{crest} is the elevation of the spillway crest. There might be extra dependence on time t is if the outflow device is opened or closed. This is a differential equation for the surface elevation itself. The procedure of solving it is called *Level-pool Routing*.

The traditional method of solving the problem, described in almost all books on hydrology, is to use an unnecessarily complicated method called the “Modified Puls” method of routing, which solves a transcendental equation for a single unknown quantity, the volume in the reservoir, at each time step. It is simpler and more fundamental to treat the problem as a differential equation (Fenton 1992). :-)

Numerical solution of the differential equation by Euler's method

Euler's method is the simplest (but least-accurate) of all methods, being of first-order accuracy only. For river engineering purposes it is usually quite good enough. However there is a good method for making it more accurate, which we will use. Euler's method is to approximate the derivative in a differential equation at a time step i by a forward difference expression in terms of a time step Δ, here applying it to equation (6.1):

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}t}\right|_i \approx \frac{\eta_{i+1}-\eta_i}{\Delta} = \frac{I(t_i)-Q(\eta_i,t_i)}{A(\eta_i)} \ ,$$

giving the scheme to calculate the value of η at t_{i+1} as

$$\eta_{i+1} = \eta_i + \Delta\frac{I(t_i)-Q(\eta_i,t_i)}{A(\eta_i)} + O\left(\Delta^2\right) , \tag{6.2}$$

where we use the notation η_i for the solution at time step i. We have shown that the error of this approximation is proportional to Δ^2. It is necessary to take small enough Δ that this is small.

Accurate results with simple methods – Richardson extrapolation

We introduce a clever device for obtaining more accurate solutions from Euler's method and others.

Consider the numerical value of any part of a computational solution for some physical quantity ϕ obtained using a time or space step Δ, such that we write $\phi(\Delta)$. Let the computational scheme be of known nth order such that the *global* error of the scheme at any point or time is proportional to

Δ^n, then if $\phi(0)$ is the exact solution, we can write the expression in terms of the error at order n:

$$\phi(\Delta) = \phi(0) + b\Delta^n + \dots \ , \tag{6.3}$$

where $\phi(0)$ is the solution for a vanishingly small time step, so that it should be exact. The b is an unknown coefficient; the neglected terms vary like Δ^{n+1}. If we have two numerical simulations or approximations with two different Δ_1 and Δ_2 giving numerical values $\phi_1 = \phi(\Delta_1)$ and $\phi_2 = \phi(\Delta_2)$ then we write (6.3) for each:

$$\begin{aligned} \phi_1 &= \phi(0) + b\Delta_1^n + \dots \ , \\ \phi_2 &= \phi(0) + b\Delta_2^n + \dots \ . \end{aligned}$$

These are two linear equations in the two unknowns $\phi(0)$ and b. Eliminating b, which is not important, between the two equations and neglecting the terms omitted, we can solve for $\phi(0)$, an approximation to the exact solution:

$$\phi(0) = \frac{\phi_2 - r^n \phi_1}{1 - r^n} + O\left(\Delta_1^{n+1}, \Delta_2^{n+1}\right) \ , \tag{6.4}$$

where $r = \Delta_2/\Delta_1$. The errors are now proportional to step size to the power $n+1$, so that we have gained a higher-order scheme without having to implement any more sophisticated numerical methods, just with a simple numerical calculation. This procedure, where n is known, is called *Richardson extrapolation to the limit*.

1. For simple Euler time-stepping solutions of ordinary differential equations, $n = 1$, and if we

perform two simulations, one with a time step Δ and then one with $\Delta/2$, we have

$$\phi(t,0) = 2\,\phi(t,\Delta/2) - \phi(t,\Delta) + O\left(\Delta^2\right), \tag{6.5}$$

where the numerical solution at time t has been shown as a function of the step. This is very simply implemented.

2. For the evaluation of an integral by the trapezoidal rule, $n = 2$.

Example 4 Consider a small detention reservoir, square in plan, with dimensions 100m by 100m, with water level at the crest of a sharp-crested weir of length of $b = 4\,\mathrm{m}$, where the outflow over the sharp-crested weir can be taken to be

$$Q\left(\eta\right) = 0.6\sqrt{g}b\eta^{3/2}, \tag{6.6}$$

where $g = 9.8\,\mathrm{ms}^{-2}$. The surrounding land has a slope (V:H) of about 1:2, so that the length of a reservoir side is $100 + 2 \times 2 \times \eta$, where η is the surface elevation relative to the weir crest, and

$$A(\eta) = \left(100 + 4\eta\right)^2.$$

The inflow hydrograph is:

$$I(t) = Q_{\min} + \left(Q_{\max} - Q_{\min}\right)\left(\frac{t}{T_{\max}}e^{1-t/T_{\max}}\right)^5, \tag{6.7}$$

where the event starts at $t = 0$ with $Q_{\min}$ and has a maximum $Q_{\max}$ at $t = T_{\max}$. This general form of inflow hydrograph mimics a typical storm, with a sudden rise and slower fall, and will be used in other places in this course. In the present example we consider a typical sudden local storm event, with $Q_{\text{min}} = 1\ \text{m}^3\text{s}^{-1}$ at $T_{\text{max}} = 1800\,\text{s}$.

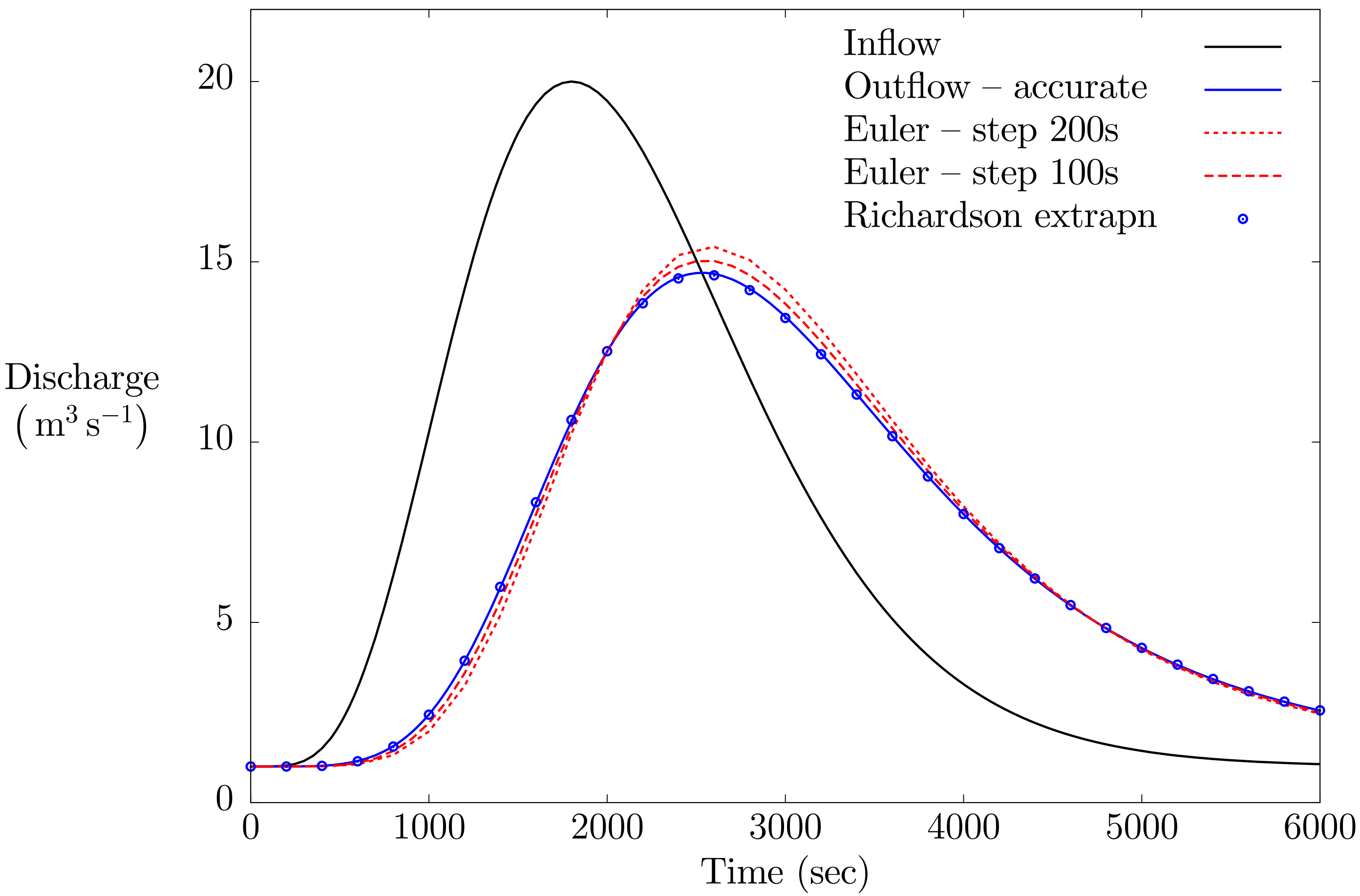

Figure 6.3: Computational results for the routing of a sudden storm through a small detention reservoir

The problem was solved with an accurate 4th-order Runge-Kutta scheme, and the results are shown as a solid blue line on figure 6.3, to provide a basis for comparison. Next, Euler's method (equation 6.2) was used with 30 steps of $200\,\text{s}$, with results that are barely acceptable. Halving the time step to $100\,\text{s}$ and taking 60 steps gave the slightly better results shown. It seems, as expected from knowledge of the behaviour of the global error of the Euler method, that it has been halved at each point. Next, applying Richardson extrapolation, equation (6.5), gave the results shown by the solid points. They almost coincide with the accurate solution, and cross the inflow hydrograph with an apparent horizontal gradient, as required, whereas the less-accurate results do not. Overall, it seems that the simplest Euler method can be used, but is better together with Richardson extrapolation. In fact, there was nothing in this example that required large time steps – a simpler approach might have been just to take rather smaller steps.

The role of the detention reservoir in reducing the maximum flow from $20\,\mathrm{m}^3\mathrm{s}^{-1}$ to $14.7\,\mathrm{m}^3\mathrm{s}^{-1}$ is clear. If one wanted a larger reduction, it would require a larger spillway. It is possible in practice that this problem might have been solved in an inverse sense, to determine the spillway length for a given maximum outflow.

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7. The one-dimensional equations of river hydraulics

These are the fundamental equations that are used to describe the propagation of floods and disturbances in rivers. They are called the *long wave equations* or the *Saint-Venant equations* and are mass and momentum conservation equations for water.

The equations are a pair of partial differential equations in the independent variables x (distance along the stream) and t (time). A typical flood routing problem is for large extra values of discharge Q to be introduced at the upstream initial point, and then for a number of time steps, to solve the equations along the channel to obtain the progress of the flood at each time.

We will also consider a mass conservation equation for soil. In their steady form, the equations describe how water level and velocities vary along a stream, and what effects boundary changes such as sand removal might have on flooding.

We make the traditional approximation that all rivers are straight. Later we will see that it is quite accurate, even for meandering streams.

The model is one-dimensional. We do not consider details of motion in the plane across the stream – all quantities are averaged across it. This does not mean that we assume they are constant. This approach requires surprisingly few approximations – the model is a good simple model of complicated reality.

It is easier to use cartesian co-ordinates, for which we use x the horizontal distance along the stream, y the horizontal transverse co-ordinate, and z the vertical, relative to some arbitrary origin.

7.1 Mass conservation of water and soil

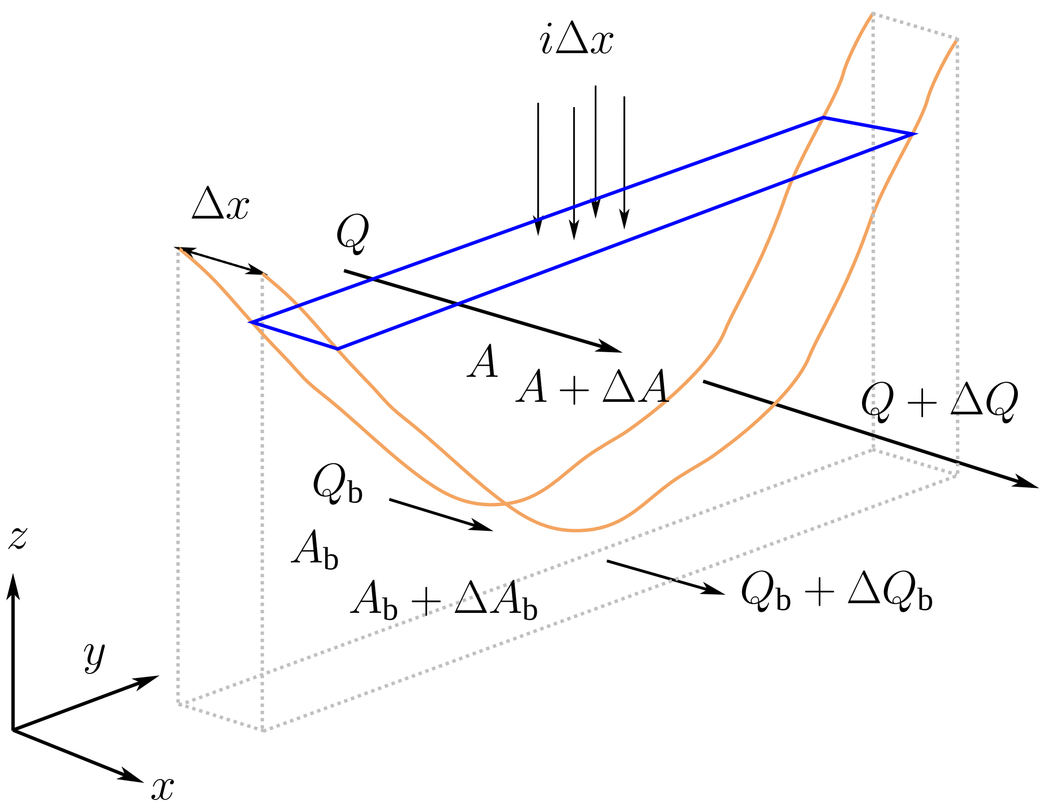

Figure 7.1: Elemental length of channel showing control volumes

Consider Figure 7.1, showing an elemental slice of channel of length Δx with two stationary vertical faces across the flow. It includes two different control volumes. The free surface and the interface between them may move. The surface shown by solid lines contains water and possibly suspended soil grains. The surface shown by dotted lines contains the soil moving as bed load and extends down into the soil such that there is no motion at its far boundaries. Each is modelled separately. We assume that the density of the fluid (water plus suspended soil particles) is constant, so that we can consider *volume* conservation.

On the upstream vertical face at any instant, there is a volume flux (rate of volume flow) Q, and that on the downstream face is $Q + \Delta Q$, so that

$$\text{Net volume flow rate of fluid } \textit{leaving} \text{ across vertical faces} = \Delta Q = \frac{\partial Q}{\partial x}\Delta x + \text{terms in } (\Delta x)^2 .$$

If rainfall, seepage, or tributaries contribute an inflow volume rate i per unit length of stream, the volume flow rate of this other fluid *entering* the control volume is $i\,\Delta x$. If the sum of the two contributions is not zero, then volume of fluid is changing inside the elemental slice, so that the water level will change in time. The rate of change with time t of fluid volume is $\partial A/\partial t \times \Delta x$. For volume to be conserved (mass, but we assume the water is incompressible) this is equal to the net rate of fluid entering the control volume, dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{7.1}$$

This is the *mass conservation equation*. Remarkably for hydraulics, it is almost exact. The only approximation has been that the channel is straight. It is also linear in the two dependent variables A and Q.

The composite bulk density ρ_b of the bed load composed of larger soil particles and is assumed to be constant. The bed has a cross-sectional area A_b, the bulk volumetric flow rate is Q_b, and there is an inflow of mass rate $\dot{m}_i$ per unit length, possibly due to deposition or erosion. Mass conservation is calculated following the same reasoning as for the channel, giving Exner's[1] equation:

$$\frac{\partial A_\mathrm{b}}{\partial t} + \frac{\partial Q_\mathrm{b}}{\partial x} = \frac{\dot{m}_i}{\rho_\mathrm{b}}. \tag{7.2}$$

The volume transport rate used here is the bulk flow rate; it is related to the volume flow rate of solid matter Q_s used in transport formulae, by $Q_\mathrm{s} = Q_\mathrm{b}\,(1-\varphi)$, where φ is the porosity.

[1] `https://de.wikipedia.org/wiki/Felix_Maria_von_Exner-Ewarten` – Austrian – Director of the Zentral Anstalt für Meteorologie und Geodynamik

Upstream Volume

The mass conservation equation (7.1) suggests the introduction of a function $V(x,t)$ which is the volume upstream of point x at time t, such that for the channel flow

$$\frac{\partial V}{\partial x} = A \quad \text{and} \quad \frac{\partial V}{\partial t} = \int^{x} i\left(x'\right)\,\mathrm{d}x' - Q. \tag{7.3}$$

The derivative of volume with respect to distance x gives the area, as shown, while the time rate of change of volume upstream is given by the rate at which the volume is increasing due to inflow, minus the rate at which volume is passing the point and therefore no longer upstream. Substituting for A and Q into equation (7.1):

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial x}\left(-\frac{\partial Q}{\partial t}\right) + \frac{\partial}{\partial x}\int^{x} i\left(x'\right)\,\mathrm{d}x' = i\ ,$$

which is identically satisfied. By introducing V we automatically satisfy one of the two conservation equations and reduce the number of unknowns from two (A and Q) to one (V). Sometimes this can be very helpful.

In the case of the bed load, a similar quantity $V_{\mathrm{b}}(x,t)$ can be introduced such that the mass conservation equation (7.2) is identically satisfied.

Use of surface elevation instead of cross-sectional area

We usually work in terms of water surface elevation ("stage") η, which is easily measurable and which is practically more important. We make a significant assumption here, but one that is usually accurate: the water surface is horizontal across the stream. Now, if the surface changes by an amount $\delta\eta$ in an increment of time δt, then the area changes by an amount $\delta A = B\,\delta\eta$, where B is the width of the stream surface. Taking the usual limit of small variations in calculus, we obtain $\partial A/\partial t = B\,\partial\eta/\partial t$, and the mass conservation equation can be written

$$B\frac{\partial\eta}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{7.4}$$

The discharge Q could be written as $Q = UA$, where U is the mean streamwise velocity over a section, and substituted into this. However, the discharge is more practical and fundamental than the velocity, and that will not be done here.

7.2 Momentum conservation equation for channel flow

The equation

The conservation of momentum principle is now applied to the mixture of water and suspended solids in the main channel for a moving and deformable control volume (White 2003, §§3.2 & 3.4). The x-component is

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho u\,\mathrm{d}V}_{\text{Unsteady term}} + \underbrace{\int_{\mathrm{CS}} \rho u\,\mathbf{u}_{\mathrm{r}}\cdot\hat{\mathbf{n}}\,\mathrm{d}S}_{\text{Fluid inertia term}} = P_x, \tag{7.5}$$

where $\mathbf{u}$ is the fluid velocity with x-component u, $\mathrm{d}V$ is an element of volume, $\mathbf{u}_{\mathrm{r}}$ is the velocity vector of the fluid relative to the local element of the control surface, which is possibly moving itself, $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards, and $\mathrm{d}S$ is an elemental area of the surface. The quantity $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}}$ is the component of relative velocity normal to the surface at any point. It is this velocity that is responsible for the transport of any quantity across the surface, momentum here. P_x is the force exerted on the fluid in the control volume by both body forces, which act on all fluid particles, and surface forces which act only on the control surface.

Hydraulic approximations

1. **Unsteady term**

   The element of volume is $\mathrm{d}V = \Delta x\,\mathrm{d}A$, and the term contribution can be written

   $$\rho\,\Delta x\,\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_A u\,\mathrm{d}A = \rho\,\Delta x\,\frac{\partial Q}{\partial t}, \tag{7.6}$$

   where the integral $\int_A u\,\mathrm{d}A$ has a simple and practical significance – it is just the discharge Q, so that the contribution of the term can be written simply as shown, but where again it has been necessary to use the partial differentiation symbol, as Q is a function of x as well. No additional approximation has been made in obtaining this term. It can be seen that the discharge Q plays a simple role in the momentum of the flow.

2. **Fluid inertia term**

   The second term on the left of equation (7.5) is $\int\limits_{\mathrm{CS}} \rho u\,\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, has its most important contributions from the stationary vertical faces perpendicular to the main flow.

   a. **Top and bottom, possibly moving surfaces:** we have chosen our control surface to coincide with these boundaries so that no fluid crosses them, $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}} = 0$ and there is no contribution.

b. **Stationary vertical faces:** on the upstream face, $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}} = -u$, giving the contribution to the term of $-\rho\int_A u^2\,\mathrm{d}A$. The downstream face at $x+\Delta x$ has a contribution of a similar nature, but positive, and where all quantities have increased over the distance Δx. The net contribution, the difference between the two, after neglecting terms like $(\Delta x)^2$, can be written

$$\rho\,\Delta x\,\frac{\partial}{\partial x}\int\limits_A u^2\,\mathrm{d}A.$$

In equation (2.12), much earlier, we approximated the integral over the cross section and with a mean in time, in terms of a Boussinesq coefficient β such that the contribution to the equation is then simply but approximately.

$$\rho\,\Delta x\,\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right). \tag{7.7}$$

It is useful to retain the β, unlike many presentations that implicitly assume it to be 1.0, as it is a signal and reminder to us that we have introduced an approximation.

c. **Lateral momentum contributions:** If there is also fluid entering or leaving from rainfall, tributaries, or seepage, there are contributions over the other faces. Their contribution to momentum exchange is small and uncertain and we will ignore them.

3. **Contributions to force** P_x

a. **Body force**: for the straight channel considered, the only body force acting is gravity; we will consider only the x-component of the momentum equation, which have chosen to be horizontal, as gravity only has a component in the $-z$ direction, there will be no contribution from gravity to our equation! The manner in which gravity acts is to cause pressure gradients in the fluid, giving rise to the following term, due to pressure variations around the control surface.

b. **Pressure forces**: these act normally to the control surface. The direction of the pressure force on the fluid at the control surface is $-\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward-directed normal; its local magnitude is $p\,\mathrm{d}S$, where p is the pressure and $\mathrm{d}S$ an elemental area of the control surface. Hence, the total pressure force is the integral $-\int_{\mathrm{CS}} p\hat{\mathbf{n}}\,\mathrm{d}S$. This is difficult to evaluate for arbitrary control surfaces, as the pressure and the non-constant unit vector have to be integrated over all the faces. A much simpler derivation is obtained if the term is evaluated using Gauss' Divergence Theorem:

$$-\int_{\mathrm{CS}} p\hat{\mathbf{n}}\,\mathrm{d}S = -\int_{\mathrm{CV}} \nabla p\,\mathrm{d}V,$$

where $\nabla p = (\partial p/\partial x, \partial p/\partial y, \partial p/\partial z)$, the vector gradient of pressure. This has turned a complicated surface integral into a volume integral of a simpler quantity.

We only need the x component $-\int_{\mathrm{CV}} \partial p/\partial x\,\mathrm{d}V$, the volume integral of the streamwise pressure gradient. The hydraulic approximation now has a problem, because we have not attempted to calculate the detailed pressure distribution throughout the flow. However, in most places in most channel flows the length of disturbances is much greater than the depth, so that streamlines in the flow are only very gently sloping and gently curved, and the pressure in the fluid is accurately given by the equivalent hydrostatic pressure, that due to a stationary column of water of the same depth. Hence we write for a point of elevation z, our equation (2.5) gives

$$p = \rho g \times \text{Depth of water above point} = \rho g(\eta - z),$$

where η is the elevation of the free surface above that point. The quantity that we need is the horizontal pressure gradient $\partial p/\partial x = \rho g\, \partial\eta/\partial x$, and so the streamwise pressure gradient is entirely due to the slope of the free surface. The contribution is

$$-\int_A \frac{\partial p}{\partial x}\,\mathrm{d}V \approx -\rho\,\Delta x\, g \int_A \frac{\partial \eta}{\partial x}\,\mathrm{d}A \approx -\rho\,\Delta x\, gA\frac{\partial \eta}{\partial x}, \tag{7.8}$$

where any variation with y has been ignored, as the surface elevation usually varies little across the channel, and so $\partial\eta/\partial x$ is constant over the section and has been able to be taken outside the integral, which is then simply evaluated.

c. **Resistance due to shear:** there is little that we can say that is exact about the shear forces. We have already considered resistance in some detail, however, and in equation (3.21) we we have

$$\frac{\tau}{\rho} = \Lambda U^2 = \Lambda \left(\frac{Q\,(x,t)}{A\,(x,t)}\right)^2 = \frac{1}{\gamma^2(\varepsilon)} \left(\frac{Q\,(x,t)}{A\,(x,t)}\right)^2,$$

where we could use a value of Λ from the figure on page 45 or from the formulae given therein, or we could use our result for the Gauckler-Manning-Strickler formula where $\gamma(\varepsilon) = 6.7/\varepsilon^{1/6}$. The value of τ is the mean around the solid boundary, so to obtain the force we multiply by the wetted perimeter P and length of the element Δx and instead of Q^2 we write $-Q\,|Q|$, to allow for possible negative Q in estuaries, as resistance always opposes the motion:

$$\text{Total horizontal shear force on control surface} = -\rho\,\Delta x\,\Lambda \frac{Q\,|Q|}{A^2} P. \tag{7.9}$$

Collection of terms and discussion

Now all contributions from the hydraulic approximations to terms in equation (7.5) are collected, using equations (7.6), (7.7), (7.8), and (7.9), and bringing all derivatives to the left and others to the right, all divided by $\rho\Delta x$, gives the momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\Lambda P \frac{Q\,|Q|}{A^2}. \tag{7.10}$$

It is convenient to generalise the resistance term so as to be able to incorporate Gauckler-Manning-Strickler resistance as well as situations where a *Rating Curve* is known from river measurements, giving a relationship between measured discharge, supposed steady and uniform, and local cross-sectional area, $Q_{\mathrm{r}}(A)$. We write the equation as

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\Omega Q\,|Q|\,, \tag{7.11}$$

where the coefficient Ω, of dimensions L^{-3}, is a function of resistance coefficient, cross-sectional area, and wetted perimeter.

$$\Omega = \begin{cases} gA\tilde{S}/Q_{\mathrm{r}}^2(A), & \text{in terms of rated discharge } Q_{\mathrm{r}}(A); \\ \Lambda P/A^2, & \text{Chézy-Weisbach, where } \Lambda = \lambda/8 = g/C^2; \\ gP^{4/3}/k_{\mathrm{St}}^2 A^{7/3}, & \text{Gauckler-Manning-Strickler.} \end{cases} \tag{7.12}$$

Example 5 Verify the use of the three resistance forms for steady uniform flow, on a uniform slope $\tilde{S} = S$.

In this case, the flow is steady so the first term in equation (7.11) is zero, and uniform so that the second is zero. The surface slope $\partial\eta/\partial x = -S$, and as Q is positive, $Q\,|Q| = Q^2$ and the momentum equation (7.11) gives $-gAS = -\Omega Q^2$, so that

$$Q = \sqrt{\frac{gAS}{\Omega}} = \begin{cases} \sqrt{\frac{gAS}{gAS}Q_{\mathrm{r}}^2} = Q_{\mathrm{r}}, & \text{in terms of rated discharge } Q_{\mathrm{r}}(A); \\ \sqrt{\frac{gA^3S}{\Lambda P}} = A\sqrt{\frac{g}{\Lambda}\frac{A}{P}S}, & \text{Chézy-Weisbach;} \\ \sqrt{\frac{gAS}{gP^{4/3}}k_{\mathrm{St}}^2 A^{7/3}} = Ak_{\mathrm{St}}\left(\frac{A}{P}\right)^{2/3}\sqrt{S}, & \text{Gauckler-Manning-Strickler.} \end{cases} \tag{7.13}$$

At this stage the non-trivial assumptions in the derivation are stated, roughly in decreasing order of importance (they are actually not very restrictive at all!):

1. Resistance to flow is modelled empirically. The Navier-Stokes equations are not being used.
2. All surface variation is sufficiently long and of small slope that the pressure throughout the flow

is given by the hydrostatic pressure corresponding to the depth of water above each point.

3. Effects of curvature of the stream course are ignored.
4. In the momentum flux term the effects of both non-uniformity of velocity over a section and turbulent fluctuations are approximated by a momentum or Boussinesq coefficient.
5. Surface elevation η across the stream is constant.

Relating surface slope $\partial \eta / \partial x$ and $\partial A / \partial x$

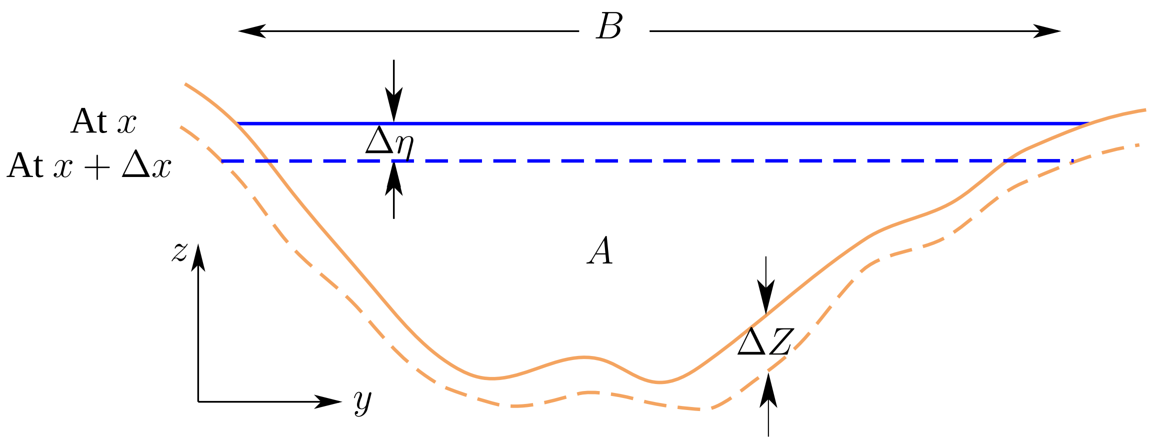

Figure 7.2: Two channel cross-sections separated by Δx

In the momentum equation (7.10) when expanded, the dependent variables are discharge Q and a mixture of derivatives of area $\partial A / \partial x$ and surface elevation $\partial \eta / \partial x$. We must relate the two, and now consider the bottom geometry in greater detail, although in practice the precise details of the bed are often not known. This will help us know when to make approximations.

The cross-section of a river in Figure 7.2 shows how ambiguous and possibly non-unique the concept of the "bottom" of the stream may be. In a distance Δx the surface elevation may change by an amount $\Delta \eta$ as shown, so that the contribution to the change in cross-section area ΔA is $B \times \Delta \eta$, where $\Delta \eta$ is usually negative as the surface drops downstream. The change in the bed is

ΔZ, which in general varies across the section, with contribution to ΔA of $-\int_B \Delta Z\, dy$, the area between the solid and dotted lines on the figure corresponding to the bed at the two locations. The minus sign is because, if the bed drops away and ΔZ is negative, as usual, the contribution to area increase is positive. Combining the two terms,

$$\Delta A = B\,\Delta\eta - \int\limits_B \Delta Z\,\mathrm{d}y \tag{7.14}$$

For the second contribution, the integral of the change in bed elevation across the stream, we introduce the symbol $\tilde{S}$ for the mean downstream bed slope across the section such that

$$\tilde{S} = -\frac{1}{B}\int\limits_B \frac{\partial Z}{\partial x}\,\mathrm{d}y, \tag{7.15}$$

where the negative sign has been introduced such that in the usual case when Z decreases with x, this mean downstream bed slope at a section is positive. In an important problem where bed details might be known, this can be evaluated. In the usual case where bed topography is poorly known, a reasonable local approximation or assumption is made. Using equations (7.14) and (7.15) we can write

$$\Delta A = B\,\Delta\eta + B\tilde{S}\,\Delta x,$$

where in a distance Δx the *mean* bed level across the channel then changes by $-\tilde{S} \times \Delta x$ under the water. In the rare case where the sides of the stream are vertical diverging or converging walls, an extra term would have to be included. Taking the usual calculus limit, we obtain

$$\frac{\partial A}{\partial x} = B \left(\frac{\partial \eta}{\partial x} + \tilde{S} \right), \tag{7.16}$$

which might have been able to have been written down without the mathematical details.

7.3 Forms of the governing equations

We use equation (7.16) to eliminate first $\partial\eta/\partial x$ and then $\partial A/\partial x$ to give two alternative forms of the momentum equation governing flows and long waves in waterways. In both cases, we restate the corresponding mass conservation equation, using (7.1) and (7.4), to give the pairs of equations:

Formulation 1 – Long wave equations in terms of area A and discharge Q

Eliminating $\partial\eta/\partial x$ gives the equations in terms of A and Q:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i, \tag{7.17a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + \frac{gA}{B}\frac{\partial A}{\partial x} = gA\tilde{S} - \Omega Q\left|Q\right|. \tag{7.17b}$$

Formulation 2 – Long wave equations in terms of stage η and discharge Q

Now eliminating $\partial A/\partial x$, but retaining A in all coefficients, as it can be calculated in terms of η:

$$\frac{\partial \eta}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = \frac{i}{B}, \tag{7.18a}$$

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2 B}{A^2}\right)\frac{\partial \eta}{\partial x} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Omega Q\left|Q\right|. \tag{7.18b}$$

These equations are the basis of computational hydraulics and flood routing. There is much commercial software written to solve them. They are actually quite simple in the form here!