Non-uniform and unsteady flows

We will be considering flows which are not uniform and those which are neither uniform nor steady. As the length scale of river flows is much longer in space than the cross-sectional dimensions and the time scale of disturbances is much longer than that of local turbulence, we will assume that the boundary stress at each place and at each time is given by the local immediate flow conditions of velocity, in terms of discharge Q and area A. From equation (3.15), and using $\Lambda = \lambda/8$ or from our presentation of the Gauckler-Manning-Strickler equation, we have

$$\frac{\tau}{\rho} = \Lambda U^2 = \Lambda \left(\frac{Q(x,t)}{A(x,t)} \right)^2 = \frac{1}{\gamma^2(\varepsilon)} \left(\frac{Q(x,t)}{A(x,t)} \right)^2, \tag{3.21}$$

where we could use a value of Λ from the figure on page 45 or from the formulae given therein, or we could use our result for the Gauckler-Manning-Strickler formula where $\gamma(\varepsilon) = 6.7/\varepsilon^{1/6}$.

4. Froude number

William Froude (1810-1879, pronounced as in "food") was a naval architect who proposed similarity rules for free-surface flows. A Froude number is a dimensionless number from a velocity scale U and a length scale L, $\mathsf{F} = U/\sqrt{gL}$. In the original definition, of a ship in deep water, the only length scale was L, the length of the ship. In river engineering it is not obvious what the length scale is. Might it be the wetted perimeter P, might it be the geometric mean depth A/B, where A is cross-sectional area and B is surface width?

In fact, the answer will usually depend on the problem. If we consider the *mean total head* of a channel flow at a section, where U is mean velocity and η is surface elevation:

$$H = \eta + \alpha\frac{U^2}{2g},$$

it does *not* appear explicitly. Neither does it appear in the momentum flux M at a section

$$M = \rho g \left(A\bar{h} + \beta U^2 A\right).$$

Here, we are going simply to define it in terms of a vertical length scale, the depth scale A/B:

$$\mathsf{F}^2 = \frac{U^2}{gA/B} = \frac{Q^2 B}{gA^3}, \tag{4.1}$$

where $Q = UA$ is discharge. For many years the lecturer was more specific, using terms of kinetic and potential energy, now all he says is that F^2 is a measure of dynamic effects relative to gravitational, the latter measured by mean depth.

Even if we were to consider rather more complicated problems such as the unsteady propagation of waves and floods, and to non-dimensionalise the equations, we would find that the Froude number F itself never appears in the equations, but always as $\alpha\mathsf{F}^2$ or $\beta\mathsf{F}^2$, depending on whether energy or momentum considerations are being used.

Flows which are fast and shallow have large Froude numbers, and those which are slow and deep have small Froude numbers. Generally F^2 is an expression of the wave-making ability of a flow, and in conversation we usually use "high/ low Froude number" as an expression of how fast a flow is. For example, consider a river or canal which is $2\ \mathrm{m}$ deep flowing at $0.5\ \mathrm{ms}^{-1}$ (make some effort to imagine it - we can well believe that it would be able to flow with little surface disturbance!). We have

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 2}} = 0.11 \quad \text{and} \quad F^2 = 0.012\,,$$

and we can imagine that the wavemaking effects are small. Now consider flow in a street gutter after rain. The velocity might also be $0.5\ \mathrm{ms}^{-1}$, while the depth might be as little as $2\ \mathrm{cm}$. The Froude number is

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 0.02}} = 1.1 \quad \text{and} \quad F^2 = 1.2\,,$$

and we can easily imagine it to have many waves and disturbances on it due to irregularities in the gutter.

Near-constancy of Froude number in a stream

It is interesting to calculate the Froude number F of a steady uniform flow given by the Chézy-Weisbach formula for discharge:

$$Q = \sqrt{\frac{gA^3}{\Lambda P} S} \, .$$

Immediately this gives

$$\mathsf{F}^2 = \frac{Q^2 B}{gA^3} = \frac{S}{\Lambda}\frac{B}{P} \, ,$$

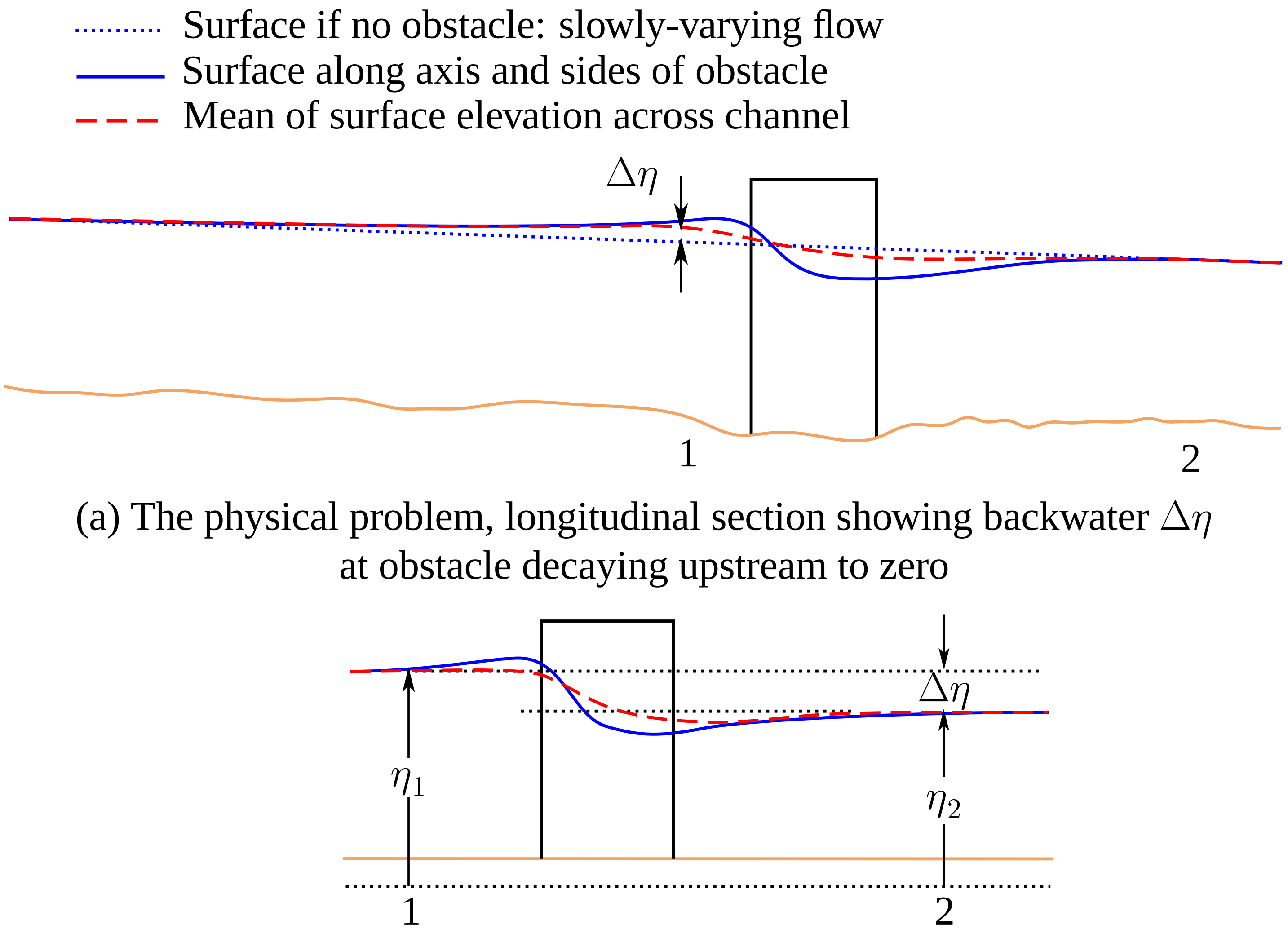
and as $B \approx P$ for wide channels, we see that the square of the Froude number is approximately equal to the ratio of bed slope S to resistance coefficient Λ, giving some significance and physical feeling for Λ. This means that for a particular reach of river, where slope S is effectively independent of flow, where B/P also does not vary much with the flow and Λ often does not vary much, the Froude number F does not change much with flow. While a flood might look more dramatic than a more-common low flow, because it is faster and higher, the Froude number is roughly the same for both.

5. The effect of obstructions on streams – an approximate method

River Traun, Bad Ischl, Oberösterreich

Structures such as weirs can almost completely block a river, but there are also other types of obstacles that are only a partial blockage, such as the piers of a bridge, blocks on the bed, Iowa vanes, *etc.* or possibly more importantly, the effects of trees placed in rivers (”Large Woody Debris”), used in their environmental rehabilitation. It might be important to know what the forces on the obstacles are, or, more importantly for us, what effects the obstacles have on the river.

The physical problem and its idealisation



(a) The physical problem, longitudinal section showing backwater $\Delta\eta$ at obstacle decaying upstream to zero

(b) The idealised problem, uniform channel with no friction or slope

Figure 5.1: A typical physical problem of flow past a bridge pier, and its idealisation for hydraulic purposes

Momentum flux in a channel

The momentum flux across a section is defined to be the sum of the pressure force, plus the mass rate of transport multiplied by the velocity. For a vertical section, the mass rate of transport is $\rho u \, \mathrm{d}A$, so the momentum flux is

$$M = \int_A \left(p + \rho u^2\right) \, \mathrm{d}A.$$

Substituting the hydrostatic pressure distribution, $p = \rho g \left(\eta - z\right)$, where η is the free surface elevation, we obtain

$$M = \rho \int_A \left(g \left(\eta - z\right) + u^2\right) \, dA.$$

- The integral $\int_A (\eta - z) \, dA$ is simply the first moment of area about a transverse horizontal axis *at the surface*, we can write it as

  $$\int_A (\eta - z) \, dA = A\bar{h}$$

  where $\bar{h}$ is the *depth* of the centroid of the section *below the surface*.
- The velocity contribution we have already evaluated as $\int_A u^2 \, dA \approx \beta U^2 A = \beta Q^2 / A$.

Collecting contributions, we have the expression for the momentum flux at a section

$$M = \rho \left(g A \bar{h} + \beta \frac{Q^2}{A} \right) . \qquad (5.1)$$

Momentum conservation

Consider the momentum conservation equation if a force P is applied in a negative direction to a flow between two sections 1 and 2:

$$P = \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2, \tag{5.2}$$

Usually one wants to calculate the effect of the obstacle on water levels. The effects of drag can be estimated by knowing the area of the object measured transverse to the flow, a, the drag coefficient C_{D}, and u, the mean fluid speed past the object:

$$P = \frac{1}{2}\rho C_{\mathrm{D}} u^2 a, \tag{5.3}$$

and so, substituting into equation (5.2) gives, after dividing by density,

$$\frac{1}{2}C_{\mathrm{D}} u^2 a = \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2. \tag{5.4}$$

We consider the velocity u on the obstacle as being proportional to the upstream velocity, such that we write

$$u^2 = \Gamma\left(\frac{Q}{A_1}\right)^2, \tag{5.5}$$

where Γ is a coefficient which recognises that the velocity which impinges on the object is generally not equal to the *mean* velocity in the flow. For a small object near the bed, Γ could be quite small; for an object near the surface it will be slightly greater than 1; for objects of a vertical scale that of the whole depth, $\Gamma \approx 1$. Equation (5.4) becomes

$$\frac{1}{2}\Gamma\, C_{\mathrm{D}}\frac{Q^2}{A_1^2}a = \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2 \tag{5.6}$$

A typical problem is where the downstream water level is given (sub-critical flow, so that the control is downstream), and we want to know by how much the water level will be raised upstream if an obstacle is installed. As both A_1 and $\overline{h}_1$ are functions of h_1, so that we would need to know in detail the geometry of the stream, and then to solve the transcendental equation for h_1. However, by *linearising* the problem, solving it approximately, we obtain a simple explicit solution that tells us rather more.

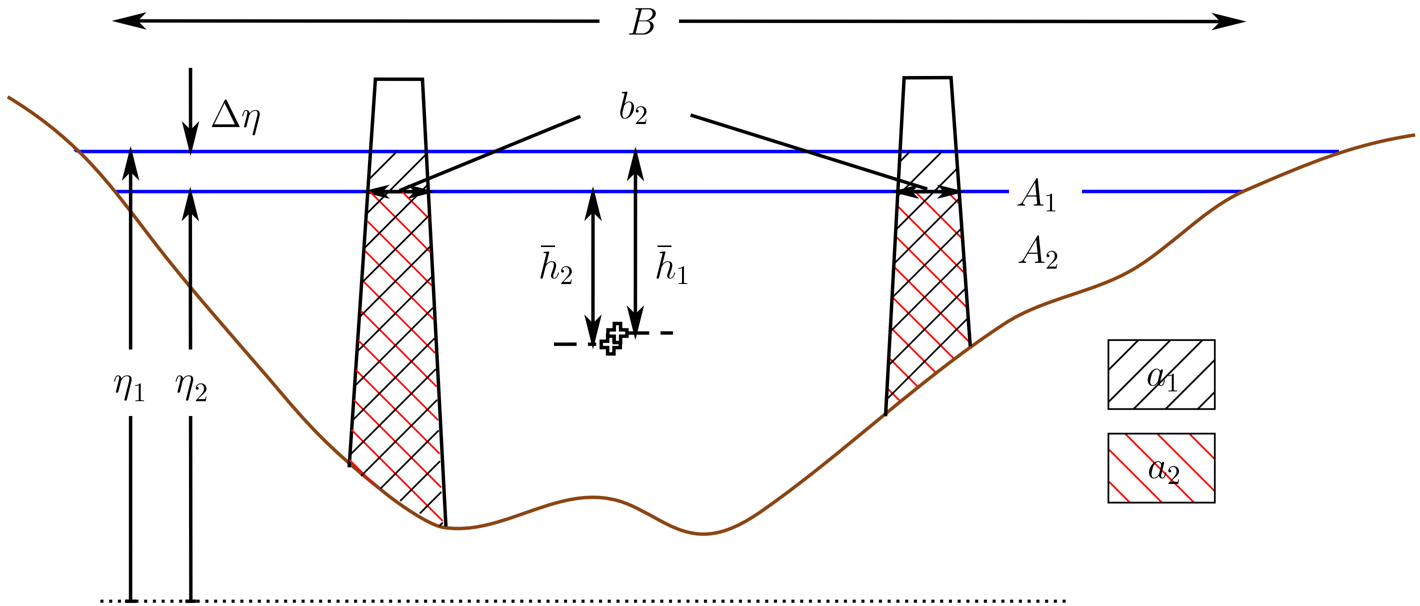


Figure 5.2: Cross-section showing dimensions for water levels at 1 and 2

Consider the stream cross-section shown in Fig. 5.2, with a small change in water level $\eta_1 = \eta_2 + \Delta\eta$. We now use geometry to obtain approximate expressions for quantities at 1 in terms of those at 2. It is easily shown that

$$A_1 = A_2 + B_2\,\Delta\eta + O\left((\Delta\eta)^2\right), \text{ and } \left(A\bar{h}\right)_1 = \left(A\bar{h}\right)_2 + A_2\,\Delta\eta + O\left((\Delta\eta)^2\right) ,$$

and similarly we write for the blockage area $a = a_2 + b_2\,\Delta\eta + O\left((\Delta\eta)^2\right)$, where b_2 is the surface width of the obstacle (which for a submerged obstacle would be zero). We have actually been using Taylor series expansions, but the physical interpretations seem simpler than the mathematical!

The momentum equation (5.6) gives us

$$\frac{1}{2}\Gamma\,C_{\mathrm{D}}\frac{Q^2}{A_1^2}a_2 = gA_2\,\Delta\eta + \beta\frac{Q^2}{A_2 + B_2\,\Delta\eta} - \beta\frac{Q^2}{gA_2}.$$

Now we use a power series expansion in $\Delta\eta$ to simplify the term in the denominator:

$$\frac{1}{A_2 + B_2\,\Delta\eta} = \frac{1}{A_2\left(1 + B_2\,\Delta\eta/A_2\right)} = \frac{1}{A_2}\left(1 + B_2\,\Delta\eta/A_2\right)^{-1} \approx \frac{1}{A_2}\left(1 - B_2\,\Delta\eta/A_2\right),$$

neglecting terms like $(\Delta\eta)^2$ (see equation A-1). The momentum equation becomes

$$\frac{1}{2}\Gamma\,C_{\mathrm{D}}\frac{Q^2}{gA_1^2}a_2 \approx \Delta\eta\,A_2\left(1 - \beta\frac{Q^2B_2}{gA_2^3}\right).$$

As $A_1 = A_2 + O(\Delta\eta)$ we replace A_1 by A_2 and introduce $\mathsf{F}_2^2 = Q^2B_2/gA_2^3$, the square of the Froude number of the downstream flow. The equation is easily solved to give an explicit approximation for the dimensionless drop across the obstacle $\Delta\eta/\left(A_2/B_2\right)$, where A_2/B_2 is the mean downstream depth:

$$\frac{\Delta\eta}{A_2/B_2} = \frac{\frac{1}{2}\,\Gamma\,C_{\mathrm{D}}\,\mathsf{F}_2^2}{1-\beta\mathsf{F}_2^2}\frac{a_2}{A_2}. \tag{5.7}$$

This explicit approximate solution has revealed the important quantities of the problem to us and how they affect the result: downstream Froude number $\mathsf{F}_2^2 = Q^2B_2/gA_2^3$ and the relative blockage area a_2/A_2. For subcritical flow $\beta\mathsf{F}_2^2 < 1$ the denominator in (3.13) is positive, and so is $\Delta\eta$, so that the surface drops from 1 to 2, as we expect. If the flow is supercritical, $\beta\mathsf{F}_2^2 > 1$, we find $\Delta\eta$ negative, and the surface rises between 1 and 2. If the flow is near critical $\beta\mathsf{F}_2^2 \approx 1$, the change in depth will be large, which is made explicit, and the theory will not be valid.

We could immediately estimate how important this is. We see that, for small Froude number $\mathsf{F}_2^2 \ll 1$, such that $1-\beta\mathsf{F}_2^2 \approx 1$, the relative change of depth is equal to $\frac{1}{2}$ times $\Gamma \approx 1$ (for a body extending the whole depth), times $C_{\mathrm{D}} \approx 1$ for cylinders *etc*., multiplied by F_2^2, usually small, multiplied by the blockage ratio a_2/A_2, which is also probably small. So, the relative result is usually small. However, the absolute value might still be finite compared with resistance losses, as will be seen below.

Another benefit of the approximate analytical solution is that it shows that such an obstacle forms a control in the channel, so that the finite sudden change in surface elevation $\Delta\eta$ is a function of Q^2, or Q a function of $\sqrt{\Delta\eta}$, in a manner analogous to a weir. In numerical river models it should ideally be included as an internal boundary condition between different reaches as if it were a type of fixed control.

The mathematical step of linearising has revealed much to us about the nature of the problem that the original momentum equation did not.

Example 3 It is proposed to build a bridge, where the bridge piers occupy about 10% of the "wetted area" of a river with Froude number 0.5 (which is quite large). How much effect will this have on the river level upstream?

As the bridge piers occupy all the depth, we have $\Gamma = 1$. A typical drag coefficient is $C_{\mathrm{D}} \approx 1$. We will use $\beta = 1$ (this is an estimate!). So we find, using equation (3.13):

$$\begin{aligned} \frac{\Delta\eta}{A_2/B_2} &= \frac{\frac{1}{2}\,\Gamma\, C_{\mathrm{D}}\, \mathsf{F}_2^2}{1-\beta\mathsf{F}_2^2}\frac{a_2}{A_2} \\ &\approx \tfrac{1}{2}\times 1 \times 1 \times 0.1 \times \frac{0.5^2}{1-0.5^2} \\ &= 0.017, \end{aligned}$$

about 2% of the mean depth. This seems small, but if the river were $2\ \mathrm{m}$ deep, there is a $4\ \mathrm{cm}$ drop across the bridge. If the slope of the river were $S = 10^{-4}$, this would correspond to the surface level change in a length of $400\ \mathrm{m}$, which can hardly be neglected.