Resistance in river and other open channel flows

The resistance to the flow of a stream is probably the most important problem in river mechanics.

Page 27: We consider a simple theory based on force balance and some classical fluid mechanics experiments to obtain a flow formula for a wide channel.

Page 29: To obtain the equivalent formula for channels of any section we consider velocity distributions in real streams and develop an approximation giving a general flow formula.

Page 34: We consider an approximation to that formula and find that we have obtained the Gauckler-Manning-Strickler formula, including a theoretical prediction of Strickler's formula for the effect of boundary grain size.

Page 37: Comparison with a series of experiments validates the approach, giving an explicit flow formula for a variety of channel boundaries.

Page 39: The common problem of calculating the water depth for a given flow rate is considered. A computational method is developed and applied.

Page 42: For more general river problems, considering the nature of the bed particles and bed forms, vegetation, meandering, and possibly obstacles, it is better to use a formulation in which forces and the mechanics are clearer: the Chézy-Weisbach flow formula.

Page 44: A large number of stream-gauging results are considered and the values of the Weisbach resistance coefficient, its dependence on grain size, and on the state of the bed are obtained. Empirical formulae are considered.

A channel flow formula from theory and experiment

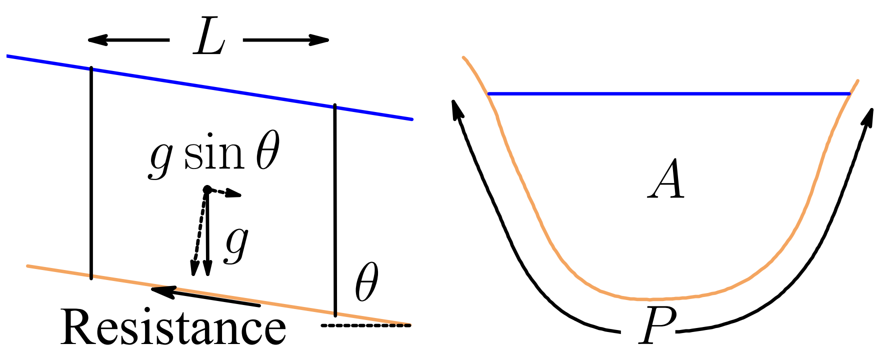


Figure 1. Uniform flow in a channel, showing resistance and gravity forces on a finite length, plus cross-section quantities

Here, as an example of the application of rational mechanics, a flow formula for steady uniform flow in channels is developed without using the empirical flow formulae of Gauckler-Manning-Strickler or Chézy-Weisbach.

Consider a horizontal length L of uniform channel flow, inclined at a small angle θ to the horizontal, with cross-sectional area A. The volume of the element is AL, the vertical gravitational force on the water is $\rho g AL$, where ρ is fluid density and g is gravitational acceleration. The component of this along the slope is $\rho g AL \sin\theta$. The resistance force along the slope, of length $L/\cos\theta$ is $\tau PL/\cos\theta$, where τ is the mean resistance shear stress, assumed uniformly distibuted around the wetted perimeter P around which it acts. Equating gravitational and resistance components gives $\tau PL/\cos\theta = \rho g AL \sin\theta$. To high accuracy for small θ, $\cos\theta \approx 1$ and $\sin\theta \approx \tan\theta = S$, the slope, giving

$$\frac{\tau}{\rho} = g\frac{A}{P}S. \tag{1}$$

Our problem is now to express shear stress τ in terms of flow quantities.

One of the most famous series of experiments in fluid mechanics was performed by Johann Nikuradse at Göttingen in the 1930s, who studied the flow of fluid over uniformly-rough sand grains. The fluid was actually air, and the sand grains were actually in circular pipes, but the results are still valid enough.

With those results, for a wide channel of depth h with sand grains of size k_{s}, the velocity distribution for fully rough flow (no effects of viscosity), the Prandtl-von Kármán *universal velocity distribution* can be written:

$$u = \frac{u_*}{\kappa} \ln \frac{30\,z}{k_{\mathrm{s}}},$$

in terms of the shear velocity $u_* = \sqrt{\tau/\rho}$, the von Kármán constant $\kappa \approx 0.4$, the vertical co-ordinate z, and where the factor of 30 is for closely-packed uniform sand grains. It varies somewhat with other types of boundary roughness. The mean velocity U is obtained by integrating between 0 and h, such that

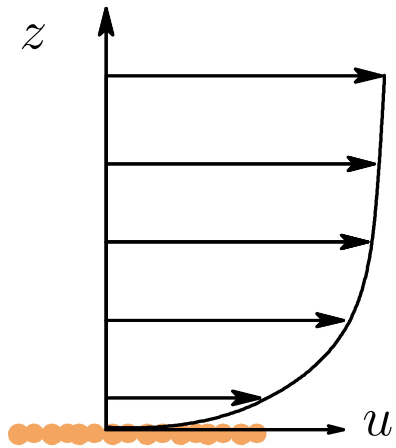


Figure 2. Idealised logarithmic velocity profile in turbulent flow over rough bed

$$U = \frac{u_*}{\kappa} \ln \frac{30/\mathrm{e}}{k_{\mathrm{s}}/h}.$$

where e is Euler's number $\exp(1) = 2.718\ldots$, obtaining the result in terms of *relative roughness* k_{s}/h. Now we replace $u_* = \sqrt{\tau/\rho}$ by the expression on the right of equation (1) to give

$$U = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln \frac{30/\mathrm{e}}{k_{\mathrm{s}}/h}\right), \qquad (2)$$

We have obtained something possibly useful – a formula for the mean flow velocity in a wide channel of constant depth h, slope S, and relative roughness k_{s}/h. We have used simple mechanics plus empirical laboratory results. Surprisingly, the formula is explicit in terms of physical quantities – we have not had to assume a value like the Strickler k_{St}!

That was for a wide channel with an idealised logarithmic velocity distribution. In nature, for channels of any general cross-section there is the problem that the velocity has a maximum at a somewhere below the surface, and in general the isovels are something like Figure 3.

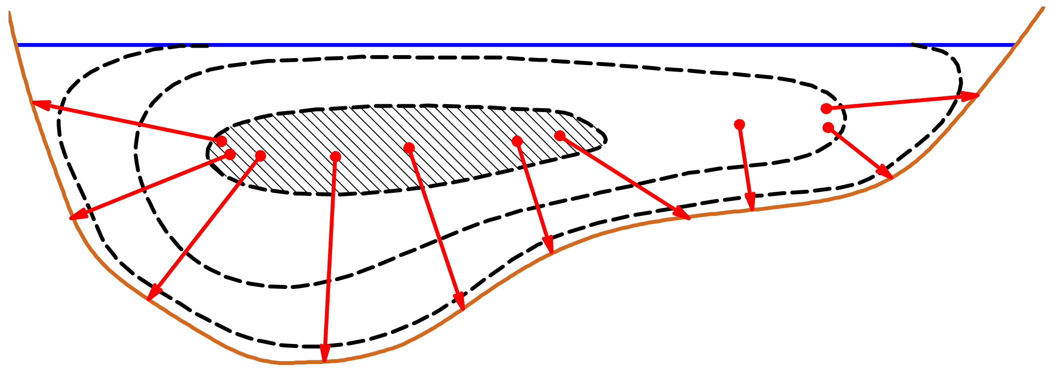


Figure 3. Cross-section of flow showing isovels and, for a number of points on the bed, where the fastest-likely fluid comes from and how far it travels, the effective length scale for resistance calculations.

To obtain a flow formula for channels of any cross-section, we hypothesise that the effective depth h for resistance calculations is the typical distance from points with the highest velocity to the nearest point on the bed, as suggested by the red arrows on the figure. If this model is correct, typical length scales as shown by the arrows are somewhat *smaller* than the overall mean depth of flow.

This is a highly approximate model, but at least it is in the spirit of modelling, that it is simple and transparent – and so far has not been obscured by mathematical detail.

Our problem is then, how can we simply approximate that distance?

We consider the experimental data for the vertical position of the locus of velocity maxima in *rectangular* channels from Yang *et al.* (2004). They presented a formula for the height above the bed of the velocity maximum as a function of position across the channel. If the mean value of this is calculated by integration, a formula for the mean elevation of the velocity maximum $z_{\max}/h$ is obtained as a function of aspect ratio (channel width B divided by depth h).

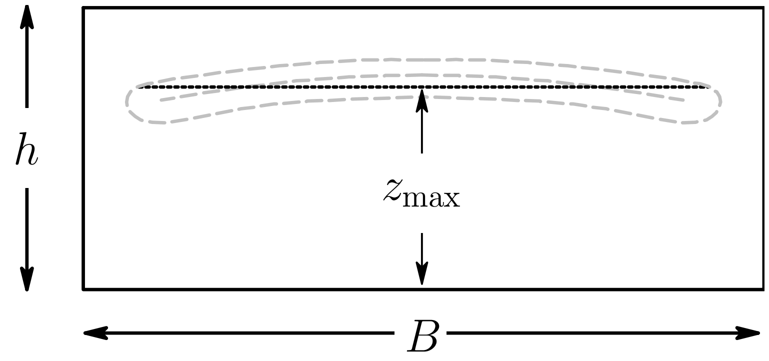


Figure 4. Experimental determination of velocity maxima in rectangular channel

The only other length scale we have above in equation (2) is the ratio of area to perimeter A/P. As $P > B$ it should be smaller than the mean depth A/B, so we will try it:

$$\frac{A/P}{h} = \frac{Bh}{(B+2h)h} = \frac{B/h}{B/h+2}. \qquad (3)$$

Both this and the experimental formula for $z_{\max}/h$ are plotted in Figure 5. Remarkably and coincidentally, the two coincide closely over a wide range of aspect ratios! We cannot claim that this is a justification as strong as it looks, but in the absence of anything else, instead of h we will use A/P for channels of any cross-section.

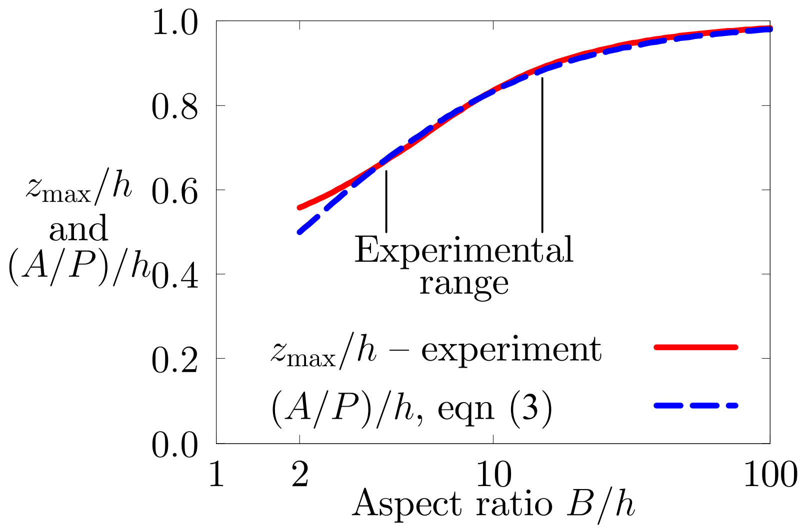


Figure 5. Rectangular channels: dimensionless mean elevation of $z_{\max}/h$ and the effective depth $(A/P)/h$

We have seen that A/P appeared naturally in the simple mechanical equilibrium calculation, and here we have found that it mimics the behaviour of z_{max}/h, which is fortunate as it is a rather simpler quantity to calculate in practice, usually with no knowledge of the flow field. We will name A/P the *hydraulic mean depth*, and we will not use the conventional and misleading term "hydraulic radius" (dt. after Strickler – "Profil- oder hydraulischer Radius"). In channels that are wide, which is most, $P \approx B$ and A/P is about the same as the geometric mean depth A/B.

For channels that are not rectangular we have presented no results. Our suggestion is that A/P will still be a plausible approximation, and it already appears in the Chézy-Weisbach and Gauckler-Manning-Strickler equations (which, officially, we do not yet know in this course!). The use in those equations was justified by Keulegan (1938), however there is much wrong with that work, mathematically correctly integrating logarithmic velocity distributions over various shapes of cross-section but without any attention to real flows in channels.

Our suggested channel flow formula, replacing h by A/P in equation (2) is

$$U = \frac{Q}{A} = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{30/\mathrm{e}}{k_{\mathrm{s}}/\left(A/P\right)}\right), \tag{4}$$

where $30/\mathrm{e} \approx 11.0$ is usually written as 12:

$$U = \frac{Q}{A} = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{12.}{k_{\mathrm{s}}/\left(A/P\right)}\right). \tag{5}$$

Generalised notation for flow formulae

We introduce the symbol ε for the relative roughness

$$\varepsilon = \frac{k}{A/P} = \frac{\text{Grain size}}{\text{Hydraulic mean depth}} \approx \frac{\text{Grain size}}{\text{Depth}},$$

also for equivalent uniform sand grain size, $\varepsilon = k_{\mathrm{s}}/(A/P)$. Our flow formula (5) is then written in generalised form

$$U = \frac{Q}{A} = \gamma\sqrt{g\frac{A}{P}S}, \tag{6}$$

in which we have already obtained the result for the shear velocity

$$u_* = \sqrt{g\frac{A}{P}S}$$

and we introduce the symbol γ for the velocity ratio

$$\frac{U}{u_*} = \gamma = \ln\frac{12.}{\varepsilon},$$

from equation (5).

Relative unimportance of grain size

In fact, γ, although all-important for us, is relatively slowly varying with grain size. Consider a small change in the relative roughness $\varepsilon\,(1+\Delta)$. The relative change in the factor γ is

$$\frac{\Delta\gamma}{\gamma} = \frac{\ln\left(12/(\varepsilon\,(1+\Delta))\right)}{\ln\left(12/\varepsilon\right)} - 1 \approx \frac{-\Delta}{\ln\left(12/\varepsilon\right)},$$

having expanded the logarithm as a power series $\ln\left(1+\Delta\right) = \Delta + \ldots$. Now for a value of $\varepsilon = 0.001$ (a $1\,\mathrm{mm}$ grain in $1\,\mathrm{m}$ of water), a relative change of $\Delta = 50\%$ gives a relative change in the factor γ in the equation of only -5%. Even for a much rougher case of $\varepsilon = 0.1$, the same relative change of 50% in grain size changes the left side by just -11%. It does not matter so much if we cannot specify the bed conditions all that accurately.

The Gauckler-Manning-Strickler formula

We now show that the G-M-S formula is an approximation to the expression we have obtained.

On Figure 6 is shown how the dimensionless factor γ varies as a function of relative roughness ε, given by equation (5) from experimental fluid mechanics. It is actually possible to approximate that curve closely using a monomial function a/ε^{μ}. The best values of a and μ can be found by performing a least-squares fit over 11 points equally-spaced in $\log \varepsilon$ between $\varepsilon = 0.001$ and 0.1. The result obtained was $\mu = 1/7.001$, which is a surprising coincidence. Now, setting $\mu = 1/7$ and determining just a by optimisation, a value of $a \approx 8.9$ was obtained:

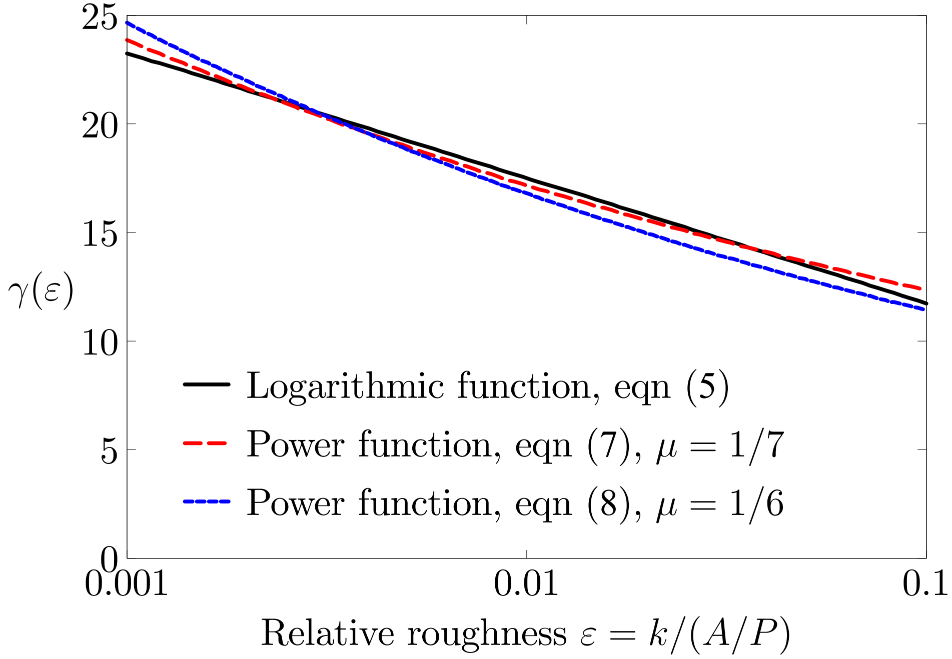


Figure 6.

$$\gamma = 8.9 \left(\frac{A/P}{k_{\mathrm{s}}} \right)^{1/7}, \tag{7}$$

with close agreement with the expression from the logarithmic velocity distribution shown in the figure. This would give us another flow formula, very similar to the Gauckler-Manning-Strickler

(G-M-S) formula. It is

$$U = \frac{Q}{A} = 8.9 \left(\frac{A/P}{k_\mathrm{s}}\right)^{1/7} \sqrt{g\frac{A}{P}S},$$

however unlike G-M-S this is explicit in terms of bed grain size. We do not want to proliferate an already crowded field, so we ignore this. It does, however, suggest the next step. We approximated the result from the logarithmic function again, this time by a function of the form $b/\varepsilon^{1/6}$, where b is a constant. We determined this constant by performing a similar least-squares fit, giving a value of $b \approx 7.8$ such that

$$\gamma = 7.8 \left(\frac{A/P}{k_\mathrm{s}}\right)^{1/6}, \tag{8}$$

with satisfactory results shown in Figure 6, showing that this is also quite a good approximation to our logarithmic function. Substituting into the flow formula, equation (6) and re-writing, we obtain

$$U = \frac{Q}{A} = \frac{7.8\sqrt{g}}{k_\mathrm{s}^{1/6}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S} = k_\mathrm{St} \left(\frac{A}{P}\right)^{2/3} \sqrt{S} = \frac{1}{n} \left(\frac{A}{P}\right)^{2/3} \sqrt{S}, \tag{9}$$

which is simply the Gauckler-Manning-Strickler equation, where k_St is the Strickler coefficient and $n = 1/k_\mathrm{St}$ is the Manning coefficient! Unlike the G-M-S equation, this has given an explicit expression for the Strickler coefficient

$$k_\mathrm{St} = \frac{7.8\sqrt{g}}{k_\mathrm{s}^{1/6}}. \tag{10}$$

A similar result was obtained by Strickler nearly a century ago without optimising software, based entirely on experiment, on boundary roughnesses of equivalent mean diameter from $D = 0.1\,\mathrm{mm}$ to $D = 300\,\mathrm{mm}$,and where that diameter was sometimes calculated from alluvial gravel with relatives lengths of the three axes 1:2:3! For the numerical coefficient he obtained a value of $4.75\sqrt{2} \approx 6.7$, giving his expression

$$k_{\mathrm{St}} = \frac{6.7\sqrt{g}}{D^{1/6}}. \tag{11}$$

The expression (10) here has been obtained by a quite different route, and the agreement between the two expressions, one based on sand grains glued to the inside of a circular pipe carrying air, is encouraging. Of course, Strickler's result (11) is to be preferred.

Sensitivity to boundary particle size

One thing we can do now is, as we did earlier, examine the effect of uncertainty or variability in the size of the boundary particles (and any perceived difference between k_{s} and D), using a power series expansion

$$\frac{\Delta k_{\mathrm{St}}}{k_{\mathrm{St}}} = \left(1 + \frac{\Delta D}{D}\right)^{-1/6} - 1 = -\frac{1}{6}\frac{\Delta D}{D} + \ldots,$$

and so a fractional change in boundary particle size gives a relative change of $1/6$ of that amount in D. Again for this form the exact particle size is actually not so important.

Test of logarithmic and G-M-S formulae

To test the accuracy of the G-M-S formula compared with the logarithmic formula we consider the results of Strickler (1923, Beilage 4), which have been interpreted as the justification for the exponent $2/3$ in the G-M-S formula, and leading to the "S" in that name. Strickler considered results from nine very different channels. For each we calculated the equivalent k_s or D, constant for each channel, by least-squares fitting of the appropriate flow formula to the points, with results shown in the figure. If anything, the Gauckler-Manning-Strickler formula gives better agreement at the lower ends than the logarithmic formula obtained from fluid mechanics experiments.

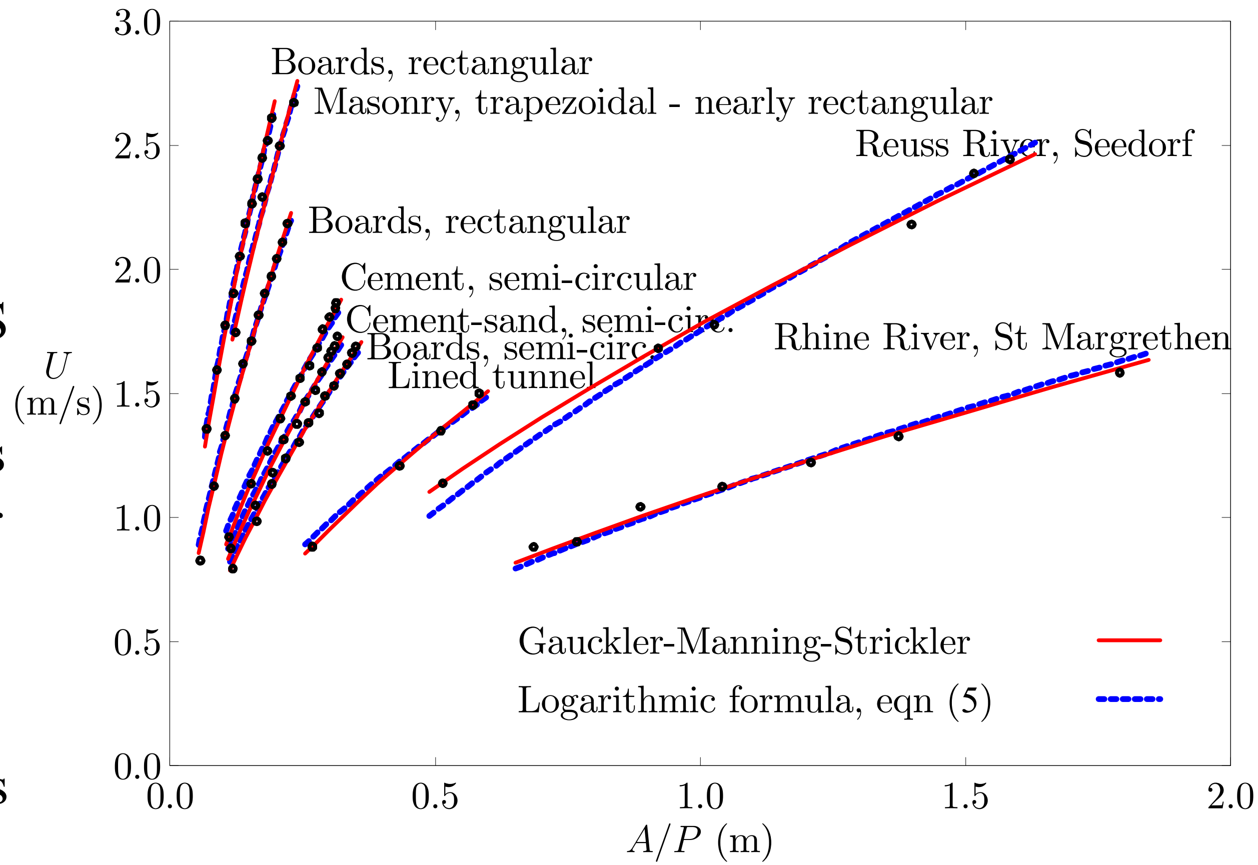


Figure 7. Strickler's results approximated by two flow formulae

Notation

In the rest of this course, we could write the Gauckler-Manning-Strickler formula in the conventional form

$$U = \frac{Q}{A} = k_{\mathrm{St}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S},$$

however if we use the Strickler expression $k_{\mathrm{St}} = 6.7\sqrt{g}/D^{1/6}$ we prefer to write it in the form which more reflects its nature and our derivation:

$$U = \frac{Q}{A} = \gamma\sqrt{g\frac{A}{P}S} \quad \text{where} \quad \gamma(\varepsilon) = \frac{6.7}{\varepsilon^{1/6}}, \text{ and } \varepsilon = \frac{D}{A/P}. \tag{12}$$

- We no longer have the problem of k_{St} with difficult units.
- The characterisation of the resistance has been reduced to that of the dimensionless relative roughness $\varepsilon = D/(A/P)$.
- If asked to estimate the flow at a particular site, we do not have to imagine a value of k_{St} or, like in an Australian water office, ring a friend to see what they used when they worked on a similar stream $20\,\mathrm{km}$ distant. We could estimate the D.
- If the bed material has a size of about $2\,\mathrm{cm}$ (Donau), then we simply use $D = 0.02\,\mathrm{m}$.
- It is much simpler and physically understandable.

Computation of normal flow

"Normal flow" is the name given to a uniform flow, and the depth is called the normal depth. If the discharge Q, slope S, resistance coefficient k_{St}, and the relationship between area and depth and perimeter and depth are known, the G-M-S formula becomes a transcendental equation for the normal depth h. To solve this is a common problem in river engineering

A numerical method

Any method for the numerical solution of transcendental equations can be used, such as Newton's method. Here we develop a simple method based on *direct iteration*, where we develop a trick, giving us rapid convergence.

In the case of wide channels, (*i.e.* channels rather wider than they are deep, a common case), the wetted perimeter P does not vary much with depth h. Similarly in the expression for the area, the width does not vary much with h. Consider the Gauckler-Manning-Strickler formula in the conventional form, written now

$$Q = k_{\mathrm{St}} \frac{A^{5/3}}{P^{2/3}} \sqrt{S}$$

we divide both sides by $h^{5/3}$, and showing functional dependence of A and P on h:

$$\frac{Q}{h^{5/3}} = k_{\mathrm{St}} \sqrt{S} \frac{(A(h)/h)^{5/3}}{P^{2/3}(h)}.$$

The term $A(h)/h$ is approximately the width of the channel, which for many channels varies little with h, as does the perimeter $P(h)$. So, the right side of the equation varies little with h, so by isolating the $h^{5/3}$ term and taking the $3/5$ power of both sides of the equation, we obtain the equation in a form suitable for direct iteration

$$h = \left(\frac{Q}{k_{\mathrm{St}}\sqrt{S}}\right)^{3/5} \times \frac{P^{2/5}(h)}{A(h)/h}, \tag{13}$$

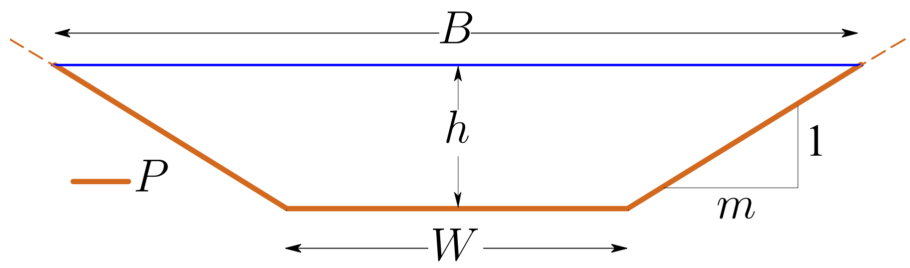
where the first term on the right is a constant for any particular problem, and the second term varies slowly with depth – a primary requirement that the direct iteration scheme be convergent and indeed be quickly convergent.

For an initial estimate we suggest making a rough estimate of the approximate width B_0 and so, making a wide channel approximation, setting $A(h)/h \approx B_0$ and $P(h) \approx B_0$, in the general scheme of (13) gives

$$h_0 = \left(\frac{Q}{k_{\mathrm{St}} B_0 \sqrt{S}}\right)^{3/5}. \tag{14}$$

Experience with typical trapezoidal sections shows that the method works well and is quickly convergent.

Trapezoidal section



Most canals are excavated to a trapezoidal section, and this is often used as a convenient approximation to river cross-sections too. In many of the problems in this course we will consider the case of trapezoidal sections. Consider the quantities shown in the figure: the bottom width is W, the depth is h, the top width is B, and the *batter slope,* defined to be the ratio of H:V dimensions is m. Geometrically, $B = W + 2mh$, area $A = h\,(W + mh)$, wetted perimeter $P = W + 2\sqrt{1+m^2}h$.

Example 1 Calculate the normal depth in a trapezoidal channel of slope 0.001, $k_{\mathrm{St}} = 25$, bottom width $W = 10\,\mathrm{m}$, with batter slopes $m = 2$, carrying a flow of $20\,\mathrm{m}^3\mathrm{s}^{-1}$. We have $A = h\,(10 + 2\,h)$, $P = 10 + 4.472\,h$. For B_0 we use $W = 10\,\mathrm{m}$. Equation (14) gives

$$h_0 = \left(\frac{Q}{k_{\mathrm{St}} B_0 \sqrt{S}}\right)^{3/5} = \left(\frac{20}{25 \times 10\sqrt{0.001}}\right)^{3/5} = 1.745\,\mathrm{m}.$$

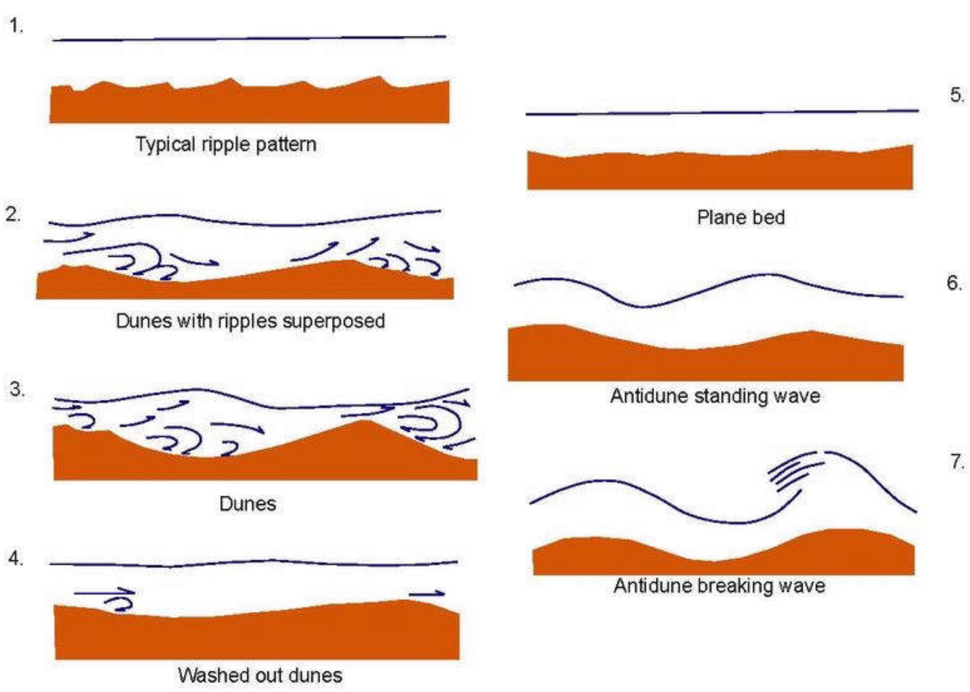
Then, equation (13) gives

$$h_{n+1} = \left(\frac{Q}{k_{\mathrm{St}}\sqrt{S_0}}\right)^{3/5} \times \frac{(10 + 4.472\,h_n)^{2/5}}{10 + 2\,h_n} = 6.948 \times \frac{(10 + 4.472\,h_n)^{2/5}}{10 + 2\,h_n}.$$

With $h_0 = 1.745$, $h_1 = 1.629$, $h_2 = 1.639$, $h_3 = 1.638\,\mathrm{m}$, and the method has converged.

General situations

In many cases the conditions in the river are more complicated than just a layer of uniform regular particles. For example:



- Irregular and variable nature of the bed particle arrangement.
- Bed forms – ripples, dunes, anti-dunes *etc*. Figure from Wikipedia **URL:** *https://en.wikipedia.org/wiki/Bedform*
- Particle movement – if the grains are moving, then the force required to move the grains appears to the water as an additional stress, whether they are moving along the bed, rolling, jumping, or carried suspended in the flow.
- Vegetation – trees standing in the water, grasses, reeds *etc*
- Meandering

Often one is required to adopt a value of resistance coefficient not given by the Strickler formula, but based on experience, knowledge, the Australian telephone method, looking at pictures in books *etc*. None of these are particularly good.

Chézy-Weisbach flow formula

A problem with the G-M-S form, using a value of k_{St} in more general situations, is that it has little basis in fluid mechanics, the coefficient has no physical significance, and it is in awkward units.

We now relate our results to another well-known standard formulae. Writing shear stress τ in terms of the result obtained from the Darcy-Weisbach formulation of flow resistance in pipes,

$$\frac{\tau}{\rho} = \tfrac{1}{8}\lambda U^2, \tag{15}$$

where λ is the Weisbach dimensionless resistance coefficient, expressing the relationship between velocity and stress. From our simple force balance we already have $\tau/\rho = \sqrt{g\,(A/P)\,S}$. Equating gives the Chézy-Weisbach flow formula

$$U = \frac{Q}{A} = \sqrt{\frac{8g}{\lambda}\frac{A}{P}S} = C\sqrt{\frac{A}{P}S}, \tag{16}$$

where $C = \sqrt{8g/\lambda}$ is the Chézy coefficient, named after the French military engineer who first wrote down such an open channel flow formula. Comparing our equation (5) we see that it is in the same form, such that

$$\frac{\text{Mean velocity}}{\text{Shear velocity}} = \gamma = \sqrt{\frac{8}{\lambda}} \tag{17}$$

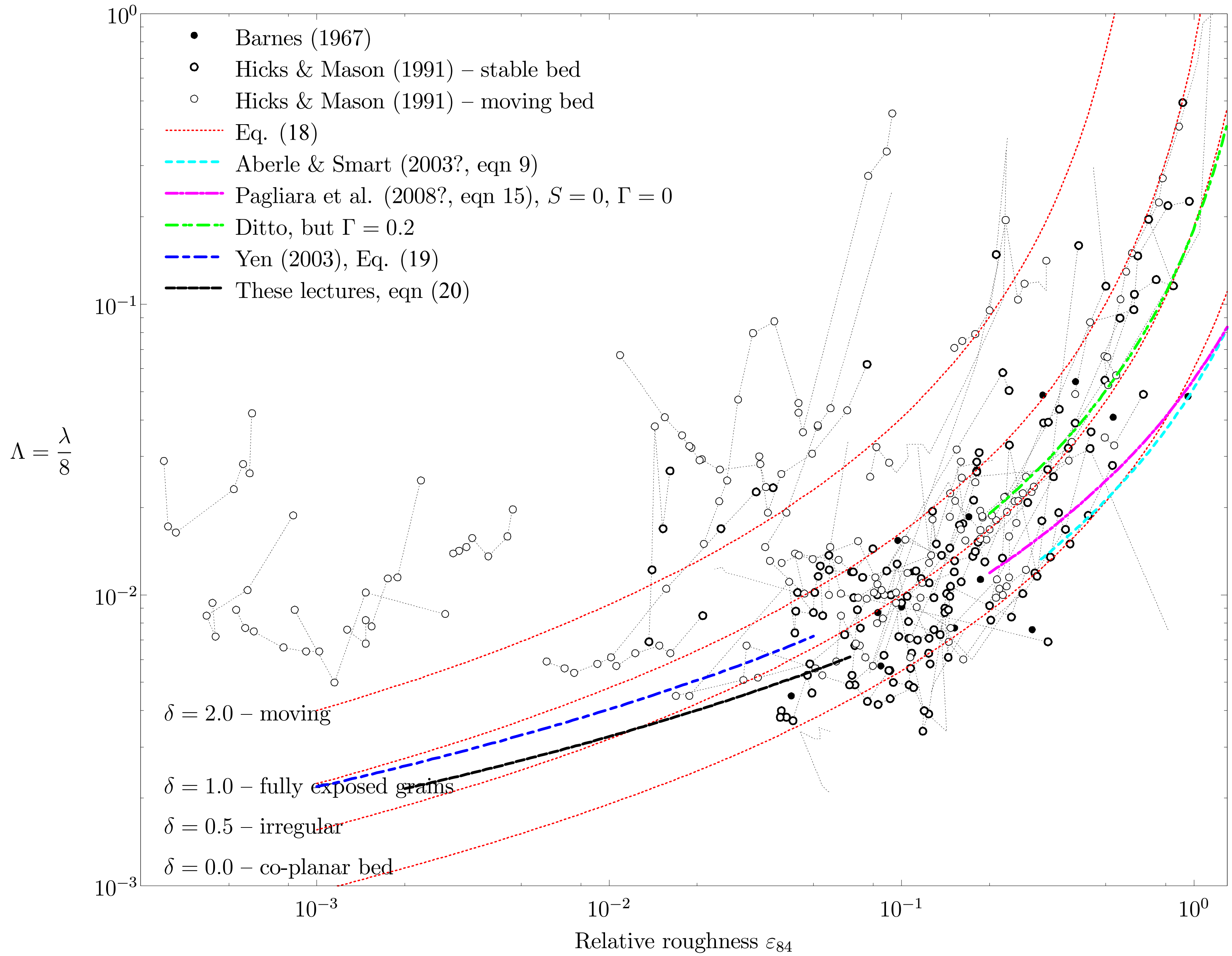
for the flows we have encountered so far in this section, steady, uniform flows in a cross-section which is relatively simple.

Where the resistance to flow is compound – where there are different parts of the stream cross-section with different resistances, where there are obstacles in a stream causing resistance, it is more consistent to use the Chézy-Weisbach formulation, which is more clearly related to forces involved and the mechanics of the problems. This is especially so when we consider the mechanics of flows and flood wave propagation.

Results for resistance coefficients in real rivers

Here we attempt to obtain understanding and a formula for the resistance coefficient using results from a number of field measurements. We considered the results of Hicks & Mason (1991), a catalogue of 558 stream-gaugings from 78 river and canal reaches in New Zealand, of which 55 were sites with grading curves for boundary material, so that particle sizes were known. Neither vegetation nor bed-form resistance can be isolated. Hicks & Mason based their approach on Barnes (1967), who provided values of Manning's resistance coefficient $n = 1/k_{\mathrm{St}}$ for a single flow at each of 50 separate river sites in the United States of America, of which boundary material details were given for 14. We also include those results here.

From both catalogues we took the values of D_{84}, the boundary particle size for which 84% of the material was finer, and from the values of A/P, calculated the relative roughness $\varepsilon_{84} = D_{84}/(A/P)$, and used the measured values of Chézy's C to calculate values of λ. Results are shown on the next page. We have plotted them for a parameter $\Lambda = \lambda/8$, which it will be more convenient for us to use later.



Barnes (1967)
Hicks & Mason (1991) – stable bed
Hicks & Mason (1991) – moving bed
Eq. (18)
Aberle & Smart (2003?, eqn 9)
Pagliara et al. (2008?, eqn 15), $S = 0$, $\Gamma = 0$
Ditto, but $\Gamma = 0.2$
Yen (2003), Eq. (19)
These lectures, eqn (20)
$\delta = 2.0$ – moving
$\delta = 1.0$ – fully exposed grains
$\delta = 0.5$ – irregular
$\delta = 0.0$ – co-planar bed
$\Lambda = \frac{\lambda}{8}$
Relative roughness ε_{84}
10^{0}
10^{-1}
10^{-2}
10^{-3}

- Many of the results from each study are for large bed material $\varepsilon_{84} > 0.1$, possibly a reflection of the hilly and mountainous nature of New Zealand and Pacific North-West of the United States of America. And Austria.
- There is a wide scatter of results. But not all that very wide if we consider that the streams range from large slow-moving rivers with extremely small grains to mountain torrents with $30\,\mathrm{cm}$ boulders. Most of the results, unless the grains are moving, fall between $\Lambda \approx 0.005$ and 0.02.
- There is, as we have seen, slow variation with relative roughness: am increase in ε by a factor of 10 leads to an increase in Λ of about 2, as we have already seen.
- The points, we believe, have a tendency to group around three of the curves shown and the rest to be bounded below by the fourth (upper) curve shown. The curves have been drawn using the expression, found by trial and error:

$$\Lambda = \frac{0.06 + 0.06\,\delta}{(1.0 - 0.6\,\delta - \ln \varepsilon_{84})^2}, \tag{18}$$

  with values of $\delta = 0$, 0.5, 1, and 2. The parameter δ is an arbitrary one that we use to identify the state of the particles making up the bed. This will now be explained.
- The first grouping of points comprises those around the bottom curve. We hypothesise that these points, having the lowest resistance, are those forming beds where the particles are relatively co-planar such that the bed is *armoured*. We assigned $\delta = 0$ to this state, and used that in equation (18) to plot the curve.
- The next grouping of points is around the second curve from the bottom, which can be seen

to substantially coincide with the a curve corresponding to exposed boulders on top of the bed occupying 0.2 of the surface area. Of course, with a number of these grains thus exposed, the resistance is greater. We assigned a value of $\delta = 0.5$ to this intermediate state.

- Substituting $\delta = 1$ in equation (18) gives the third curve on the figure, passing through what we believe is the third grouping of particles. This is probably the state for the maximum resistance for a stable bed corresponding to exposed grains occupying something like 50% of the surface area. Any more such grains will cause shielding of particles, the bed will start to resemble the co-planar case, and resistance will actually be reduced.
- Further evidence supporting our assertions is obtained from the expression proposed by Yen (2002, eqn 19), who considered results from a number of experimental studies using fixed impermeable beds. We used his formula, converted to $\Lambda = \lambda/8$, used an infinite Reynolds number, and converted his equivalent sand roughness $\varepsilon_s = 2\varepsilon_{84}$. It can be seen that the curve passes (left to right) from our curve $\delta = 1$ for small particles, which are unlikely to have the tops levelled so that particles are exposed, to the second curve for larger particles, more likely to be levelled in the laboratory experiments, with $\delta = 0$. Yen obtained the approximation for λ:

$$\lambda = \frac{1}{4.}\left(-\log_{10}\left(\frac{1}{12.}\frac{k_{\mathrm{s}}}{A/P} + \frac{1.95}{\mathsf{R}^{0.9}}\right)\right)^{-2}, \tag{19}$$

where R is the channel Reynolds number $\mathsf{R} = (Q/P)/\nu$, in which ν is the kinematic viscosity.

- The logarithmic formula we obtained quite simply, equation (5), leads to, if we use $\varepsilon_s = 2\varepsilon_{84}$,

$$\Lambda = (6 - 2.5\ln(2\varepsilon_{84}))^{-2}, \tag{20}$$

quite similar to the results from Yen's formula.

- For points above the third curve almost all experimental points had shear stresses greater than the critical one necessary for movement. If particles move, not only do many particles protrude above others, increasing the stress, but there is the additional force required to maintain the sliding and rolling and jostling of all the particles. Hence, the resistance is greater. And, if there is a need to maintain particles in suspension, that will contribute also to resistance. We have shown the fourth curve as drawn for $\delta = 2$.

Hopefully the figure and approximating curves have given us an idea of the magnitudes and variation of the quantities, and maybe even some results for use in practice.