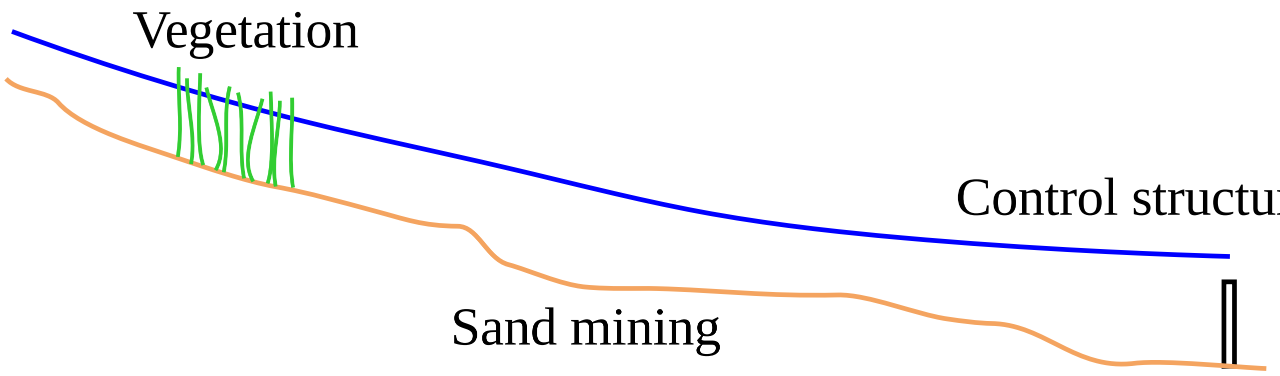
8. Steady flow



- A common task in river engineering is to calculate the free surface elevation along a steadily flowing stream.
- Simply the solution of a first-order differential equation – often obscured in writings.
- Flow is usually sub-critical, so the control / boundary condition is at the downstream end and one computes upstream.
- Alternative approach suggested here, using cross-sectional area as the dependent variable, requiring little knowledge of the details of the underwater topography.
- Traditional textbook methods are unsatisfactory: the "Standard Step" method is unnecessarily complicated and the "Direct Step" method is incorrect.
- Application of simple explicit numerical methods is described.
- If A is not used, for non-prismatic streams all methods require much data. Often that is not available. An approximate linearised model of flow in a river is made. This gives us insight into the nature of the problem, as well as simple approximate answers.

8.1 The gradually-varied flow equation (GVFE)

Use of area A and application to streams of unknown bathymetry

For steady flow where Q is constant so that $\partial Q/\partial t$ and $\partial Q/\partial x$ are zero, the long wave momentum equation (7.17b) on page 82 in terms of cross-sectional area A, gives one version of the GVFE in terms of area A:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = B\frac{\tilde{S} - Q^2/K^2}{1-\beta\mathsf{F}^2} = B\frac{\tilde{S} - Q^2P^{4/3}/k_{\mathrm{St}}^2A^{10/3}}{1-\beta Q^2B/gA^3}\,. \tag{8.1}$$

In the resistance term we are using the conveyance K, which is a function of section properties and the Strickler coefficient k_{St}

$$K = k_{\mathrm{St}}\frac{A^{5/3}}{P^{2/3}}\,, \tag{8.2}$$

such that for uniform flow, $\tilde{S} = S =$ constant, $Q_{\mathrm{r}} = K\sqrt{S}$.

The ordinary differential equation (8.1) is valid also for non-prismatic channels. The mean bed slope at a section $\tilde{S}$, can be variable but is, usually poorly known and is often just estimated, like the other parameters of the problem; β might be something like 1.1. The coefficient k_{St} is also often poorly known.

In the differential equation there are strongly-varying functions of the dependent variable itself, A^3 and possibly $A^{10/3}$, plus the usually slowly-varying functions $B(A)$ and $P(A)$. This suggests that using the GVFE in terms of A has an important advantage: one needs few details of the under-water topography. It is not necessary to know the precise details of the underwater bathymetry other than those weakly-varying functions $B(A)$ and $P(A)$. The obvious approximation could be made that they are constant and equal; river width often does not vary much.

To start numerical solution, one would need to know the area at a control where surface elevation might be known. The solution in terms of area might be enough, to give an idea of how far upstream the effects of a structure or channel changes extend. It is surprising that we can do so much with so little information. However, if one needed a value of surface elevation η at a certain value of x, one would then need cross-sectional details there to go from the computed A to η.

Customary use of a quantity h called the "water depth"

The long wave momentum equation (7.18b) in terms of surface elevation η, for Q constant so that $\partial Q/\partial t$ and $\partial Q/\partial x$ are zero gives another version of the GVFE:

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{\tilde{S}\beta\mathsf{F}^2 - Q^2/K^2}{1-\beta\mathsf{F}^2} \left(\approx -Q^2/K^2 \text{ for } \mathsf{F}^2 \text{ small, the common case}\right)$$

The tradition is not to use η, but instead a depth-like quantity $h = \eta - Z_0$, where Z_0 is the elevation

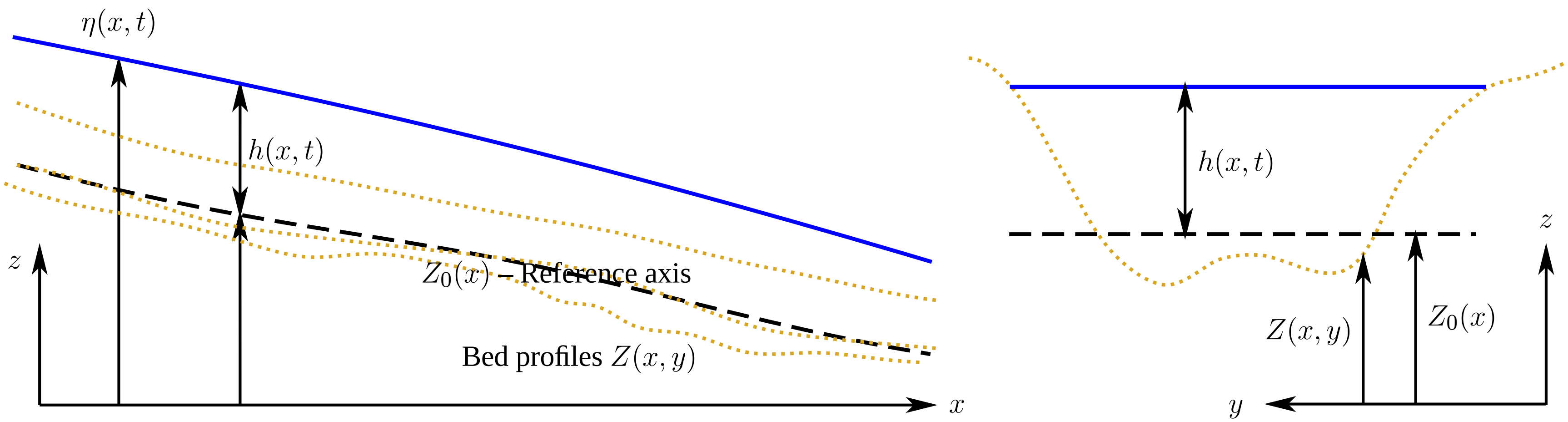


Figure 8.1: Complicated reality

of a longitudinal axis, almost always the supposed bed of the channel. The GVFE becomes

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 + \beta\left(\tilde{S} - S_0\right)\mathsf{F}^2 - Q^2/K^2}{1 - \beta\mathsf{F}^2},$$

where $S_0 = -\mathrm{d}Z_0/\mathrm{d}x$, the slope of the reference axis, positive downwards. We almost never know the details of $\tilde{S}$ so here we assume that $\tilde{S} = S_0$, which we now write as S, giving

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S - Q^2/K^2}{1 - \beta\mathsf{F}^2}$$

where in general both K and F are functions of both x and h, while in a prismatic channel, functions just of h.

Because of our use of h, we pretend that we know the bed in great detail, or, that our channel looks like this:

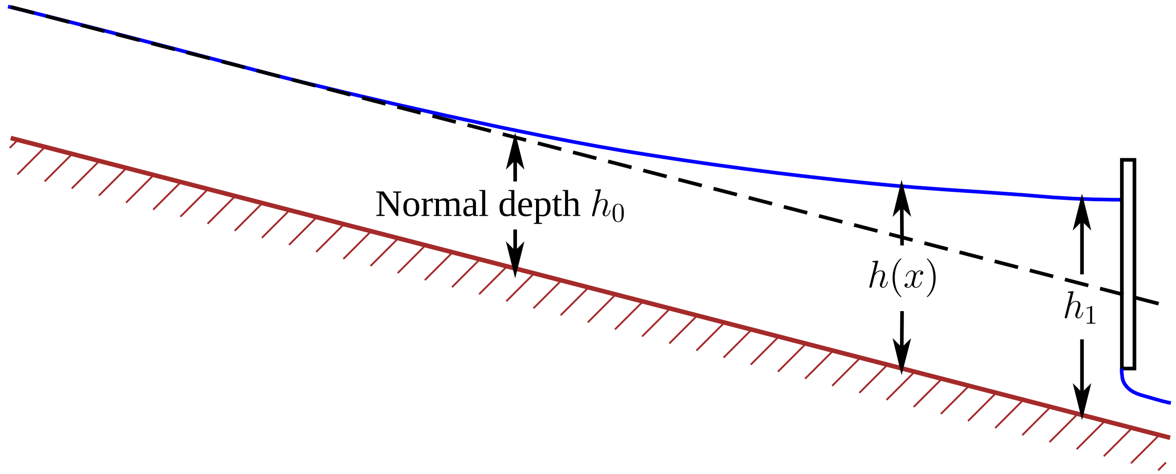


Figure 8.2: Our simpler model

This shows a typical subcritical flow retarded by a structure, showing the free surface disturbance decaying upstream, and if the channel is prismatic, to constant normal depth.

8.2 Traditional textbook methods – some old, complicated, and wrong

The "Standard" step method

The almost trivial energy derivation, ignoring non-prismatic effects, is that the rate of change of total head H is given by the empirical expression for the energy gradient

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -Q^2/K^2\,(x,h) \quad \text{where} \quad H = Z_0(x) + h + \alpha\frac{Q^2}{2gA^2(x,h)}.$$

The computational approximation scheme is

$$\frac{H_{i+1}\,(h_{i+1}) - H_i\,(h_i)}{x_{i+1} - x_i} = -\tfrac{1}{2}Q^2\left(\frac{1}{K^2\,(x_i, h_i)} + \frac{1}{K^2\,(x_{i+1}, h_{i+1})}\right)$$

J. Fenton, Australia 1966;
H. Honsowitz, Austria, 1970?
solving transcendental equations

- The method advocated by Chow (1959) in a pre-computer era and still suggested by textbooks.
- $H(h)$ and $K(x, h)$ are both complicated geometrical functions of h, the unknown h_{i+1} is deep inside left and right sides.
- Requires numerical solution of a transcendental equation at each time step.

The "Direct" step method – distance calculated from depth

- Applied by taking steps in the water depth and calculating the corresponding step in x.
- It has some advantages: iterative methods are not necessary ("Direct").
- Practical disadvantages are:
  - It is applicable only to prismatic sections
  - Results are not obtained at specified points in x
  - As uniform flow is approached the steps become infinitely large
  - AND, it is wrong, as we now show

Consider the "specific head", the head relative to the local channel bottom, denoted here by H_0:

$$H_0(h) = H(h) - Z = h + \alpha \frac{Q^2}{2gA^2(h)}.$$

The differential equation becomes, after inverting each side

$$\frac{\mathrm{d}x}{\mathrm{d}H_0(h)} = \frac{1}{S - Q^2/K^2}.$$

A mistake and a correction

- The differential equation is now approximated, the left side by a finite difference expression $(x_i - x_{i+1}) \,/\, (H_{0,i} - H_{0,i+1})$.

- For the right side the numerical method as set out in textbooks is to take the mean of just the *denominator* at beginning and end points, and so to write

$$x_{i+1} = x_i + \frac{H_{0,i+1} - H_{0,i}}{\frac{1}{2}\left(S_i - Q^2/K_i^2 + S_{i+1} - Q^2/K_{i+1}^2\right)}$$

where the red shows the quantity that is a supposed mean value.

- While this is a plausible approximation, it is not mathematically consistent. What should be done is to use the mean value at beginning and end points of the *whole* right side of the differential equation, to give a trapezoidal approximation of the right side, which leads to

$$x_{i+1} = x_i + (H_{0,i+1} - H_{0,i})\,\frac{1}{2}\left(\frac{1}{S_i - Q^2/K_i^2} + \frac{1}{S_{i+1} - Q^2/K_{i+1}^2}\right).$$

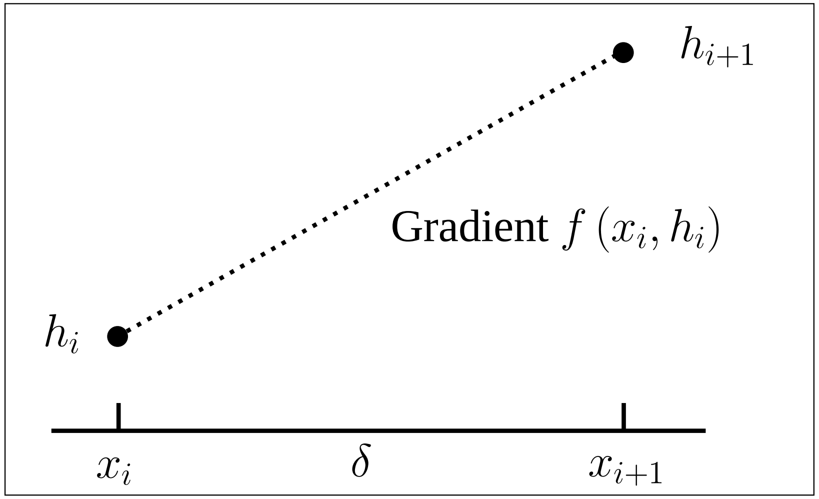
8.3 Standard simple numerical methods for differential equations

We write the differential equation as

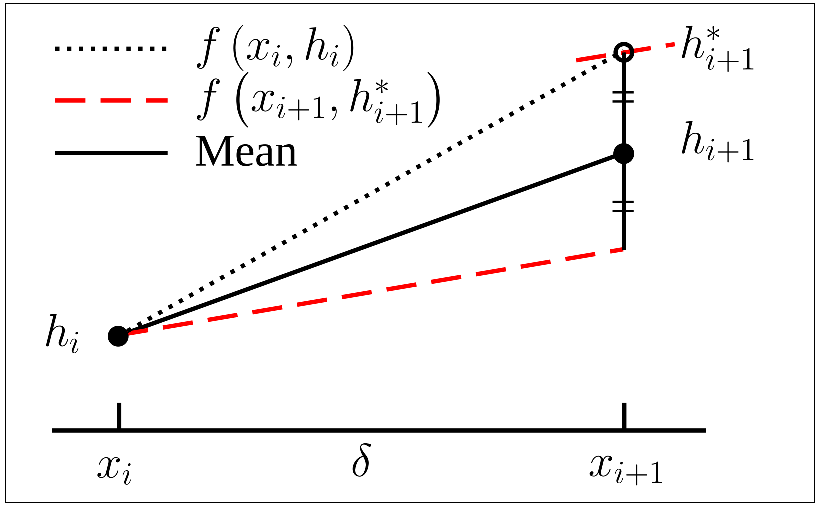
$$\frac{\mathrm{d}h}{\mathrm{d}x} = f\left(x,h\right) = \frac{S(x) - Q^2/K^2\left(x,h\right)}{1-\beta \mathsf{F}^2\left(x,h\right)}$$

The two simplest numerical methods are:

Euler



Heun



$$h_{i+1} \approx h_i + \delta\, f\left(x_i,h_i\right) + O\left(\delta^2\right)$$

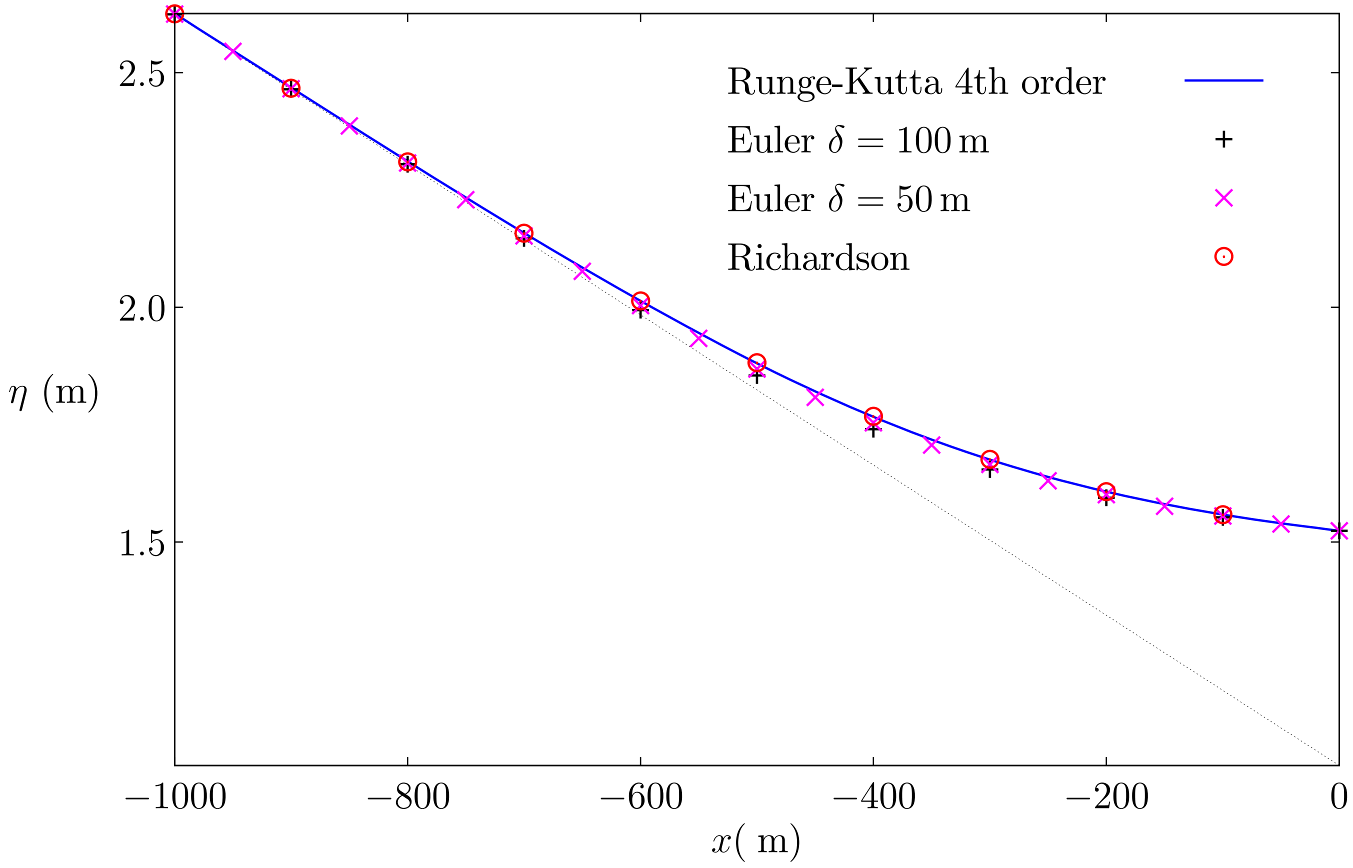
$$h^*_{i+1} \approx h_i + \delta\, f\left(x_i,h_i\right),$$

$$h_{i+1} \approx h_i + \frac{\delta}{2}\left(f\left(x_i,h_i\right) + f\left(x_{i+1},h^*_{i+1}\right)\right) + O\left(\delta^3\right)$$

- Euler's method is the simplest but least accurate – yet it might be appropriate for open channel problems where quantities may only be known approximately
- One can use simple modifications such as Heun's method to gain better accuracy, or use Richardson extrapolation, or even more simply, just take smaller steps δ
- For greater accuracy one can use the Trapezoidal method, simply repeating the second Heun step several times, setting $h^*_{i+1} = h_{i+1}$ each time
- Often these two methods are not presented in hydraulics textbooks as alternatives, yet they are simple and flexible, and reveal the nature of what we are doing
- The step δ can be varied at will, to suit possible irregularly spaced cross-sectional data
- In many situations, where $\mathsf{F}^2 \ll 1$, we can ignore the $\beta\mathsf{F}^2$ term in the denominators, giving a notationally simpler scheme

Comparison of schemes

Example 7 A flow of $11.33\,\mathrm{m}^3\,\mathrm{s}^{-1}$ passes down a trapezoidal channel of gradient $S = 0.0016$, bed width $6.10\,\mathrm{m}$ and channel side slopes $H : V = 2$, $\alpha = \beta = 1.1$, and $k_{\mathrm{St}} = 40$. At $x = 0$ the flow is backed up to a depth of $1.524\,\mathrm{m}$. Compute the backwater curve for $1000\,\mathrm{m}$ in 10 steps and then 20, then perform Richardson extrapolation for a more accurate estimate.



Convergence of numerical schemes

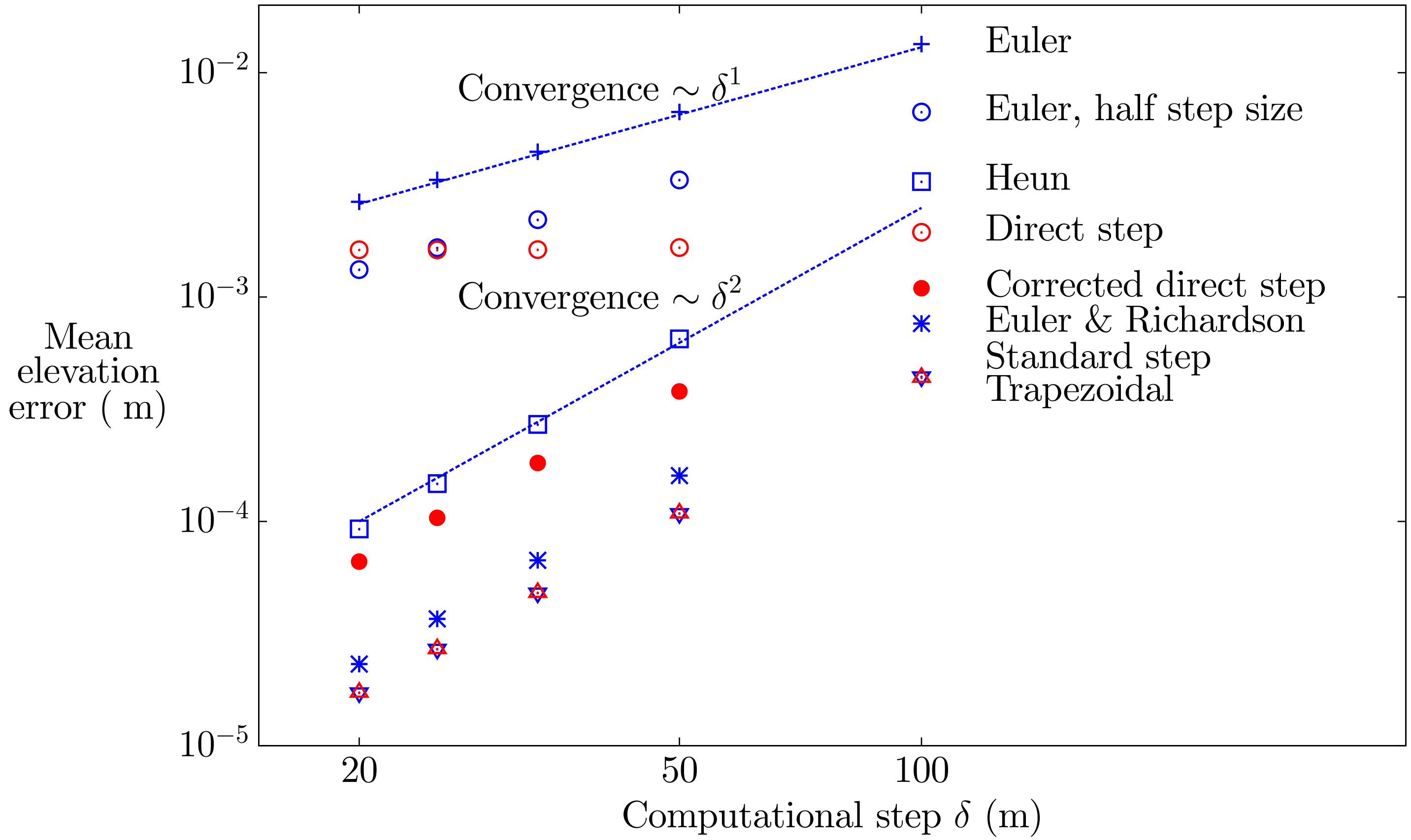


Figure 8.3: Comparison of accuracy - logarithmic scales

- Using Euler, then applying Richardson extrapolation, gave the third most accurate of all the methods, more than enough for practical purposes
- The most accurate were the Standard step method and the Trapezoidal method
- There *is* something wrong with the conventional Direct step method as we have suggested, while the corrected scheme is highly accurate

8.4 A simple model of steady flow in a river

- Often the precise details of a stream are not known, and it is quite legitimate to make approximations
- These might give us more insight and understanding of the problem
- Now a model is made where the GVFE is linearised and a general solution obtained
- Simple deductions as to the length of backwater effects can be made
- One can calculate an approximate solution for a whole stream if the variation in the resistance coefficient and geometry are known or can be estimated
- There is more of a balance between what we know (usually little) and the (un)sophistication of the model

The GVFE is

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S - Q^2/K^2(x,h)}{1 - \beta\mathsf{F}^2(x,h)}$$

We linearise the problem (similar to obtaining the Telegraph equation) and consider small perturbations about an underlying uniform flow of slope S_0 and depth h_0, such that we write

$$h = h_0 + \varepsilon h_1(x) + \ldots,$$

where ε is a small quantity expressing the magnitude of deviations from uniform.

Similarly we also let the possible non-constant slope be

$$S = S_0 + \varepsilon S_1(x) + \ldots .$$

In a real stream varying along its length, both K and F are functions of x and h. We write the series:

$$K = K_0 + \varepsilon K_1(x) + \varepsilon h_1(x) K_{h0} + O\left(\varepsilon^2\right) \ ,$$

where K_1 is a change caused by a change in the channel properties in x, whether the resistance coefficient or the cross-section, and $K_{h0} = \mathrm{d}K/\mathrm{d}h|_0$ expresses the change of conveyance with water depth. We also write

$$\mathsf{F}^2 = \mathsf{F}_0^2 + O\left(\varepsilon\right) + \ldots ,$$

in which we will find that terms in ε are not necessary.

Multiplying through by $1 - \beta\mathsf{F}^2$, setting dh_0/dx to zero for uniform flow and neglecting terms in ε^2:

$$\varepsilon\left(1 - \beta\mathsf{F}_0^2\right)\frac{\mathrm{d}h_1(x)}{\mathrm{d}x} = S_0 + \varepsilon S_1(x) - \frac{Q^2}{K_0^2}\left(1 - 2\varepsilon\frac{K_1(x)}{K_0} - 2\varepsilon h_1(x)\frac{K_{h0}}{K_0}\right) \ .$$

At zeroeth order ε^0 we obtain

$$S_0 - Q^2/K_0^2 \ ,$$

an expression of whichever flow formula is being used, and is identically satisfied.

At ε^1, we obtain the linear differential equation

$$\frac{\mathrm{d}h_1}{\mathrm{d}x} - \gamma h_1 = \phi(x)$$

where γ is a constant:

$$\gamma = 2\frac{S_0 K_{h0}/K_0}{1-\beta \mathsf{F}_0^2} = \frac{S_0}{1-\beta \mathsf{F}_0^2} \times \begin{cases} 2\dfrac{\mathrm{d}K/\mathrm{d}h|_0}{K_0}, & \text{General expression;} \\ 3\dfrac{B_0}{A_0} - \dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Chézy-Weisbach;} \\ \frac{10}{3}\dfrac{B_0}{A_0} - \frac{4}{3}\dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Gauckler-Manning;} \end{cases} \tag{8.3}$$

and the forcing term on the right is

$$\phi(x) = \frac{S_0}{1-\beta \mathsf{F}_0^2}\left(\frac{S_1(x)}{S_0} + \frac{2K_1(x)}{K_0}\right), \tag{8.4}$$

showing the effects of fractional changes in slope and conveyance K.

Solving the differential equation

The differential equation is in *integrating factor* form, and can be solved by multiplying both sides by $e^{-\gamma x}$ and writing the result

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\gamma x} h_1\right) = e^{-\gamma x}\phi\left(x\right),$$

which can be integrated to give

$$h_1 = e^{\gamma x} \left(\int^x e^{-\gamma x'} \phi(x') \, dx' + \text{Constant} \right),$$

where x' is a dummy variable. Returning to physical variables, $h = h_0 + \varepsilon h_1$ gives the solution

$$h = h_0 + He^{\gamma x} + \int^x e^{\gamma(x-x')} \phi(x') \, dx'$$

The part of the solution $He^{\gamma x}$ is that obtained by Samuels (1989), giving the solution for backwater level in a uniform channel by evaluating the constant of integration using a downstream boundary condition $h = H$ at $x = 0$. The solution shows how the surface decays upstream at a rate $e^{\gamma x}$, as x becomes increasingly negative, because γ is positive,

- For a wide channel, the terms in dP/dh in the formulae for γ are unimportant (and are often not well known), so that $A_0/B_0 \approx h_0$, the channel depth, and for small Froude number this gives

$$\gamma \approx 3 \frac{S_0}{h_0}, \tag{8.5}$$

  showing that the rate of exponential decay is small for gently sloping and deep streams and greatest for steep and shallow ones.

- Consider the distance $x_{1/2}$ upstream for the effect of a downstream surface elevation to diminish by a factor of $1/2$. Then $\exp\left(-\gamma x_{1/2}\right) = 1/2$, or

$$x_{1/2} = \frac{\ln 2}{\gamma} \approx \frac{\ln 2}{3}\frac{h_0}{S_0} \approx 0.2\frac{h_0}{S_0}$$

So for a gently-sloping river $S_0 = 10^{-4}$ and $2\ \mathrm{m}$ deep, the effect of any backwater decreases by $1/2$ in a distance of $4\ \mathrm{km}$. To diminish to $1/16$, say, the distance is $16\ \mathrm{km}$. For a steeper river, say $S_0 = 0.0016$ from the example simulated above, where $h_0 \approx 1\ \mathrm{m}$, the "half-length" is about $150\ \mathrm{m}$. This is roughly in agreement with the computed results in Example 7 above.

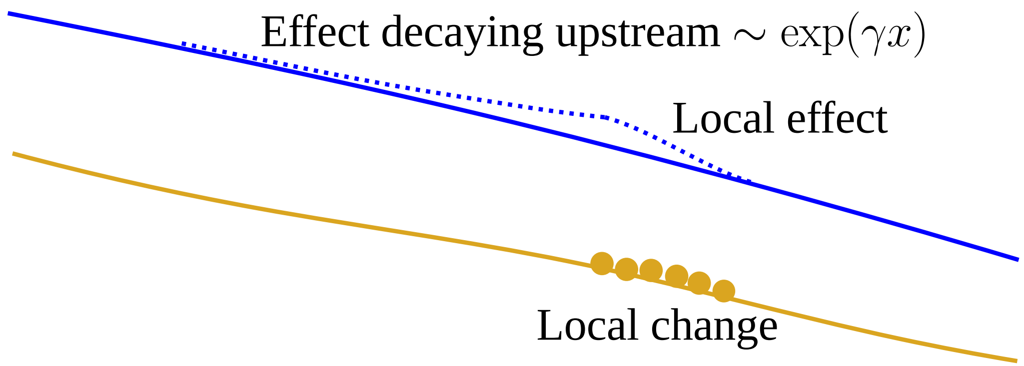
- If the approximate exponential decay solution were shown on that figure, it would not agree closely with the computed results, because the checked-up disturbance is as large as 50% of the depth, when the linear solution is not all that accurate. The beauty of Samuels' result is in its ability to give a quick estimate and an appreciation of the quantities that affect the length of backwater.

General solution for channel

Here we neglect any boundary conditions and consider just the solution due to the forcing function ϕ due to changes in the channel:

$$h = h_0 - \int_x^{\infty} e^{\gamma(x-x')} \phi\left(x'\right)\, dx' \tag{8.6}$$

This is a simple result: at any point x in subcritical flow, any disturbance is due to the integrated effects of the disturbance function ϕ for all downstream points, from x to ∞, weighted according to the exponential decay function.



Example 8 The effect on a river of a finite length of greater resistance

Consider, as an example, a case where over a finite length L of river, the carrying capacity is reduced by the conveyance K decreasing by a relative amount $K_1/K_0 = -\delta$, such as by local deposition of material, between $x = 0$ and $x = L$, and constant in that interval. Assume F_0^2 negligible and the river wide.

The forcing function form equation (8.4) is:

$$\phi(x) = \begin{cases} 0, & \text{if } x \leqslant 0; \\ -S_0\delta, & \text{if } 0 \leqslant x \leqslant L; \\ 0, & \text{if } x \geqslant L. \end{cases}$$

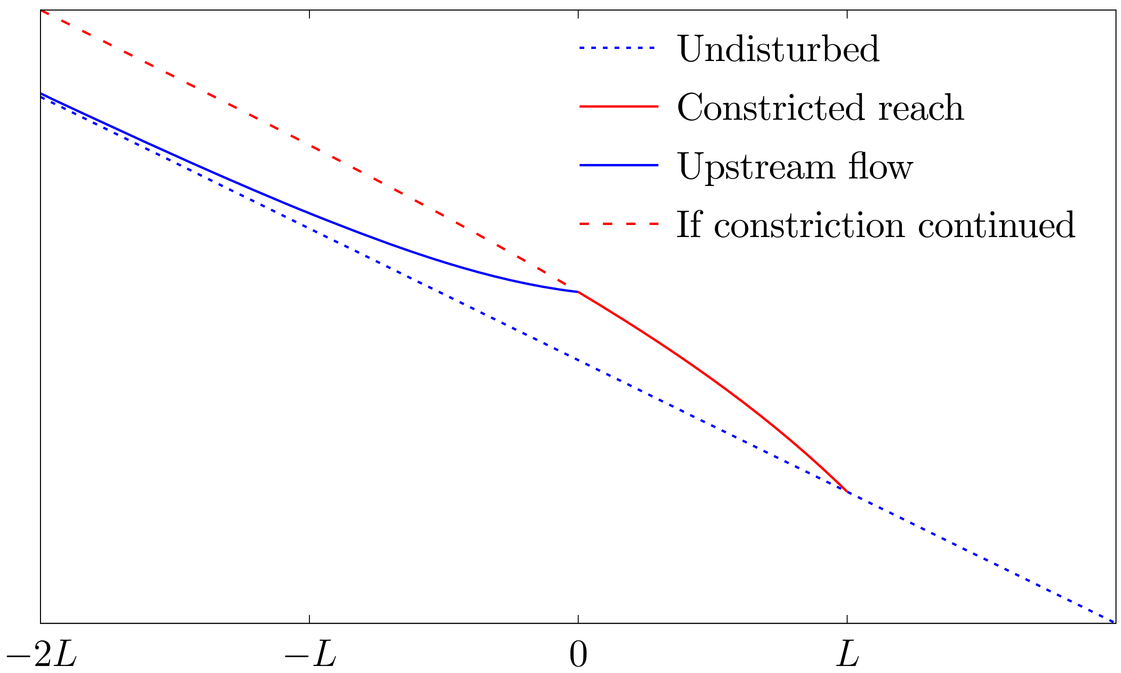
For x downstream, $x \geqslant L$, $\phi(x) = 0$, and $h = h_0$, which is correct in this sub-critical flow, there are no downstream effects.

For x in the section where the changes occur, $0 \leqslant x \leqslant L$, the solution is

$$h = h_0 + S_0\delta \int_x^L e^{\gamma(x-x')}\, dx' = h_0 + \frac{S_0\delta}{\gamma}\left(1 - e^{\gamma(x-L)}\right).$$

For x upstream, $x \leqslant 0$, where there is no extra resistance,

$$h = h_0 + S_0\delta e^{\gamma x} \int_0^L e^{-\gamma x'}\, dx' = h_0 + \frac{S_0\delta}{\gamma} e^{\gamma x}\left(1 - e^{-\gamma L}\right).$$



These solutions are all shown in the figure with an arbitrary vertical scale such that the slope is exaggerated. The calculations were performed for $S_0 = 0.0005$, $h_0 = 1\,\mathrm{m}$, and with a constricted length of $L = 1000\,\mathrm{m}$, with a 10% increase in resistance there, such that $\delta = 0.1$. Using these figures, and with $\gamma = 3S_0/h_0$, the computed backwater at the beginning of the constriction calculated according to the formula was $2.6\,\mathrm{cm}$.

In the reach of increased resistance the surface is raised, as one expects and shows an exponential approach to the changed depth $S_0\delta/\gamma$ if $L \to \infty$.

The abrupt changes of gradient violate our physical assumptions of the long wave equations, but they give us a clear picture of what happens, possibly obvious in retrospect, but hopefully of assistance.

We have obtained an approximate solution to the problem, with little input data necessary.