River Engineering

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1. Introduction

1.1 The nature of what we will and will not do – illuminated by some aphorisms and some people

"There is nothing so practical as a good theory" – stated in 1951 by Kurt Lewin (D-USA, 1890-1947): this is essentially the guiding principle behind these lectures. We want to solve practical problems, both in professional practice and research, and to do this it is a big help to have a theoretical understanding and a framework.

"The purpose of computing is insight, not numbers" – the motto of a 1973 book on numerical methods for practical use by the mathematician Richard Hamming (USA, 1915-1998). That statement has excited the opinions of many people (search any three of the words in the Internet!). However, numbers *are* often important in engineering, whether for design, control, or other aspects of the practical world. A characteristic of many engineers, however, is that they are often blinded by the numbers, and do not seek the physical understanding that can be a valuable addition to the numbers. In this course we are not going to deal with many numbers. Instead we will deal with the methods by which numbers could be obtained in practice, and will try to obtain insight into those methods. Hence we might paraphrase simply: "The purpose of this course is insight into the behaviour of rivers; with that insight, numbers can be often be obtained more simply and reliably".

“It is EXACT, Jane” – a story told to the lecturer by a botanist colleague. The most important river in Australia is the Murray River, $2\,375\,\mathrm{km}$ (Danube $2\,850\,\mathrm{km}$), maximum recorded flow $3\,950\ \mathrm{m^3s^{-1}}$ (Danube at Iron Gate Dam: $15\,400\ \mathrm{m^3s^{-1}}$). It has many tributaries, flow measurement in the system is approximate and intermittent, there is huge biological and fluvial diversity and irregularity. My colleague, non-numerical by training, had just seen the demonstration by an hydraulic engineer of a one-dimensional computational model of the river. She asked: “Just how accurate is your model?”. The engineer replied intensely: "It is EXACT, Jane".

Nothing in these lectures will be exact. We are talking about the *modelling* of complex physical systems.

A further example of the sort of thinking that we would like to avoid: in the area of palaeo-hydraulics, some Australian researchers made a survey to obtain the heights of floods at individual trees. This showed that the palaeo-flood reached a maximum height on the River Murray at a certain position of $18.01\ \mathrm{m}$ (*sic*), Having measured the cross-section of the river, they applied the Gauckler-Manning-Strickler Equation to determine the discharge of the prehistoric flood, stated to be $7\,686\ \mathrm{m^3s^{-1}}$ ...

William of Ockham (England, c1288-c1348): Ockham’s razor is the principle that can be popularly stated as “when you have two competing theories that make similar predictions, the simpler one is the better”. The term razor refers to the act of shaving away unnecessary assumptions to get to the simplest explanation, attributed to 14th-century English logician and Franciscan friar, William of Ockham. The explanation of any phenomenon should make as few assumptions as possible, eliminating those that make no difference in the observable predictions of the explanatory hypothesis or theory. When competing hypotheses are equal in other respects, the principle recommends selection of the hypothesis that introduces the fewest assumptions and postulates the fewest entities *while still sufficiently answering the question*. That is, we should not *over*-simplify our approach.

In general, model complexity involves a trade-off between simplicity and accuracy of the model. Occam's Razor is particularly relevant to modelling. While added complexity usually improves the fit of a model, it can make the model difficult to understand and work with.

The principle has inspired numerous expressions including "parsimony of postulates", the "principle of simplicity", the "KISS principle" (Keep It Simple, Stupid). Other common restatements are:

Leonardo da Vinci (I, 1452–1519, world's most famous hydraulician, also an artist): his variant short-circuits the need for sophistication by equating it to simplicity "Simplicity is the ultimate sophistication".

Wolfang A. Mozart (A, 1756–1791): "Gewaltig viel Noten, lieber Mozart", soll Kaiser Josef II. über die erste der großen Wiener Opern, die "Entführung", gesagt haben, und Mozart antwortete: "Gerade so viel, Eure Majestät, als nötig ist." (Emperor Joseph II said about the first of the great Vienna operas, "Die Entführung aus dem Serail", "Far too many notes, dear Mozart", to which Mozart replied "Your Majesty, there are just as many notes as are necessary"). The truthfulness of the story is questioned – Josef was more sophisticated than that ...

Albert Einstein (D-USA,1879-1955): "Make everything as simple as possible, but not simpler." This is a better and shorter statement than Ockham!

Karl Popper (A-UK, 1902-1994) argued that we prefer simpler theories to more complex ones "because their empirical content is greater; and because they are better testable". In other words, a simple theory applies to more cases than a more complex one, and is thus more easily falsifiable. Popper coined the term critical rationalism to describe his philosophy. The term indicates his rejection of classical empiricism, and of the classical observationalist-inductivist account of science that had grown out of it. Logically, no number of positive outcomes at the level of experimental testing can confirm a scientific theory (Hume's "Problem of Induction"), but a single counterexample is logically decisive: it shows the theory, from which the implication is derived, to be false. For example, consider the inference that "all swans we have seen are white, and therefore all swans are white", before the discovery of black swans in Australia. Popper's account of the logical asymmetry between verification and falsifiability lies at the heart of his philosophy of science. It also inspired him to take falsifiability as his criterion of demarcation between what is and is not genuinely scientific: a theory should be considered scientific if and only if it is falsifiable. This led him to attack the claims of both psychoanalysis and contemporary Marxism to scientific status, on the basis that the theories enshrined by them are not falsifiable.

Thomas Kuhn (USA, 1922-1996): In *The Structure of Scientific Revolutions* argued that scientists work in a series of paradigms, and found little evidence of scientists actually following a falsificationist methodology. Kuhn argued that as science progresses, explanations tend to become more complex before a sudden *paradigm shift* offers radical simplification. For example Newton's classical mechanics is an approximated model of the real world. Still, it is quite sufficient for most ordinary-life situations. Popper's student Imre Lakatos (H-UK, 1922-1974) attempted to reconcile Kuhn's work with falsificationism by arguing that science progresses by the falsification of research programs rather than the more specific universal statements of naive falsificationism.

Another of Popper's students Paul Feyerabend (A-USA, 1924-1994) ultimately rejected any prescriptive methodology, and argued that the only universal method characterising scientific progress was "anything goes!"

1.2 Summary

- We will use theory, but we will try to keep things simple, rather simpler than is often the case in this field, especially in numerical methods.
- Often our knowledge of physical quantities is limited, and approximation is justified.
- We will recognise that we are modelling.
- An approximate model can often reveal to us more about the problem.
- It might be thought that the lectures show a certain amount of inconsistency – in occasional places the lecturer will develop a more generalised and "accurate" model, paradoxically to emphasise that we are just modelling.
- We will attempt to obtain insight and understanding – and a sense of criticality.

1.3 Types of channel flow to be studied

An important part of this course will be the study of different types of channel flow.

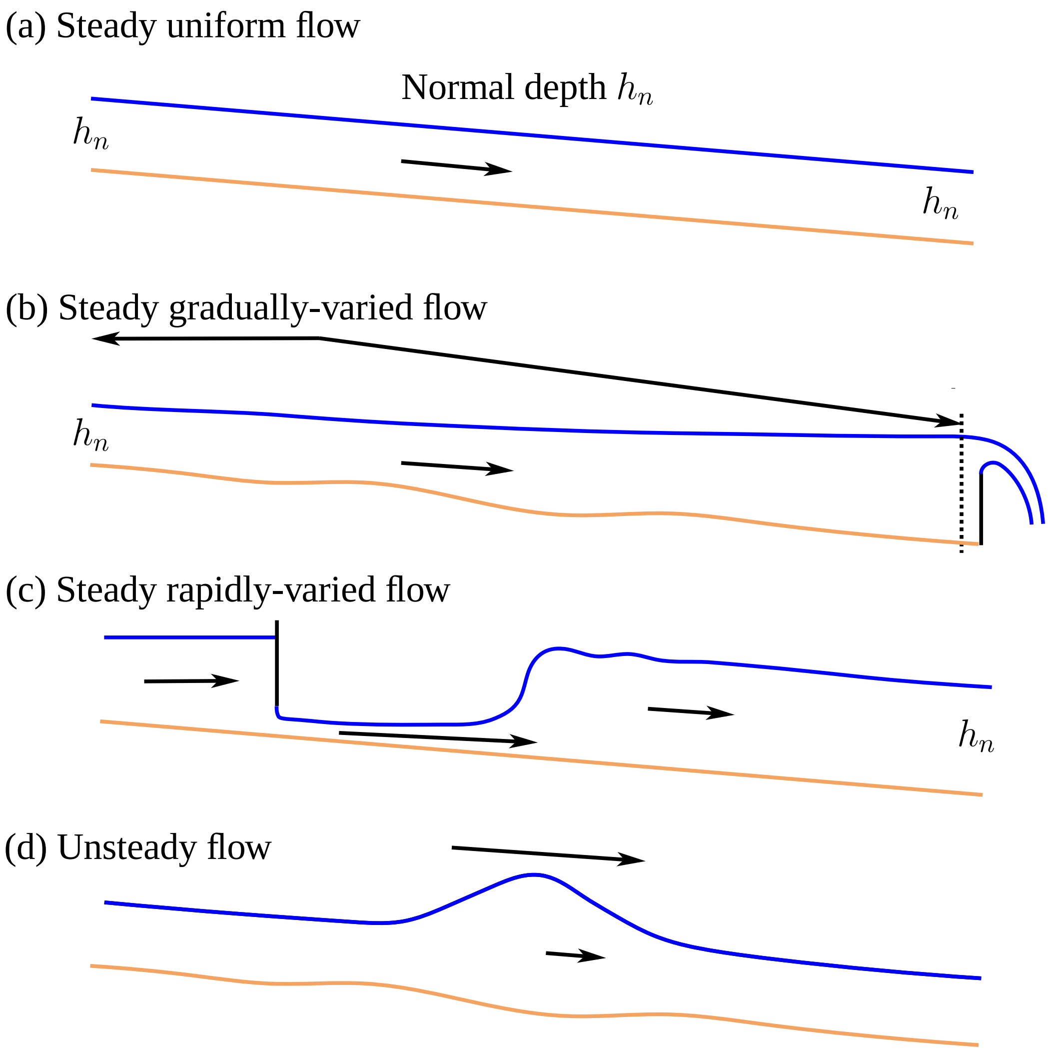


Figure 1.1: Different types of flow in an open channel

Case (a) – Steady uniform flow:

Steady flow is where there is no change with time, $\partial/\partial t \equiv 0$. Distant from control structures, gravity and resistance are in balance, and if the cross-section is constant, the flow is uniform, $\partial/\partial x \equiv 0$. This is the simplest model, and often is used as the basis and a first approximation for others.

Case (b) – Steady gradually-varied flow:

Where all inputs are steady but where channel properties may vary and/or a control may be introduced which imposes a water level at a certain point. The height of the surface varies along the channel. For this case we will study the governing differential equation that describes how conditions vary along the waterway, and we will obtain an approximate mathematical solution to solve general problems approximately.

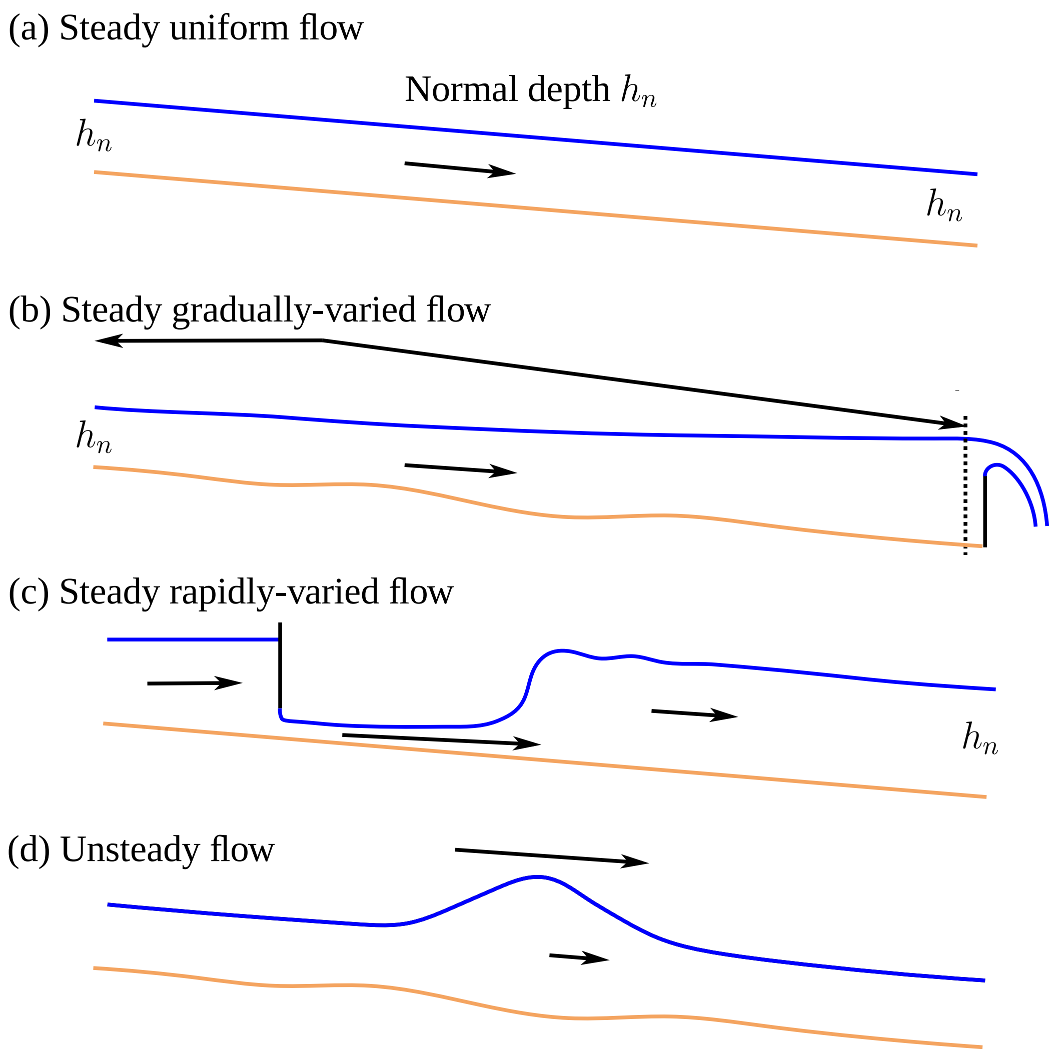


Figure 1.2: Different types of flow in an open channel

Case (c) – Steady rapidly-varied flow:

Figure (c) shows three separate gradually-varied flow regions separated by two rapidly-varied regions: (1) flow under a sluice gate and (2) a hydraulic jump. The basic hydraulic approximation that variation is gradual breaks down in those regions. We can analyse them by considering energy or momentum conservation locally. In this course we will not be considering these – earlier courses at TUW have.

Case (d) – Unsteady flow:

Here conditions vary with time and position as a flood wave traverses the waterway. We will consider flood wave motion at some length.

2. Conservation of mass, momentum and energy

2.1 Some possibly-surprising results

Effects of turbulence on dynamics

Where the fluid flow fluctuates in time, apparently randomly, about some mean condition, *e.g.* the flow of wind, water in pipes, water in a river. In practice we tend to work with mean flow properties, however in this course we will adopt empirical means of incorporating some of the effects of turbulence. Consider the x component of velocity at u a point written as a sum of the mean ($\bar{u}$) and fluctuating (u') components:

$$u = \bar{u} + u'.$$

By definition, the mean of the fluctuations, which we write as $\overline{u'}$, is

$$\overline{u'} = \frac{1}{T}\int_0^T u'\,dt = 0, \tag{2.1}$$

where T is some time period much longer than the fluctuations.

Now let us compute the mean value of the *square* of the velocity, such as we might find in computing the mean pressure on an object in the flow:

$$\begin{aligned}
\overline{u^2} &= \overline{(\bar{u}+u')^2} = \overline{\bar{u}^2 + 2\bar{u}u' + u'^2}, \text{ expanding,} \\
&= \overline{\bar{u}^2} + \overline{2\bar{u}\,u'} + \overline{u'^2}, \text{ considering each term in turn,} \\
&= \bar{u}^2 + 2\bar{u}\,\overline{u'} + \overline{u'^2}, \text{ but, as } \overline{u'} = 0 \text{ from(2.1),} \\
&= \bar{u}^2 + \overline{u'^2}.
\end{aligned} \tag{2.2}$$

hence we see that the mean of the square of the fluctuating velocity is not equal to the square of the mean of the fluctuating velocity, but that there is also a component $\overline{u'^2}$, the mean of the fluctuating components. We will need to incorporate this.

Pressure in open channel flow – effects of resistance on flows over steep slopes

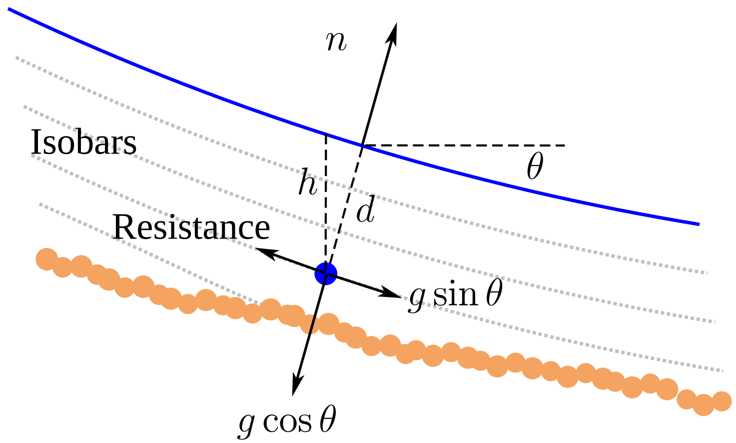


Figure 2.1: Channel flow showing isobars and forces per unit mass on a fluid particle

An almost-universal assumption in river engineering is that the pressure distribution is *hydrostatic*, that of water which is not moving, such that pressure p at a point is given by the height of water above, $p = \rho g h$, where ρ is fluid density ($\approx 1000\,\mathrm{kg\,m^{-3}}$ for fresh water), $g \approx 9.8\,\mathrm{ms^{-2}}$ is gravitational acceleration, and h is the vertical height of the surface above the point. This is not necessarily the case in flowing water, and needs to be known for cases such as spillways or block ramps, which are steep.

Consider figure 2.1 showing an open channel flow with forces per unit mass acting on a particle. The figure is drawn, showing that in general, the depth is not constant, and the bed is not parallel to the free surface. It is an isobar, a line of constant pressure, $p = 0$. In the flow, other isobars will generally be parallel to this, while the channel bed is not necessarily an isobar. We consider the vector Euler equation for the motion of a fluid particle

$$\text{Acceleration} = -\frac{1}{\rho} \times \text{Pressure gradient} + \text{Body forces per unit mass}$$

In a direction parallel to the free surface, the pressure is constant and there is no pressure gradient. The acceleration of the particle will be given by the difference between the component of gravity $g \sin\theta$ and the resistance force per unit mass. We usually do not know the details of that, so there is little that we can say. Now considering a direction perpendicular to that, given by the co-ordinate n on the figure, there is very little acceleration, so we assume it to be zero, and so we obtain the result

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} - g\cos\theta.$$

Now integrating this with respect to n between a general point, such as $n = -d$ at the particle shown, and $n = 0$ on the surface where $p = 0$ we obtain

$$p = \rho g \cos\theta \times d.$$

It is much more convenient to measure all elevations vertically, and so we use h, such that $d = h\cos\theta$, and we obtain the general expression for pressure

$$p = \rho g h \cos^2\theta.$$

This result (for steady uniform flow) however it and its implications for general flows seems to have been forgotten by many. While that is nice to know, we do not need it now, because, like in the uniform flow section, *in almost all open channels the slope is small enough* such that $\cos^2\theta \approx 1$, and we can use the *hydrostatic approximation*, obtained from a static fluid, where the surface is horizontal,

$$p = \rho g h. \tag{2.3}$$

Substituting $h = \eta - z$, where η is the free surface elevation and z is the elevation of an arbitrary point in the fluid,

$$p = \rho g \left(\eta - z\right). \tag{2.4}$$

From this we have *at a specific vertical cross-section*,

$$p + \rho g z = \rho g \eta, \tag{2.5}$$

so that anywhere on a vertical section $p + \rho g z$ is constant, given by the free surface elevation.

2.2 Flux of volume, mass, momentum and energy across a surface

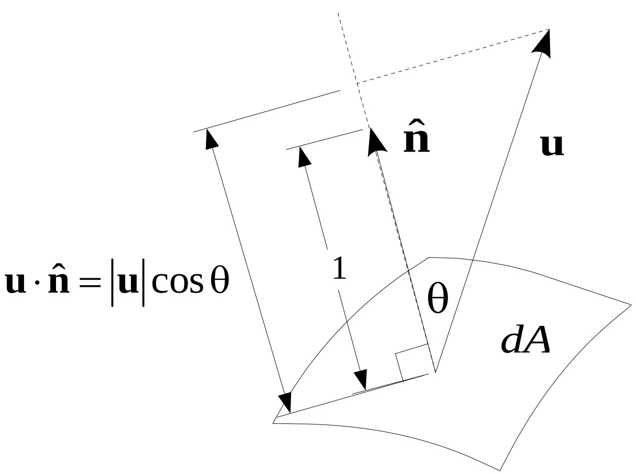


Figure 2.2: Element of surface dA with local velocity vector $\mathbf{u}$ showing how the velocity component normal to the surface is $\mathbf{u} \cdot \hat{\mathbf{n}}$.

It is necessary for us to be able to calculate the total quantity of fluid and integral quantities such as mass, momentum, and energy flowing across an arbitrary surface in space, which we will then apply to the rather more simple case of control surfaces. Consider an element of an arbitrary surface shown in Figure 2.2 through which fluid flows at velocity $\mathbf{u}$. The velocity component perpendicular to the surface is $|\mathbf{u}| \cos\theta = \mathbf{u} \cdot \hat{\mathbf{n}}$. In a time dt the volume of fluid which passes across the surface is $\mathbf{u} \cdot \hat{\mathbf{n}}\, dt\, dA$, or, the *rate* of volume transport is $\mathbf{u} \cdot \hat{\mathbf{n}}\, dA$. Other quantities easily follow from this: multiplying by density ρ gives the rate of *mass* transport, multiplying by velocity $\mathbf{u}$ gives the rate of transport of *momentum* due to fluid inertia (there is another contribution due to pressure for total momentum), and if e is the energy per unit mass, multiplying by e gives the rate of energy transport across the element. By integrating over the whole surface A, not necessarily closed, gives the transport of each of the quantities, so that we can write

$$\text{Rate of} \left\{ \begin{array}{c} \text{volume} \\ \text{mass} \\ \text{inertial momentum} \\ \text{energy} \end{array} \right\} \text{transport across surface } A = \int_A \begin{bmatrix} 1 \\ \rho \\ \rho\mathbf{u} \\ \rho e \end{bmatrix} \mathbf{u} \cdot \hat{\mathbf{n}}\, dA. \tag{2.6}$$

Note that as $\mathbf{u}\cdot\hat{\mathbf{n}}$ is a scalar there is no problem in multiplying this simply by either a vector or a scalar. In hydraulic practice such integrals are usually evaluated more easily. For example, across a pipe or channel which is locally straight, to calculate the rates of transport we choose a surface perpendicular to the flow.

Flux across solid boundaries: There can be no velocity component normal to a solid boundary, such that every solid boundary satisfies the boundary condition $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ and so from equation (2.6) there can be no volume, mass, momentum, or energy transfer across solid boundaries.

2.3 Mass and volume flux

Now we consider successively mass, momentum, and energy, and the effects of turbulence will be incorporated. In most hydraulics the density of water varies very little and so ρ can be assumed to be a constant and we can often just consider volume transport..

If the flow through each flow boundary cuts the boundary at right angles, we can write the velocity as $\mathbf{u} = \pm u\,\hat{\mathbf{n}}$, such that $\mathbf{u}\cdot\hat{\mathbf{n}} = \pm u$, where the plus/minus sign is taken when the flow leaves/enters the control volume. Then across any section of area A we have the contribution $\int_A \mathbf{u}\cdot\hat{\mathbf{n}}\,dA = \pm\int_A u\,dA$, which is $\pm Q$, the *volume flow rate* or *discharge* across the section. Sometimes it is convenient to express this in terms of U, the mean velocity, such that

$$\text{Rate of volume transport across surface } = \int_A u\,dA = Q = U A.$$

2.4 Momentum flux

Formulation

Newton's second law states that the net rate of change of momentum is equal to the force applied. In equation (2.6) we obtained

$$\text{Momentum transport across a surface} = \int_A \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA. \tag{2.7}$$

There are two main contributions to the force applied. One is due to surface forces, the pressure p over the surface. On an element of the control surface with area dA and outwardly-directed normal $\hat{\mathbf{n}}$ the pressure force on the fluid in the control surface has a magnitude of $p\, dA$ (simply pressure multiplied by area) and a direction $-\hat{\mathbf{n}}$, because the pressure acts normal to the surface and the direction of the force on the fluid is directed inwards to the control volume.

The other contribution is the sum of all the body forces, which will be usually due to gravity. We let these be denoted by $\mathbf{F}_{\text{body}}$. Equating the rate of change of momentum to the applied forces and taking the pressure force over to the other side we obtain the *integral momentum theorem* for steady flow

$$\int_{\text{CS}} \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA + \int_{\text{CS}} p\, \hat{\mathbf{n}}\, dA = \mathbf{F}_{\text{body}}. \tag{2.8}$$

This form enables us to solve a number of problems yielding the force of fluid on objects and structures. Now the integrals in equation (2.8) will be separated into those over surfaces through which fluid flows and solid surfaces:

$$\sum_{\text{Fluid surfaces}} \left(\int_A (\rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}} + p\, \hat{\mathbf{n}})\ dA \right) + \sum_{\text{Solid surfaces}} \left(\int_A \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA + \int_A p\, \hat{\mathbf{n}}\, dA \right) = \mathbf{F}_{\text{body}}.$$

However, as $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ on all solid surfaces, there are no contributions. Also on the solid surfaces, unless we know all details of the flow field, we do not know the pressure p. However the sum of all those contributions is the total force $\mathbf{P}$ of the fluid on the surrounding structure. Hence we have the theorem in a more practical form for calculating the force on objects:

$$\text{Total force on solid surfaces} = \mathbf{P} = -\sum_{\text{Fluid surfaces}} \left(\underbrace{\int_A (\rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}} + p\, \hat{\mathbf{n}})\ dA}_{\text{Momentum flux}} \right) + \mathbf{F}_{\text{body}}. \qquad (2.9)$$

Note the use of the term *momentum flux* for the integral shown – it includes contributions from the inertial momentum flux and from pressure.

Inertial momentum flux

Here we evaluate the integral describing the transport of momentum by fluid velocity. In many situations *we can choose the control surface such that on each part where the fluid crosses it, the local surface element is planar, and the velocity crosses it at right angles.* We write

$\mathbf{u} = \pm u\hat{\mathbf{n}}$,where u is the fluid speed, whose magnitude might vary over that part of the control surface through which it passes, but whose direction is perpendicular either in the direction of the unit normal or opposite to it. Hence

$$\int_A \rho\mathbf{u}\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dA = \int_A \rho\left(\pm u\hat{\mathbf{n}}\right)\,(\pm u)\;dA = +\hat{\mathbf{n}}\rho\int_A u^2\,dA, \tag{2.10}$$

where, as $(\pm)\times(\pm)$ is always positive, the surprising result has been obtained that the contribution to momentum flux is always in the direction of the outwardly-directed normal, whether the fluid is entering or leaving the control volume. Also we have assumed that the area A is planar such that $\hat{\mathbf{n}}$ is constant, so that we have been able to take the $\hat{\mathbf{n}}$ outside the integral sign.

Approximation of the integral allowing for turbulence and boundary layers

Although it has not been written explicitly, it is understood that equation (2.10) is evaluated in a time mean sense. In equation (2.2) we saw that if a flow is turbulent, then $\overline{u^2} = \bar{u}^2 + \overline{u'^2}$, such that the time mean of the square of the velocity is greater than the square of the mean velocity. In this way, we should include the effects of turbulence in the inertial momentum flux by writing the integral on the right of equation (2.10) as

$$\int_A u^2\,dA = \int_A \left(\bar{u}^2 + \overline{u'^2}\right)\;dA. \tag{2.11}$$

Usually we do not know the nature of the turbulence structure, or even the actual velocity distribution across the flow, so that we approximate this in a simple sense by recognising that the time mean velocity at any point and the magnitude of the turbulent fluctuations are all of the scale of the mean velocity in the flow in a time and spatial mean sense, $\bar{U} = Q/A$, such that we write for the integral in space of the time mean of the squared velocities:

$$\int\limits_A u^2\,dA = \int\limits_A \left(\bar{u}^2 + \overline{u'^2}\right)\,dA \approx \beta\bar{U}^2 A = \beta\left(\frac{Q}{A}\right)^2 A = \beta\frac{Q^2}{A}. \tag{2.12}$$

The coefficient β is called a Boussinesq coefficient, after the French engineer who introduced it to allow for the spatial variation of velocity. Allowing for the effects of time variation, turbulence, has been a recent addition.

The coefficients β have typical values of 1.05 to something like 1.5 or more in channels of irregular cross-section. Almost all textbooks introduce this quantity for open channel flow (without turbulence) but then assume it is equal to 1. Surprisingly, for pipe flow it seems not to have been used at all. In this course we consider it important and will include it.

2.5 Energy flux and conservation

The energy equation in integral form can be written for a control volume CV bounded by a control surface CS, where there is no heat added or work done on the fluid inside the control volume:

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho e \, \mathrm{d}V}_{\text{Rate of change of energy inside CV}} + \underbrace{\int_{\mathrm{CS}} \rho e \, \mathbf{u}\cdot\hat{\mathbf{n}} \, \mathrm{d}S}_{\text{Flux of energy across CS}} + \underbrace{\int_{\mathrm{CS}} p \, \mathbf{u}\cdot\hat{\mathbf{n}} \, \mathrm{d}S}_{\text{Rate of work done by pressure}} = 0, \tag{2.13}$$

where t is time, ρ is density, $\mathrm{d}V$ is an element of volume, ρe is the internal energy per unit volume of fluid, ignoring nuclear, electrical, magnetic, surface tension, and intrinsic energy due to molecular spacing, leaving the sum of the potential and kinetic energies such that the internal energy *per unit mass* is

$$e = gz + \tfrac{1}{2}\left(u^2 + v^2 + w^2\right), \tag{2.14}$$

where the velocity vector $\mathbf{u} = (u, v, w)$ in a cartesian coordinate system, the co-ordinate z is vertically upwards, p is pressure, and dS is the elemental area of the control surface.

Here *steady* flow is considered, at least in a time-mean sense, so that the first term in equation (2.13) is zero. The equation becomes, after dividing by density ρ and taking the long-term time mean, denoted by an overbar:

$$\overline{\int_{\mathrm{CS}} \left(p + \rho g z + \frac{\rho}{2}\left(u^2 + v^2 + w^2\right)\right) \mathbf{u}\cdot\hat{\mathbf{n}} \, \mathrm{d}S} = 0. \tag{2.15}$$

The energy flux over a section of area A is then $\pm\dot{E}$, depending on whether the flow is leaving/entering the control volume, where

$$\dot{E} = \overline{\int_A \left(p + \rho g z + \frac{\rho}{2}\left(u^2 + v^2 + w^2\right)\right) u \, \mathrm{d}A},$$

Now we consider the individual contributions to this integral.

The pressure distribution in an open channel (river, canal, drain, *etc.*) is usually very close to hydrostatic (streamlines have a very small slope), so that $p/\rho + gz$ is constant over a section. Hence we can take the first two terms of the integral outside the integral sign and use the result that $\int_A u \; dA = Q$ to give

$$\dot{E} = (p + \rho g z)\, Q + \frac{\rho}{2} \overline{\int_A \left(u^2 + v^2 + w^2\right) u \, \mathrm{d}A}, \tag{2.16}$$

where p and z outside the integral are the pressure and elevation at any point on a particular section. *Our treatment is not entirely satisfactory – we have ignored turbulent contributions in the nonlinear term* pQ*, as we almost never know anything about turbulent pressure fluctuations. It really should be written* $\bar{p}\bar{Q} + \overline{p'Q'}$.

Now, in the same spirit as for momentum, when we introduced a coefficient β to allow for a non-constant velocity distribution, we introduce a coefficient α such that it allows for the variation of the kinetic energy term across the section and in time and we write

$$\overline{\int_A \left(u^2+v^2+w^2\right)\, u\, \mathrm{d}A} = \alpha U^3 A = \alpha \frac{Q^3}{A^2}, \tag{2.17}$$

where $U = Q/A$ is the mean velocity. Obviously, α is defined by

$$\alpha = \frac{A^2}{Q^3} \overline{\int_A \left(u^2+v^2+w^2\right)\, u\, \mathrm{d}A} = \frac{1}{U^3 A} \overline{\int_A \left(u^2+v^2+w^2\right)\, u\, \mathrm{d}A} \tag{2.18}$$

which is "the kinetic energy term divided by the value of the term if the flow had a single velocity component constant in time and space". Such a coefficient was introduced by Coriolis, a French military engineer who neglected the other two velocity components and turbulence. Strangely, textbooks even today just use his original definition and take the first component under the integral sign, neglect turbulence, and write

$$\alpha = \frac{A^2}{Q^3} \int_A u^3\, \mathrm{d}A = \frac{1}{U^3 A} \int_A u^3\, \mathrm{d}A, \tag{2.19}$$

where α is the *Coriolis* coefficient, for which a typical value is $\alpha \approx 1.05 – 1.5$. With equation (2.16) and the definition of α from equation (2.18):

$$\text{Rate of energy transport across a section} = \dot{E} = (p + \rho g z)\, Q + \alpha \frac{\rho}{2} \frac{Q^3}{A^2}.$$

This can be written in a factorised form

$$\dot{E} = \underbrace{\rho Q}_{\text{Mass rate of flow}} \times \left(\underbrace{\frac{p}{\rho} + gz + \frac{\alpha}{2}\frac{Q^2}{A^2}}_{\text{Energy per unit mass}} \right).$$

The energy per unit mass has units of Joules per kilogram, $\mathrm{J\,kg^{-1}}$. It is common in civil and environmental engineering problems to factor out gravitational acceleration and to write

$$\dot{E} = \rho g Q \times \left(\underbrace{\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}}_{\text{Mean total head of the flow}} \right), \tag{2.20}$$

where the quantity in the brackets has units of length, corresponding to elevation, and is termed the *Mean total head of the flow* H, the mean energy per unit mass divided by g. This form is more convenient, because often in hydraulics elevations are more important and useful than actual energies. For example, the height of a reservoir surface, or the height of a levee bank on a river might be known and govern design calculations.

Energy conservation equation for a control surface

Now, evaluating the integral energy equation (2.15) using these approximations over each of the parts of the control surface through which fluid flows, numbered 1, 2, 3, … gives an energy conservation equation

$$\pm \dot{E}_1 \pm \dot{E}_2 \pm \dot{E}_3 \pm \ldots = \pm \rho_1 g Q_1 H_1 \pm \rho_2 g Q_2 H_2 \pm \rho_3 g Q_3 H_3 \pm \ldots = 0, \tag{2.21}$$

where the positive/negative sign is taken for fluid leaving/entering the control volume. In almost all hydraulics problems the density can be assumed to be constant, and so dividing through by the common density and gravity, we have

$$\pm Q_1 H_1 \pm Q_2 H_2 \pm Q_3 H_3 \pm \ldots = 0. \tag{2.22}$$

If we consider a control surface enclosing a junction of two pipes or channels, it seems we must include the volume rates of flow as shown.

We will consider a length of pipe or channel in which water enters at only one point 1 and leaves at another 2, so that $-Q_1 H_1 + Q_2 H_2 = 0$, but by volume conservation $Q_1 = Q_2$ so we obtain our familiar result $H_1 = H_2$ or

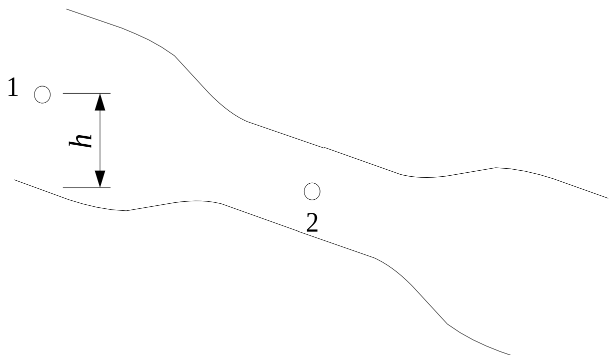
$$\left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_1 = \left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_2. \tag{2.23}$$

However, to give more accurate practical results, an empirical allowance is usually made for energy losses, and in most applications the equation is used in the form

$$H_1 = H_2 + \Delta H,$$

where ΔH is a head loss. In many situations it is given as an empirical coefficient times the kinetic head.

Example 1 Application of the integral form of the energy equation – The Venturi meter, where we will see that the fact that a coefficient which is necessary to give correct results is not due to head loss, but just due to $\alpha \neq 1$.



Consider the gradual constriction in the pipe shown, where there are pressure tapping points in the horizontal side of the pipe at point 1 before the constriction and point 2 in the constriction. Using a control volume crossing the pipe at 1 and 2 and applying equation (2.23), the integral energy theorem for a single inlet and outlet,

$$\frac{p_1}{\rho g} + z_1 + \frac{\alpha}{2g}\frac{Q^2}{A_1^2} = \frac{p_2}{\rho g} + z_2 + \frac{\alpha}{2g}\frac{Q^2}{A_2^2}.$$

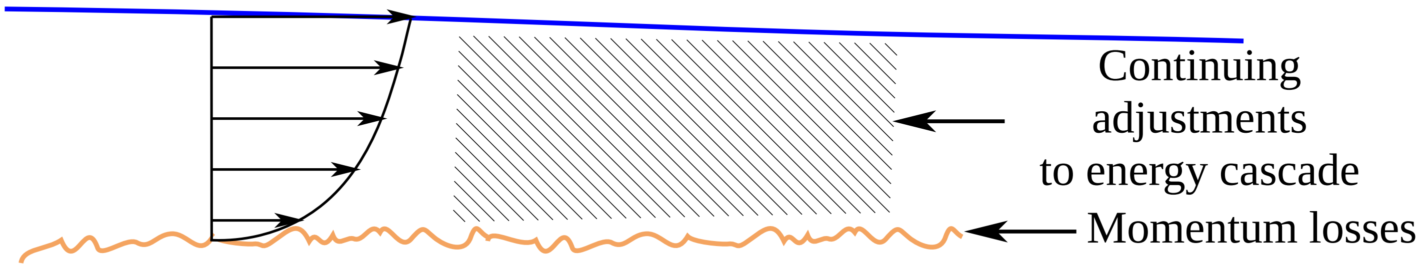
Solving for Q and using $z_1 = z_2 + h$ gives

$$Q = \sqrt{\frac{2}{\alpha}\frac{\frac{p_1-p_2}{\rho} + gh}{\frac{1}{A_2^2} - \frac{1}{A_1^2}}}.$$

Hence we see that by measuring the pressure change between 1 and 2 we can obtain an approximation to the discharge. It is interesting that other presentations introduce a coefficient $C\ (< 1)$ in front to agree with experiment. In our presentation, by including the parameter $\alpha\ (> 1)$ we may have gone some way to quantifying that.

Comments and conclusions from our discussion of energy

- We have considered energy flux and conservation in *integral* form. We have *not* considered Bernoulli's equation, which is only valid along a streamline. It is easier to derive the energy principle in integral form, it is what we use in practice anyway, *and* a little-known fact, Bernoulli's equation is actually based on conservation of momentum. It does not look like it!
- In this course of River Engineering, we will rarely use energy conservation. Momentum conservation is simpler because momentum losses only occur at boundaries, such as the bed of a channel, or at an obstacle. Energy losses due to shear layers, turbulence, viscosity *etc* are diffused through the fluid, which are more complicated than the momentum losses.



- So, why did we consider energy at some length? Knowledge is never wasted; we never know when we might need energy; the treatment of momentum and energy was similar so it was a good time to do it; *and*
- We showed that in two cases, the strangely limited treatment of α and β without including turbulence or other velocity components *and* our demonstration of the nature of the formula for the Venturi meter, the general principle: *no-one can be trusted, neither textbooks nor lecturers* ...

3. Resistance in river and other open channel flows

The resistance to the flow of a stream is probably the most important problem in river mechanics.

Page 27: We consider a simple theory based on force balance and some classical fluid mechanics experiments to obtain a flow formula for a wide channel.

Page 29: To obtain the equivalent formula for channels of any section we consider velocity distributions in real streams and develop an approximation giving a general flow formula.

Page 34: We consider an approximation to that formula and find that we have obtained the Gauckler-Manning-Strickler formula, including a theoretical prediction of Strickler's formula for the effect of boundary grain size.

Page 37: Comparison with a series of experiments validates the approach, giving an explicit flow formula for a variety of channel boundaries.

Page 39: The common problem of calculating the water depth for a given flow rate is considered. A computational method is developed and applied.

Page 42: For more general river problems, considering the nature of the bed particles and bed forms, vegetation, meandering, and possibly obstacles, it is better to use a formulation in which forces and the mechanics are clearer: the Chézy-Weisbach flow formula.

Page 44: A large number of stream-gauging results are considered and the values of the Weisbach resistance coefficient, its dependence on grain size, and on the state of the bed are obtained. Empirical formulae are considered.

3.1 A channel flow formula from theory and experiment

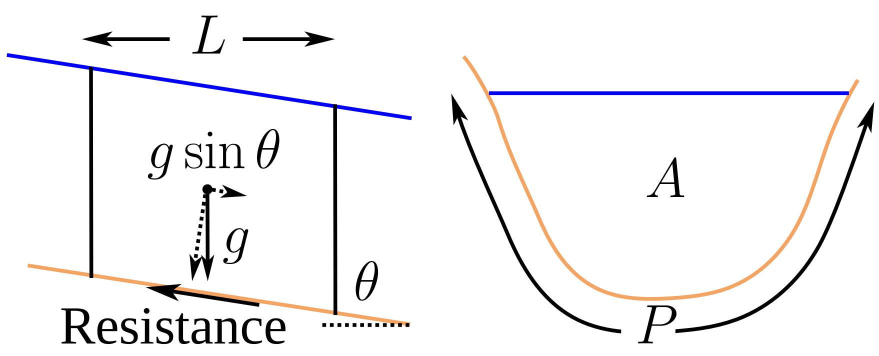


Figure 3.1: Uniform flow in a channel, showing resistance and gravity forces on a finite length, plus cross-section quantities

Here, as an example of the application of rational mechanics, a flow formula for steady uniform flow in channels is developed without using the empirical flow formulae of Gauckler-Manning-Strickler or Chézy-Weisbach.

Consider a horizontal length L of uniform channel flow, inclined at a small angle θ to the horizontal, with cross-sectional area A. The volume of the element is AL, the vertical gravitational force on the water is $\rho g AL$, where ρ is fluid density and g is gravitational acceleration. The component of this along the slope is $\rho g AL \sin\theta$. The resistance force along the slope, of length $L/\cos\theta$ is $\tau PL/\cos\theta$, where τ is the mean resistance shear stress, assumed uniformly distibuted around the wetted perimeter P around which it acts. Equating gravitational and resistance components gives $\tau PL/\cos\theta = \rho g AL \sin\theta$. To high accuracy for small θ, $\cos\theta \approx 1$ and $\sin\theta \approx \tan\theta = S$, the slope, giving

$$\frac{\tau}{\rho} = g\frac{A}{P}S. \tag{3.1}$$

Our problem is now to express shear stress τ in terms of flow quantities.

One of the most famous series of experiments in fluid mechanics was performed by Johann Nikuradse at Göttingen in the 1930s, who studied the flow of fluid over uniformly-rough sand grains. The fluid was actually air, and the sand grains were actually in circular pipes, but the results are still valid enough.

With those results, for a wide channel of depth h with sand grains of size k_s, the velocity distribution for fully rough flow (no effects of viscosity), the Prandtl-von Kármán *universal velocity distribution* can be written:

$$u = \frac{u_*}{\kappa} \ln \frac{30\,z}{k_s},$$

in terms of the shear velocity $u_* = \sqrt{\tau/\rho}$, the von Kármán constant $\kappa \approx 0.4$, the vertical co-ordinate z, and where the factor of 30 is for closely-packed uniform sand grains. It varies somewhat with other types of boundary roughness. The mean velocity U is obtained by integrating between 0 and h, such that

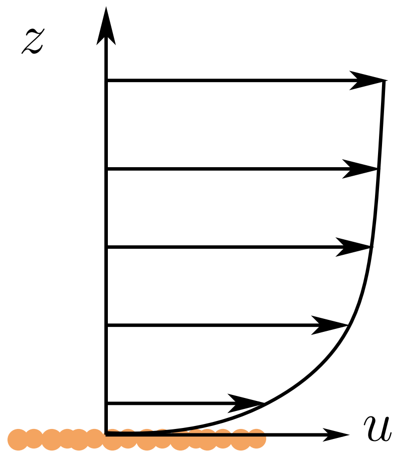


Figure 3.2: Idealised logarithmic velocity profile in turbulent flow over rough bed

$$U = \frac{u_*}{\kappa} \ln \frac{30/\mathrm{e}}{k_s/h}.$$

where e is Euler's number $\exp(1) = 2.718\ldots$, obtaining the result in terms of *relative roughness* k_s/h. Now we replace $u_* = \sqrt{\tau/\rho}$ by the expression on the right of equation (3.1) to give

$$U = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln \frac{30/\mathrm{e}}{k_s/h}\right), \tag{3.2}$$

We have obtained something possibly useful – a formula for the mean flow velocity in a wide channel of constant depth h, slope S, and relative roughness k_s/h. We have used simple mechanics plus empirical laboratory results. Surprisingly, the formula is explicit in terms of physical quantities – we have not had to assume a value like the Strickler k_{St}!

That was for a wide channel with an idealised logarithmic velocity distribution. In nature, for channels of any general cross-section there is the problem that the velocity has a maximum at a somewhere below the surface, and in general the isovels are something like Figure 3.3.

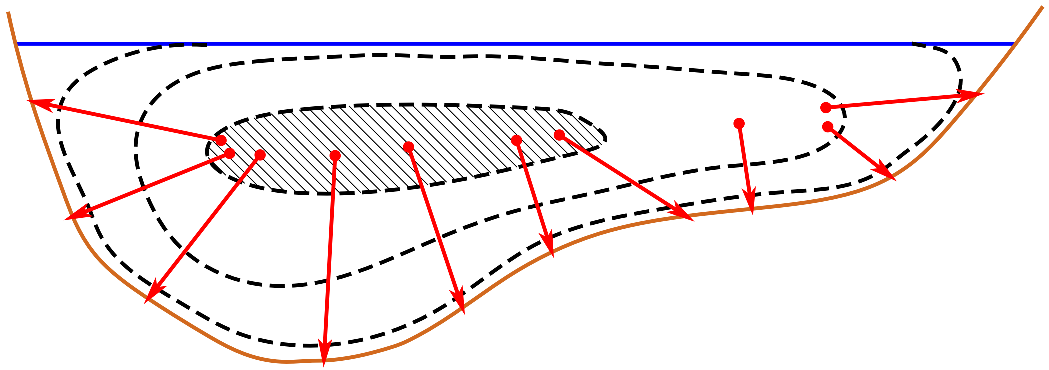


Figure 3.3: Cross-section of flow showing isovels and, for a number of points on the bed, where the fastest-likely fluid comes from and how far it travels, the effective length scale for resistance calculations.

To obtain a flow formula for channels of any cross-section, we hypothesise that the effective depth h for resistance calculations is the typical distance from points with the highest velocity to the nearest point on the bed, as suggested by the red arrows on the figure. If this model is correct, typical length scales as shown by the arrows are somewhat *smaller* than the overall mean depth of flow.

This is a highly approximate model, but at least it is in the spirit of modelling, that it is simple and transparent – and so far has not been obscured by mathematical detail.

Our problem is then, how can we simply approximate that distance?

We consider the experimental data for the vertical position of the locus of velocity maxima in *rectangular* channels from Yang, Tan & Lim (2004). They presented a formula for the height above the bed of the velocity maximum as a function of position across the channel. If the mean value of this is calculated by integration, a formula for the mean elevation of the velocity maximum $z_{\max}/h$ is obtained as a function of aspect ratio (channel width B divided by depth h).

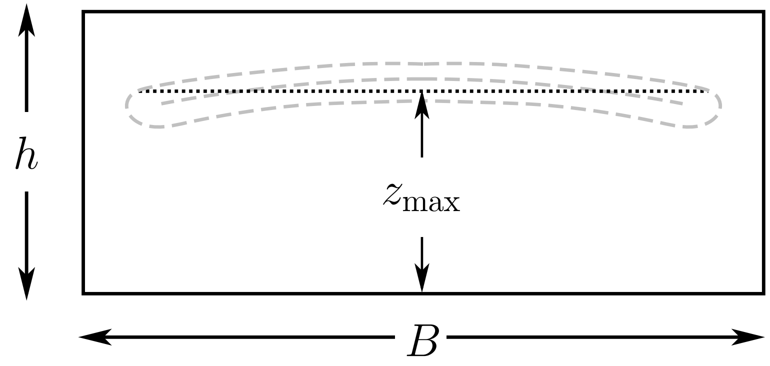


Figure 3.4: Experimental determination of velocity maxima in rectangular channel

The only other length scale we have above in equation (3.2) is the ratio of area to perimeter A/P. As $P > B$ it should be smaller than the mean depth A/B, so we will try it:

$$\frac{A/P}{h} = \frac{Bh}{(B+2h)h} = \frac{B/h}{B/h+2}. \tag{3.3}$$

Both this and the experimental formula for $z_{\max}/h$ are plotted in Figure 3.5. Remarkably and coincidentally, the two coincide closely over a wide range of aspect ratios! We cannot claim that this is a justification as strong as it looks, but in the absence of anything else, instead of h we will use A/P for channels of any cross-section.

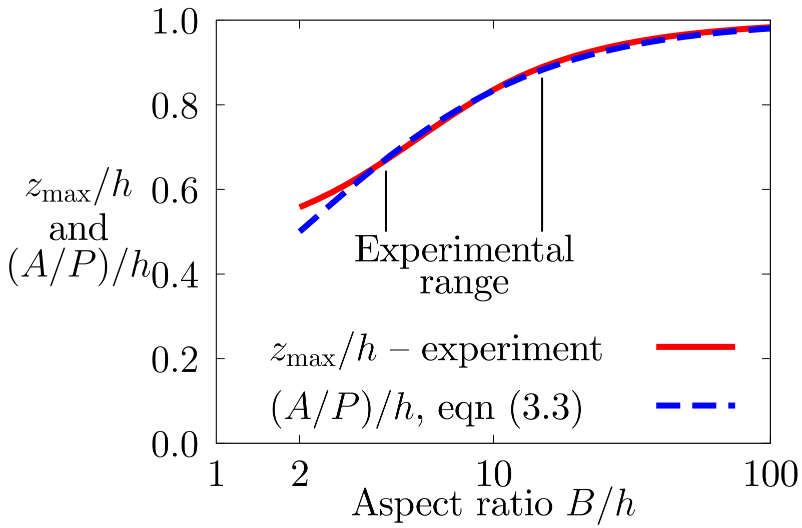


Figure 3.5: Rectangular channels: dimensionless mean elevation of $z_{\max}/h$ and the effective depth $(A/P)/h$

We have seen that A/P appeared naturally in the simple mechanical equilibrium calculation, and here we have found that it mimics the behaviour of z_{max}/h, which is fortunate as it is a rather simpler quantity to calculate in practice, usually with no knowledge of the flow field. We will name A/P the *hydraulic mean depth*, and we will not use the conventional and misleading term "hydraulic radius" (dt. after Strickler – "Profil- oder hydraulischer Radius"). In channels that are wide, which is most, $P \approx B$ and A/P is about the same as the geometric mean depth A/B.

For rectangular channels that are not rectangular we have presented no results. Our suggestion is that A/P will still be a plausible approximation, and it already appears in the Chézy-Weisbach and Gauckler-Manning-Strickler equations (which, officially, we do not yet know in this course!). The use in those equations was justified by Keulegan (1938), however there is much wrong with that work, mathematically correctly integrating logarithmic velocity distributions over various shapes of cross-section but without any attention to real flows in channels.

Our suggested channel flow formula, replacing h by A/P in equation (3.2) is

$$U = \frac{Q}{A} = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{30/\mathrm{e}}{k_{\mathrm{s}}/\left(A/P\right)}\right), \tag{3.4}$$

where $30/\mathrm{e} \approx 11.0$ is usually written as 12:

$$U = \frac{Q}{A} = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{12.}{k_{\mathrm{s}}/\left(A/P\right)}\right). \tag{3.5}$$

3.2 Generalised notation for flow formulae

We introduce the symbol ε for the relative roughness

$$\varepsilon = \frac{k}{A/P} = \frac{\text{Grain size}}{\text{Hydraulic mean depth}} \approx \frac{\text{Grain size}}{\text{Depth}},$$

also for equivalent uniform sand grain size, $\varepsilon = k_{\mathrm{s}}/(A/P)$. Our flow formula (3.5) is then written in generalised form

$$U = \frac{Q}{A} = \gamma\sqrt{g\frac{A}{P}S}, \tag{3.6}$$

in which we have already obtained the result for the shear velocity

$$u_* = \sqrt{g\frac{A}{P}S}$$

and we introduce the symbol γ for the velocity ratio

$$\frac{U}{u_*} = \gamma = \ln\frac{12.}{\varepsilon},$$

from equation (3.5).

Relative unimportance of grain size

In fact, γ, although all-important for us, is relatively slowly varying with grain size. Consider a small change in the relative roughness $\varepsilon\,(1+\Delta)$. The relative change in the factor γ is

$$\frac{\Delta\gamma}{\gamma} = \frac{\ln\left(12/(\varepsilon\,(1+\Delta))\right)}{\ln\left(12/\varepsilon\right)} - 1 \approx \frac{-\Delta}{\ln\left(12/\varepsilon\right)},$$

having expanded the logarithm as a power series $\ln(1+\Delta) = \Delta + \ldots$. Now for a value of $\varepsilon = 0.001$ (a $1\,\mathrm{mm}$ grain in $1\,\mathrm{m}$ of water), a relative change of $\Delta = 50\%$ gives a relative change in the factor γ in the equation of only -5%. Even for a much rougher case of $\varepsilon = 0.1$, the same relative change of 50% in grain size changes the left side by just -11%. It does not matter so much if we cannot specify the bed conditions all that accurately.

3.3 The Gauckler-Manning-Strickler formula

We now show that the G-M-S formula is an approximation to the expression we have obtained.

On Figure 3.6 is shown how the dimensionless factor γ varies as a function of relative roughness ε, given by equation (3.5) from experimental fluid mechanics. It is actually possible to approximate that curve closely using a monomial function a/ε^{μ}. The best values of a and μ can be found by performing a least-squares fit over 11 points equally-spaced in $\log\varepsilon$ between $\varepsilon = 0.001$ and 0.1. The result obtained was $\mu = 1/7.001$, which is a surprising coincidence. Now, setting $\mu = 1/7$ and determining just a by optimisation, a value of $a \approx 8.9$ was obtained:

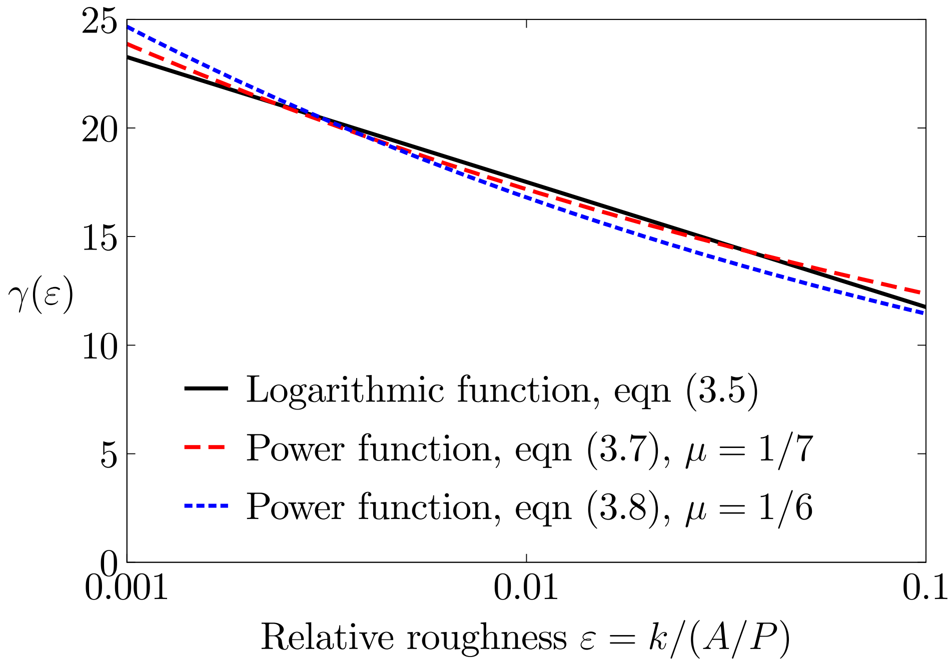


Figure 3.6:

$$\gamma = 8.9 \left(\frac{A/P}{k_{\mathrm{s}}}\right)^{1/7}, \tag{3.7}$$

with close agreement with the expression from the logarithmic velocity distribution shown in the figure. This would give us another flow formula, very similar to the Gauckler-Manning-Strickler

(G-M-S) formula. It is

$$U = \frac{Q}{A} = 8.9 \left(\frac{A/P}{k_{\mathrm{s}}}\right)^{1/7} \sqrt{g \frac{A}{P} S},$$

however unlike G-M-S this is explicit in terms of bed grain size. We do not want to proliferate an already crowded field, so we ignore this. It does, however, suggest the next step. We approximated the result from the logarithmic function again, this time by a function of the form $b/\varepsilon^{1/6}$, where b is a constant. We determined this constant by performing a similar least-squares fit, giving a value of $b \approx 7.8$ such that

$$\gamma = 7.8 \left(\frac{A/P}{k_{\mathrm{s}}}\right)^{1/6}, \tag{3.8}$$

with satisfactory results shown in Figure 3.6, showing that this is also quite a good approximation to our logarithmic function. Substituting into the flow formula, equation (3.6) and re-writing, we obtain

$$U = \frac{Q}{A} = \frac{7.8\sqrt{g}}{k_{\mathrm{s}}^{1/6}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S} = k_{\mathrm{St}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S} = \frac{1}{n} \left(\frac{A}{P}\right)^{2/3} \sqrt{S}, \tag{3.9}$$

which is simply the Gauckler-Manning-Strickler equation, where k_{St} is the Strickler coefficient and $n = 1/k_{\mathrm{St}}$ is the Manning coefficient! Unlike the G-M-S equation, this has given an explicit expression for the Strickler coefficient

$$k_{\mathrm{St}} = \frac{7.8\sqrt{g}}{k_{\mathrm{s}}^{1/6}}. \tag{3.10}$$

A similar result was obtained by Strickler nearly a century ago without optimising software, based entirely on experiment, on boundary roughnesses of equivalent mean diameter from $D = 0.1\,\mathrm{mm}$ to $D = 300\,\mathrm{mm}$,and where that diameter was sometimes calculated from alluvial gravel with relatives lengths of the three axes 1:2:3! For the numerical coefficient he obtained a value of $4.75\sqrt{2} \approx 6.7$, giving his expression

$$k_{\mathrm{St}} = \frac{6.7\sqrt{g}}{D^{1/6}}. \tag{3.11}$$

The expression (3.10) here has been obtained by a quite different route, and the agreement between the two expressions, one based on sand grains glued to the inside of a circular pipe carrying air, is encouraging. Of course, Strickler's result (3.11) is to be preferred.

Sensitivity to boundary particle size

One thing we can do now is, as we did earlier, examine the effect of uncertainty or variability in the size of the boundary particles (and any perceived difference between k_{s} and D), using a power series expansion

$$\frac{\Delta k_{\mathrm{St}}}{k_{\mathrm{St}}} = \left(1 + \frac{\Delta D}{D}\right)^{-1/6} - 1 = -\frac{1}{6}\frac{\Delta D}{D} + \ldots,$$

and so a fractional change in boundary particle size gives a relative change of $1/6$ of that amount in D. Again for this form the exact particle size is actually not so important.

Test of logarithmic and G-M-S formulae

To test the accuracy of the G-M-S formula compared with the logarithmic formula we consider the results of Strickler (1923, Beilage 4), which have been interpreted as the justification for the exponent $2/3$ in the G-M-S formula, and leading to the "S" in that name. Strickler considered results from nine very different channels. For each we calculated the equivalent k_s or D, constant for each channel, by least-squares fitting of the appropriate flow formula to the points, with results shown in the figure. If anything, the Gauckler-Manning-Strickler formula gives better agreement at the lower ends than the logarithmic formula obtained from fluid mechanics experiments.

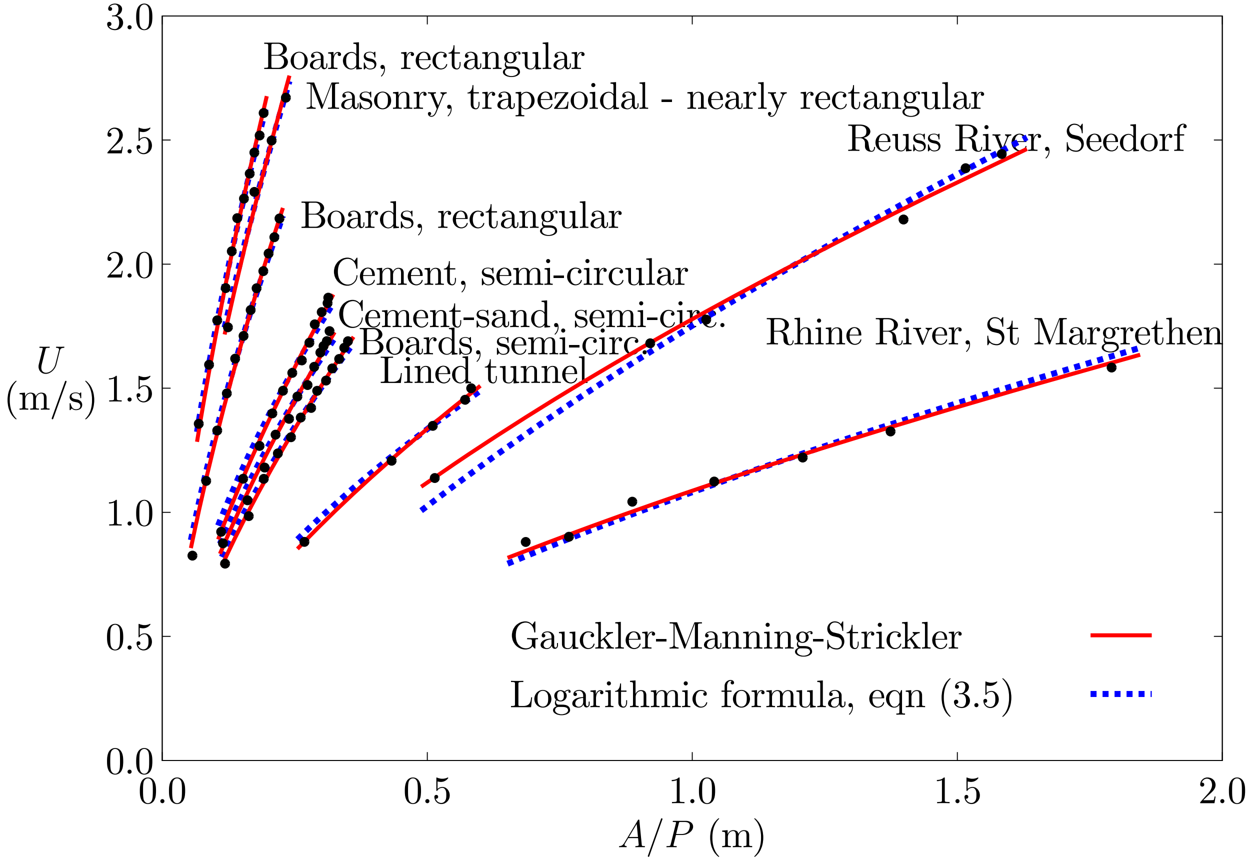


Figure 3.7: Strickler's results approximated by two flow formulae

3.4 Notation

In the rest of this course, we could write the Gauckler-Manning-Strickler formula in the conventional form

$$U = \frac{Q}{A} = k_{\mathrm{St}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S},$$

however if we use the Strickler expression $k_{\mathrm{St}} = 6.7\sqrt{g}/D^{1/6}$ we prefer to write it in the form which more reflects its nature and our derivation:

$$U = \frac{Q}{A} = \gamma \sqrt{g \frac{A}{P} S} \quad \text{where} \quad \gamma(\varepsilon) = \frac{6.7}{\varepsilon^{1/6}}, \text{ and } \varepsilon = \frac{D}{A/P}. \tag{3.12}$$

- We no longer have the problem of k_{St} with difficult units.
- The characterisation of the resistance has been reduced to that of the dimensionless relative roughness $\varepsilon = D/(A/P)$.
- If asked to estimate the flow at a particular site, we do not have to imagine a value of k_{St} or, like in an Australian water office, ring a friend to see what they used when they worked on a similar stream $20\,\mathrm{km}$ distant. We could estimate the D.
- If the bed material has a size of about $2\,\mathrm{cm}$ (Donau), then we simply use $D = 0.02\,\mathrm{m}$.
- It is much simpler and physically understandable.

3.5 Computation of normal flow

"Normal flow" is the name given to a uniform flow, and the depth is called the normal depth. If the discharge Q, slope S, resistance coefficient k_{St}, and the relationship between area and depth and perimeter and depth are known, the G-M-S formula becomes a transcendental equation for the normal depth h. To solve this is a common problem in river engineering

A numerical method

Any method for the numerical solution of transcendental equations can be used, such as Newton's method. Here we develop a simple method based on *direct iteration*, where we develop a trick, giving us rapid convergence.

In the case of wide channels, (*i.e.* channels rather wider than they are deep, a common case), the wetted perimeter P does not vary much with depth h. Similarly in the expression for the area, the width does not vary much with h. Consider the Gauckler-Manning-Strickler formula in the conventional form, written now

$$Q = k_{\mathrm{St}} \frac{A^{5/3}}{P^{2/3}} \sqrt{S}$$

we divide both sides by $h^{5/3}$, and showing functional dependence of A and P on h:

$$\frac{Q}{h^{5/3}} = k_{\mathrm{St}} \sqrt{S} \frac{(A(h)/h)^{5/3}}{P^{2/3}(h)}.$$

The term $A(h)/h$ is approximately the width of the channel, which for many channels varies little with h, as does the perimeter $P(h)$. So, the right side of the equation varies little with h, so by isolating the $h^{5/3}$ term and taking the $3/5$ power of both sides of the equation, we obtain the equation in a form suitable for direct iteration

$$h = \left(\frac{Q}{k_{\mathrm{St}}\sqrt{S}}\right)^{3/5} \times \frac{P^{2/5}(h)}{A(h)/h}, \tag{3.13}$$

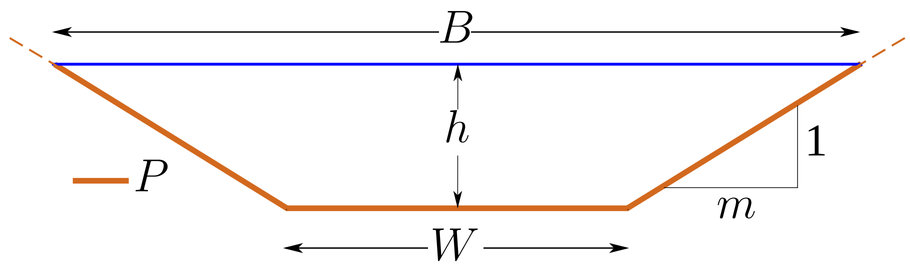
where the first term on the right is a constant for any particular problem, and the second term varies slowly with depth – a primary requirement that the direct iteration scheme be convergent and indeed be quickly convergent.

For an initial estimate we suggest making a rough estimate of the approximate width B_0 and so, making a wide channel approximation, setting $A(h)/h \approx B_0$ and $P(h) \approx B_0$, in the general scheme of (3.13) gives

$$h_0 = \left(\frac{Q}{k_{\mathrm{St}}B_0\sqrt{S}}\right)^{3/5}. \tag{3.14}$$

Experience with typical trapezoidal sections shows that the method works well and is quickly convergent.

Trapezoidal section



Most canals are excavated to a trapezoidal section, and this is often used as a convenient approximation to river cross-sections too. In many of the problems in this course we will consider the case of trapezoidal sections. Consider the quantities shown in the figure: the bottom width is W, the depth is h, the top width is B, and the *batter slope*, defined to be the ratio of H:V dimensions is m. Geometrically, $B = W + 2mh$, area $A = h\,(W + mh)$, wetted perimeter $P = W + 2\sqrt{1+m^2}h$.

Example 2 Calculate the normal depth in a trapezoidal channel of slope 0.001, $k_{\mathrm{St}} = 25$, bottom width $W = 10\,\mathrm{m}$, with batter slopes $m = 2$, carrying a flow of $20\,\mathrm{m}^3\mathrm{s}^{-1}$. We have $A = h\,(10 + 2\,h)$, $P = 10 + 4.472\,h$. For B_0 we use $W = 10\,\mathrm{m}$. Equation (3.14) gives

$$h_0 = \left(\frac{Q}{k_{\mathrm{St}} B_0 \sqrt{S}}\right)^{3/5} = \left(\frac{20}{25 \times 10\sqrt{0.001}}\right)^{3/5} = 1.745\,\mathrm{m}.$$

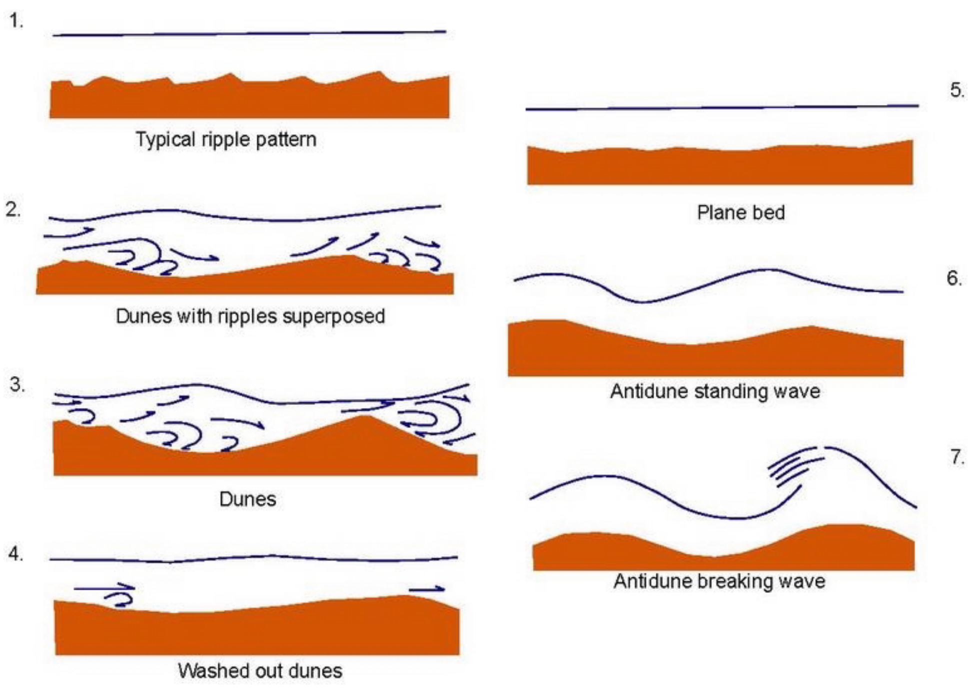
Then, equation (3.13) gives

$$h_{n+1} = \left(\frac{Q}{k_{\mathrm{St}} \sqrt{S_0}}\right)^{3/5} \times \frac{(10 + 4.472\,h_n)^{2/5}}{10 + 2\,h_n} = 6.948 \times \frac{(10 + 4.472\,h_n)^{2/5}}{10 + 2\,h_n}.$$

With $h_0 = 1.745$, $h_1 = 1.629$, $h_2 = 1.639$, $h_3 = 1.638\,\mathrm{m}$, and the method has converged.

3.6 General situations

In many cases the conditions in the river are more complicated than just a layer of uniform regular particles. For example:



- Irregular and variable nature of the bed particle arrangement.
- Bed forms – ripples, dunes, anti-dunes *etc*. Figure from Wikipedia **URL:** *https://en.wikipedia.org/wiki/Bedform*
- Particle movement – if the grains are moving, then the force required to move the grains appears to the water as an additional stress, whether they are moving along the bed, rolling, jumping, or carried suspended in the flow.
- Vegetation – trees standing in the water, grasses, reeds *etc*

Often one is required to adopt a value of resistance coefficient not given by the Strickler formula, but based on experience, knowledge, the Australian telephone method, looking at pictures in books *etc*. None of these are particularly good.

3.7 Chézy-Weisbach flow formula

A problem with the G-M-S form, using a value of k_{St} in more general situations, is that it has little basis in fluid mechanics, the coefficient has no physical significance, and it is in awkward units.

We now relate our results to another well-known standard formulae. Writing shear stress τ in terms of the result obtained from the Darcy-Weisbach formulation of flow resistance in pipes,

$$\frac{\tau}{\rho} = \tfrac{1}{8}\lambda U^2, \tag{3.15}$$

where λ is the Weisbach dimensionless resistance coefficient, expressing the relationship between velocity and stress. From our simple force balance we already have $\tau/\rho = \sqrt{g\,(A/P)\,S}$. Equating gives the Chézy-Weisbach flow formula

$$U = \frac{Q}{A} = \sqrt{\frac{8g}{\lambda}\frac{A}{P}S} = C\sqrt{\frac{A}{P}S}, \tag{3.16}$$

where $C = \sqrt{8g/\lambda}$ is the Chézy coefficient, named after the French military engineer who first wrote down such an open channel flow formula. Comparing our equation (3.5) we see that it is in the same form, such that

$$\frac{\text{Mean velocity}}{\text{Shear velocity}} = \gamma = \sqrt{\frac{8}{\lambda}} \tag{3.17}$$

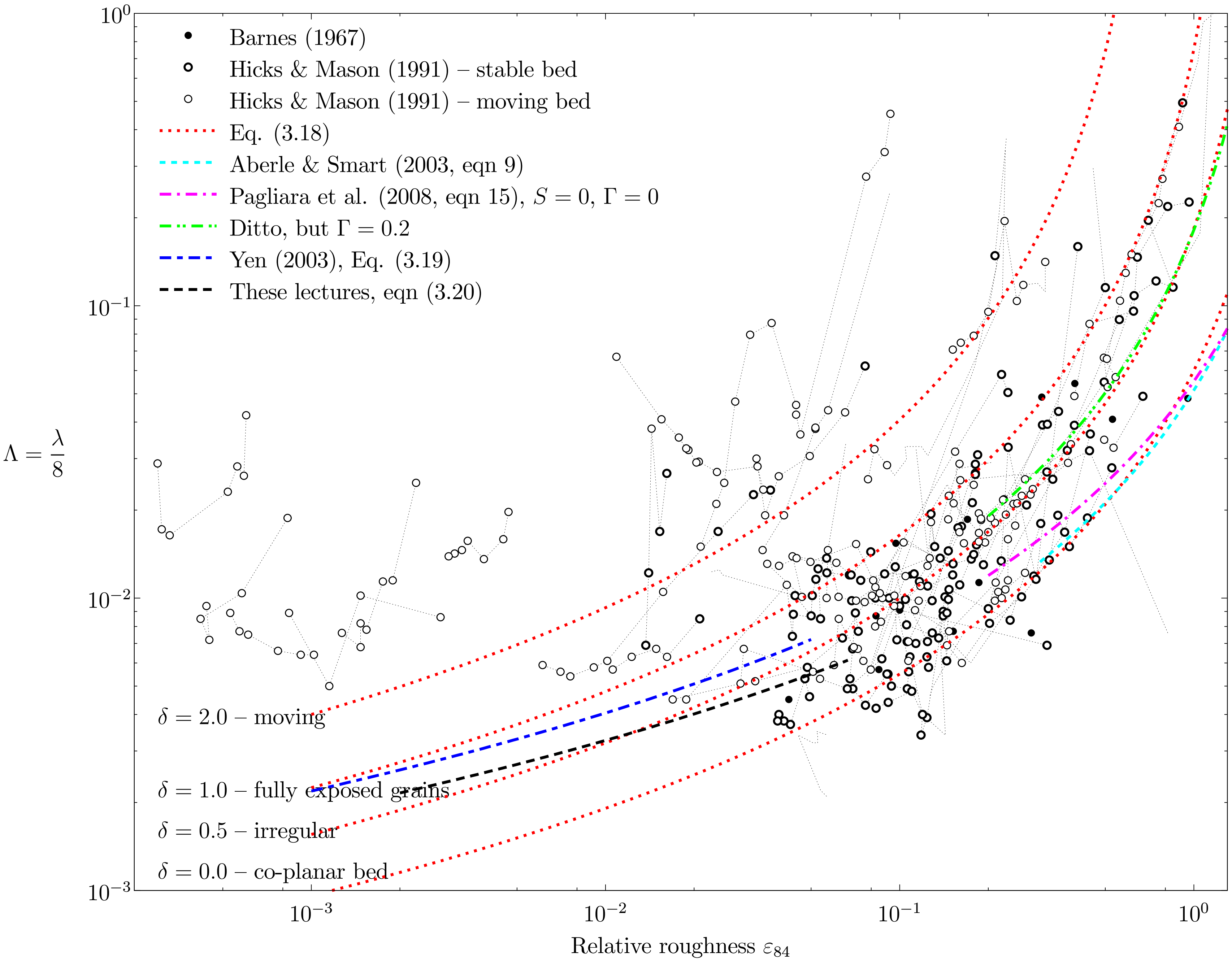
for the flows we have encountered so far in this section, steady, uniform flows in a cross-section which is relatively simple.

Where the resistance to flow is compound – where there are different parts of the stream cross-section with different resistances, where there are obstacles in a stream causing resistance, it is more consistent to use the Chézy-Weisbach formulation, which is more clearly related to forces involved and the mechanics of the problems. This is especially so when we consider the mechanics of flows and flood wave propagation.

Results for resistance coefficients in real rivers

Here we attempt to obtain understanding and a formula for the resistance coefficient using results from a number of field measurements. We considered the results of Hicks & Mason (1991), a catalogue of 558 stream-gaugings from 78 river and canal reaches in New Zealand, of which 55 were sites with grading curves for boundary material, so that particle sizes were known. Neither vegetation nor bed-form resistance can be isolated. Hicks & Mason based their approach on Barnes (1967), who provided values of Manning's resistance coefficient $n = 1/k_{\mathrm{St}}$ for a single flow at each of 50 separate river sites in the United States of America, of which boundary material details were given for 14. We also include those results here.

From both catalogues we took the values of D_{84}, the boundary particle size for which 84% of the material was finer, and from the values of A/P, calculated the relative roughness $\varepsilon_{84} = D_{84}/(A/P)$, and used the measured values of Chézy's C to calculate values of λ. Results are shown on the next page. We have plotted them for a parameter $\Lambda = \lambda/8$, which it will be more convenient for us to use later.


Barnes (1967)
Hicks & Mason (1991) – stable bed
Hicks & Mason (1991) – moving bed
Eq. (3.18)
Aberle & Smart (2003, eqn 9)
Pagliara et al. (2008, eqn 15), $S = 0$, $\Gamma = 0$
Ditto, but $\Gamma = 0.2$
Yen (2003), Eq. (3.19)
These lectures, eqn (3.20)
$\delta = 2.0$ – moving
$\delta = 1.0$ – fully exposed grains
$\delta = 0.5$ – irregular
$\delta = 0.0$ – co-planar bed
$\Lambda = \frac{\lambda}{8}$
Relative roughness ε_{84}
10^0
10^{-1}
10^{-2}
10^{-3}

- Many of the results from each study are for large bed material $\varepsilon_{84} > 0.1$, possibly a reflection of the hilly and mountainous nature of New Zealand and Pacific North-West of the United States of America. And Austria.
- There is a wide scatter of results. But not all that very wide if we consider that the streams range from large slow-moving rivers with extremely small grains to mountain torrents with $30\,\mathrm{cm}$ boulders. Most of the results, unless the grains are moving, fall between $\Lambda \approx 0.005$ and 0.02.
- There is, as we have seen, slow variation with relative roughness: am increase in ε by a factor of 10 leads to an increase in Λ of about 2, as we have already seen.
- The points, we believe, have a tendency to group around three of the curves shown and the rest to be bounded below by the fourth (upper) curve shown. The curves have been drawn using the expression, found by trial and error:

$$\Lambda = \frac{0.06 + 0.06\,\delta}{\left(1.0 - 0.6\,\delta - \ln \varepsilon_{84}\right)^2}, \tag{3.18}$$

  with values of $\delta = 0$, 0.5, 1, and 2. The parameter δ is an arbitrary one that we use to identify the state of the particles making up the bed. This will now be explained.
- The first grouping of points comprises those around the bottom curve. We hypothesise that these points, having the lowest resistance, are those forming beds where the particles are relatively co-planar such that the bed is *armoured*. We assigned $\delta = 0$ to this state, and used that in equation (3.18) to plot the curve.
- The next grouping of points is around the second curve from the bottom, which can be seen

to substantially coincide with the a curve corresponding to exposed boulders on top of the bed occupying 0.2 of the surface area. Of course, with a number of these grains thus exposed, the resistance is greater. We assigned a value of $\delta = 0.5$ to this intermediate state.

- Substituting $\delta = 1$ in equation (3.18) gives the third curve on the figure, passing through what we believe is the third grouping of particles. This is probably the state for the maximum resistance for a stable bed corresponding to exposed grains occupying something like 50% of the surface area. Any more such grains will cause shielding of particles, the bed will start to resemble the co-planar case, and resistance will actually be reduced.
- Further evidence supporting our assertions is obtained from the expression proposed by Yen (2002, eqn 19), who considered results from a number of experimental studies using fixed impermeable beds. We used his formula, converted to $\Lambda = \lambda/8$, used an infinite Reynolds number, and converted his equivalent sand roughness $\varepsilon_s = 2\varepsilon_{84}$. It can be seen that the curve passes (left to right) from our curve $\delta = 1$ for small particles, which are unlikely to have the tops levelled so that particles are exposed, to the second curve for larger particles, more likely to be levelled in the laboratory experiments, with $\delta = 0$. Yen obtained the approximation for λ:

$$\lambda = \frac{1}{4.}\left(-\log_{10}\left(\frac{1}{12.}\frac{k_\mathrm{s}}{A/P} + \frac{1.95}{\mathsf{R}^{0.9}}\right)\right)^{-2}, \tag{3.19}$$

  where R is the channel Reynolds number $\mathsf{R} = (Q/P)/\nu$, in which ν is the kinematic viscosity.
- The logarithmic formula we obtained quite simply, equation (3.5), leads to, if we use $\varepsilon_s = 2\varepsilon_{84}$,

$$\Lambda = (6 - 2.5\ln(2\varepsilon_{84}))^{-2}, \tag{3.20}$$

quite similar to the results from Yen's formula.

- For points above the third curve almost all experimental points had shear stresses greater than the critical one necessary for movement. If particles move, not only do many particles protrude above others, increasing the stress, but there is the additional force required to maintain the sliding and rolling and jostling of all the particles. Hence, the resistance is greater. And, if there is a need to maintain particles in suspension, that will contribute also to resistance. We have shown the fourth curve as drawn for $\delta = 2$.

Hopefully the figure and approximating curves have given us an idea of the magnitudes and variation of the quantities, and maybe even some results for use in practice.