River Engineering

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1. Introduction

1.1 The nature of what we will and will not do – illuminated by some aphorisms and some people

“There is nothing so practical as a good theory” – stated in 1951 by Kurt Lewin (D-USA, 1890-1947): this is essentially the guiding principle behind these lectures. We want to solve practical problems, both in professional practice and research, and to do this it is a big help to have a theoretical understanding and a framework.

“The purpose of computing is insight, not numbers” – the motto of a 1973 book on numerical methods for practical use by the mathematician Richard Hamming (USA, 1915-1998). That statement has excited the opinions of many people (search any three of the words in the Internet!). However, numbers *are* often important in engineering, whether for design, control, or other aspects of the practical world. A characteristic of many engineers, however, is that they are often blinded by the numbers, and do not seek the physical understanding that can be a valuable addition to the numbers. In this course we are not going to deal with many numbers. Instead we will deal with the methods by which numbers could be obtained in practice, and will try to obtain insight into those methods. Hence we might paraphrase simply: "The purpose of this course is insight into the behaviour of rivers; with that insight, numbers can be often be obtained more simply and reliably".

“It is EXACT, Jane” – a story told to the lecturer by a botanist colleague. The most important river in Australia is the Murray River, $2\,375\,\mathrm{km}$ (Danube $2\,850\,\mathrm{km}$), maximum recorded flow $3\,950\ \mathrm{m^3s^{-1}}$ (Danube at Iron Gate Dam: $15\,400\ \mathrm{m^3s^{-1}}$). It has many tributaries, flow measurement in the system is approximate and intermittent, there is huge biological and fluvial diversity and irregularity. My colleague, non-numerical by training, had just seen the demonstration by an hydraulic engineer of a one-dimensional computational model of the river. She asked: “Just how accurate is your model?”. The engineer replied intensely: "It is EXACT, Jane".

Nothing in these lectures will be exact. We are talking about the *modelling* of complex physical systems.

A further example of the sort of thinking that we would like to avoid: in the area of palaeo-hydraulics, some Australian researchers made a survey to obtain the heights of floods at individual trees. This showed that the palaeo-flood reached a maximum height on the River Murray at a certain position of $18.01\ \mathrm{m}$ (*sic*), Having measured the cross-section of the river, they applied the Gauckler-Manning-Strickler Equation to determine the discharge of the prehistoric flood, stated to be $7\,686\ \mathrm{m^3s^{-1}}$ ...

William of Ockham (England, c1288-c1348): Ockham’s razor is the principle that can be popularly stated as “when you have two competing theories that make similar predictions, the simpler one is the better”. The term razor refers to the act of shaving away unnecessary assumptions to get to the simplest explanation, attributed to 14th-century English logician and Franciscan friar, William of Ockham. The explanation of any phenomenon should make as few assumptions as possible, eliminating those that make no difference in the observable predictions of the explanatory hypothesis or theory. When competing hypotheses are equal in other respects, the principle recommends selection of the hypothesis that introduces the fewest assumptions and postulates the fewest entities *while still sufficiently answering the question*. That is, we should not *over*-simplify our approach.

In general, model complexity involves a trade-off between simplicity and accuracy of the model. Occam's Razor is particularly relevant to modelling. While added complexity usually improves the fit of a model, it can make the model difficult to understand and work with.

The principle has inspired numerous expressions including "parsimony of postulates", the "principle of simplicity", the "KISS principle" (Keep It Simple, Stupid). Other common restatements are:

Leonardo da Vinci (I, 1452–1519, world's most famous hydraulician, also an artist): his variant short-circuits the need for sophistication by equating it to simplicity "Simplicity is the ultimate sophistication".

Wolfang A. Mozart (A, 1756–1791): "Gewaltig viel Noten, lieber Mozart", soll Kaiser Josef II. über die erste der großen Wiener Opern, die "Entführung", gesagt haben, und Mozart antwortete: "Gerade so viel, Eure Majestät, als nötig ist." (Emperor Joseph II said about the first of the great Vienna operas, "Die Entführung aus dem Serail", "Far too many notes, dear Mozart", to which Mozart replied "Your Majesty, there are just as many notes as are necessary"). The truthfulness of the story is questioned – Josef was more sophisticated than that ...

Albert Einstein (D-USA,1879-1955): "Make everything as simple as possible, but not simpler." This is a better and shorter statement than Ockham!

Karl Popper (A-UK, 1902-1994) argued that we prefer simpler theories to more complex ones "because their empirical content is greater; and because they are better testable". In other words, a simple theory applies to more cases than a more complex one, and is thus more easily falsifiable. Popper coined the term critical rationalism to describe his philosophy. The term indicates his rejection of classical empiricism, and of the classical observationalist-inductivist account of science that had grown out of it. Logically, no number of positive outcomes at the level of experimental testing can confirm a scientific theory (Hume's "Problem of Induction"), but a single counterexample is logically decisive: it shows the theory, from which the implication is derived, to be false. For example, consider the inference that "all swans we have seen are white, and therefore all swans are white", before the discovery of black swans in Australia. Popper's account of the logical asymmetry between verification and falsifiability lies at the heart of his philosophy of science. It also inspired him to take falsifiability as his criterion of demarcation between what is and is not genuinely scientific: a theory should be considered scientific if and only if it is falsifiable. This led him to attack the claims of both psychoanalysis and contemporary Marxism to scientific status, on the basis that the theories enshrined by them are not falsifiable.

Thomas Kuhn (USA, 1922-1996): In *The Structure of Scientific Revolutions* argued that scientists work in a series of paradigms, and found little evidence of scientists actually following a falsificationist methodology. Kuhn argued that as science progresses, explanations tend to become more complex before a sudden *paradigm shift* offers radical simplification. For example Newton's classical mechanics is an approximated model of the real world. Still, it is quite sufficient for most ordinary-life situations. Popper's student Imre Lakatos (H-UK, 1922-1974) attempted to reconcile Kuhn's work with falsificationism by arguing that science progresses by the falsification of research programs rather than the more specific universal statements of naive falsificationism.

Another of Popper's students Paul Feyerabend (A-USA, 1924-1994) ultimately rejected any prescriptive methodology, and argued that the only universal method characterising scientific progress was "anything goes!"

1.2 Summary

- We will use theory, but we will try to keep things simple, rather simpler than is often the case in this field, especially in numerical methods.
- Often our knowledge of physical quantities is limited, and approximation is justified.
- We will recognise that we are modelling.
- An approximate model can often reveal to us more about the problem.
- It might be thought that the lectures show a certain amount of inconsistency – in occasional places the lecturer will develop a more generalised and "accurate" model, paradoxically to emphasise that we are just modelling.
- We will attempt to obtain insight and understanding – and a sense of criticality.

1.3 Types of channel flow to be studied

An important part of this course will be the study of different types of channel flow.

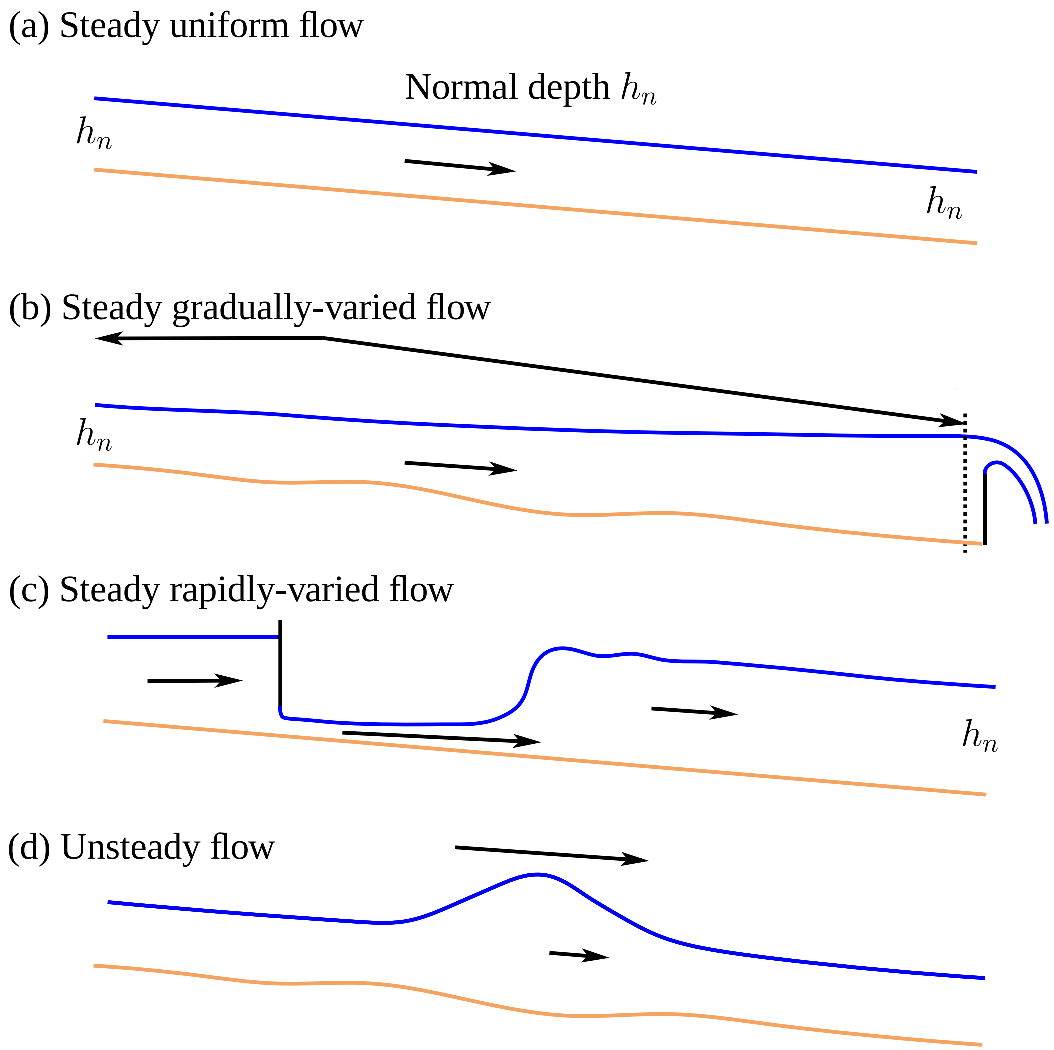


Figure 1.1: Different types of flow in an open channel

Case (a) – Steady uniform flow:

Steady flow is where there is no change with time, $\partial/\partial t \equiv 0$. Distant from control structures, gravity and resistance are in balance, and if the cross-section is constant, the flow is uniform, $\partial/\partial x \equiv 0$. This is the simplest model, and often is used as the basis and a first approximation for others.

Case (b) – Steady gradually-varied flow:

Where all inputs are steady but where channel properties may vary and/or a control may be introduced which imposes a water level at a certain point. The height of the surface varies along the channel. For this case we will study the governing differential equation that describes how conditions vary along the waterway, and we will obtain an approximate mathematical solution to solve general problems approximately.

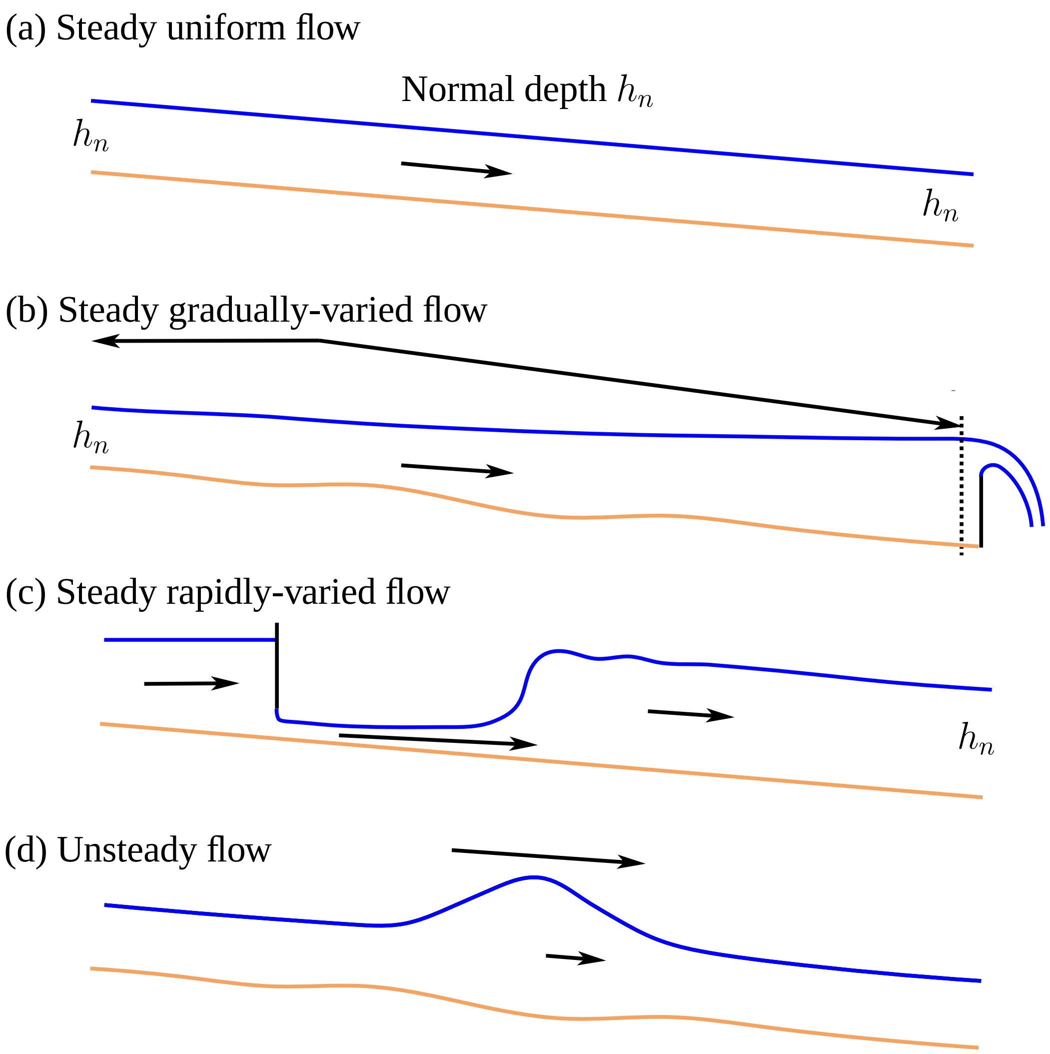


Figure 1.2: Different types of flow in an open channel

Case (c) – Steady rapidly-varied flow:

Figure (c) shows three separate gradually-varied flow regions separated by two rapidly-varied regions: (1) flow under a sluice gate and (2) a hydraulic jump. The basic hydraulic approximation that variation is gradual breaks down in those regions. We can analyse them by considering energy or momentum conservation locally. In this course we will not be considering these – earlier courses at TUW have.

Case (d) – Unsteady flow:

Here conditions vary with time and position as a flood wave traverses the waterway. We will consider flood wave motion at some length.

2. Conservation of mass, momentum and energy

2.1 Some possibly-surprising results

Effects of turbulence on dynamics

Where the fluid flow fluctuates in time, apparently randomly, about some mean condition, *e.g.* the flow of wind, water in pipes, water in a river. In practice we tend to work with mean flow properties, however in this course we will adopt empirical means of incorporating some of the effects of turbulence. Consider the x component of velocity at u a point written as a sum of the mean ($\bar{u}$) and fluctuating (u') components:

$$u = \bar{u} + u'.$$

By definition, the mean of the fluctuations, which we write as $\overline{u'}$, is

$$\overline{u'} = \frac{1}{T}\int_0^T u'\,dt = 0, \tag{2.1}$$

where T is some time period much longer than the fluctuations.

Now let us compute the mean value of the *square* of the velocity, such as we might find in computing the mean pressure on an object in the flow:

$$
\begin{aligned}
\overline{u^2} &= \overline{(\bar{u}+u')^2} = \overline{\bar{u}^2 + 2\bar{u}u' + u'^2}, \text{ expanding,}\\
&= \overline{\bar{u}^2} + \overline{2\bar{u}\,u'} + \overline{u'^2}, \text{ considering each term in turn,}\\
&= \bar{u}^2 + 2\bar{u}\,\overline{u'} + \overline{u'^2}, \text{ but, as } \overline{u'} = 0 \text{ from(2.1),}\\
&= \bar{u}^2 + \overline{u'^2}. \qquad (2.2)
\end{aligned}
$$

hence we see that the mean of the square of the fluctuating velocity is not equal to the square of the mean of the fluctuating velocity, but that there is also a component $\overline{u'^2}$, the mean of the fluctuating components. We will need to incorporate this.

Pressure in open channel flow – effects of resistance on flows over steep slopes

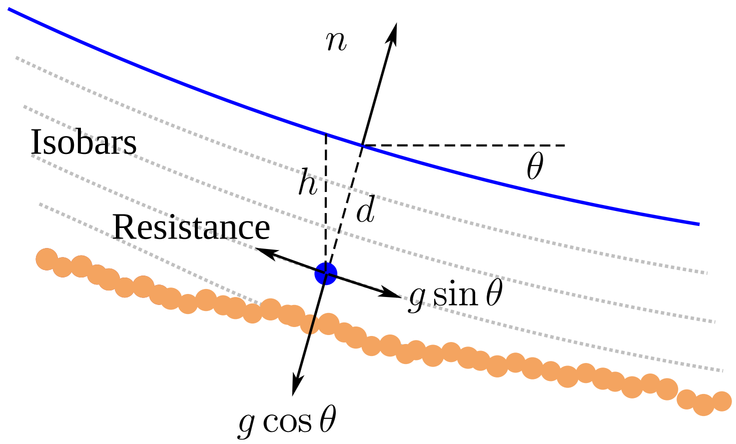


Figure 2.1: Channel flow showing isobars and forces per unit mass on a fluid particle

An almost-universal assumption in river engineering is that the pressure distribution is *hydrostatic*, that of water which is not moving, such that pressure p at a point is given by the height of water above, $p = \rho g h$, where ρ is fluid density ($\approx 1000\,\mathrm{kg\,m^{-3}}$ for fresh water), $g \approx 9.8\,\mathrm{ms^{-2}}$ is gravitational acceleration, and h is the vertical height of the surface above the point. This is not necessarily the case in flowing water, and needs to be known for cases such as spillways or block ramps, which are steep.

Consider figure 2.1 showing an open channel flow with forces per unit mass acting on a particle. The figure is drawn, showing that in general, the depth is not constant, and the bed is not parallel to the free surface. It is an isobar, a line of constant pressure, $p = 0$. In the flow, other isobars will generally be parallel to this, while the channel bed is not necessarily an isobar. We consider the vector Euler equation for the motion of a fluid particle

$$\text{Acceleration} = -\frac{1}{\rho} \times \text{Pressure gradient} + \text{Body forces per unit mass}$$

In a direction parallel to the free surface, the pressure is constant and there is no pressure gradient. The acceleration of the particle will be given by the difference between the component of gravity $g \sin\theta$ and the resistance force per unit mass. We usually do not know the details of that, so there is little that we can say. Now considering a direction perpendicular to that, given by the co-ordinate n on the figure, there is very little acceleration, so we assume it to be zero, and so we obtain the result

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} - g\cos\theta.$$

Now integrating this with respect to n between a general point, such as $n = -d$ at the particle shown, and $n = 0$ on the surface where $p = 0$ we obtain

$$p = \rho g \cos\theta \times d.$$

It is much more convenient to measure all elevations vertically, and so we use h, such that $d = h\cos\theta$, and we obtain the general expression for pressure

$$p = \rho g h \cos^2\theta.$$

This result (for steady uniform flow) however it and its implications for general flows seems to have been forgotten by many. While that is nice to know, we do not need it now, because, like in the uniform flow section, *in almost all open channels the slope is small enough* such that $\cos^2\theta \approx 1$, and we can use the *hydrostatic approximation*, obtained from a static fluid, where the surface is horizontal,

$$p = \rho g h. \tag{2.3}$$

Substituting $h = \eta - z$, where η is the free surface elevation and z is the elevation of an arbitrary point in the fluid,

$$p = \rho g \left(\eta - z\right). \tag{2.4}$$

From this we have *at a specific vertical cross-section*,

$$p + \rho g z = \rho g \eta, \tag{2.5}$$

so that anywhere on a vertical section $p + \rho g z$ is constant, given by the free surface elevation.

2.2 Flux of volume, mass, momentum and energy across a surface

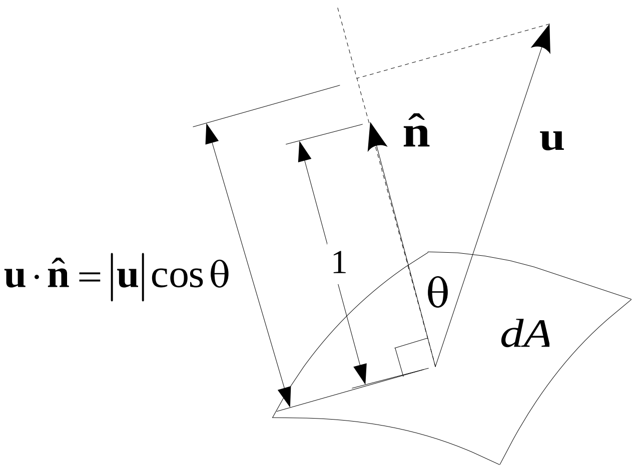


Figure 2.2: Element of surface dA with local velocity vector $\mathbf{u}$ showing how the velocity component normal to the surface is $\mathbf{u}\cdot\hat{\mathbf{n}}$.

It is necessary for us to be able to calculate the total quantity of fluid and integral quantities such as mass, momentum, and energy flowing across an arbitrary surface in space, which we will then apply to the rather more simple case of control surfaces. Consider an element of an arbitrary surface shown in Figure 2.2 through which fluid flows at velocity $\mathbf{u}$. The velocity component perpendicular to the surface is $|\mathbf{u}|\cos\theta = \mathbf{u}\cdot\hat{\mathbf{n}}$. In a time dt the volume of fluid which passes across the surface is $\mathbf{u}\cdot\hat{\mathbf{n}}\,dt\,dA$, or, the *rate* of volume transport is $\mathbf{u}\cdot\hat{\mathbf{n}}\,dA$. Other quantities easily follow from this: multiplying by density ρ gives the rate of *mass* transport, multiplying by velocity $\mathbf{u}$ gives the rate of transport of *momentum* due to fluid inertia (there is another contribution due to pressure for total momentum), and if e is the energy per unit mass, multiplying by e gives the rate of energy transport across the element. By integrating over the whole surface A, not necessarily closed, gives the transport of each of the quantities, so that we can write

$$\text{Rate of}\left\{\begin{array}{c}\text{volume}\\ \text{mass}\\ \text{inertial momentum}\\ \text{energy}\end{array}\right\}\text{transport across surface } A = \int_A \begin{bmatrix}1\\ \rho\\ \rho\mathbf{u}\\ \rho e\end{bmatrix}\mathbf{u}\cdot\hat{\mathbf{n}}\,dA. \tag{2.6}$$

Note that as $\mathbf{u}\cdot\hat{\mathbf{n}}$ is a scalar there is no problem in multiplying this simply by either a vector or a scalar. In hydraulic practice such integrals are usually evaluated more easily. For example, across a pipe or channel which is locally straight, to calculate the rates of transport we choose a surface perpendicular to the flow.

Flux across solid boundaries: There can be no velocity component normal to a solid boundary, such that every solid boundary satisfies the boundary condition $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ and so from equation (2.6) there can be no volume, mass, momentum, or energy transfer across solid boundaries.

2.3 Mass and volume flux

Now we consider successively mass, momentum, and energy, and the effects of turbulence will be incorporated. In most hydraulics the density of water varies very little and so ρ can be assumed to be a constant and we can often just consider volume transport..

If the flow through each flow boundary cuts the boundary at right angles, we can write the velocity as $\mathbf{u} = \pm u\,\hat{\mathbf{n}}$, such that $\mathbf{u}\cdot\hat{\mathbf{n}} = \pm u$, where the plus/minus sign is taken when the flow leaves/enters the control volume. Then across any section of area A we have the contribution $\int_A \mathbf{u}\cdot\hat{\mathbf{n}}\,dA = \pm\int_A u\,dA$, which is $\pm Q$, the *volume flow rate* or *discharge* across the section. Sometimes it is convenient to express this in terms of U, the mean velocity, such that

$$\text{Rate of volume transport across surface } = \int_A u\,dA = Q = UA.$$

2.4 Momentum flux

Formulation

Newton's second law states that the net rate of change of momentum is equal to the force applied. In equation (2.6) we obtained

$$\text{Momentum transport across a surface} = \int_A \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA. \tag{2.7}$$

There are two main contributions to the force applied. One is due to surface forces, the pressure p over the surface. On an element of the control surface with area dA and outwardly-directed normal $\hat{\mathbf{n}}$ the pressure force on the fluid in the control surface has a magnitude of $p\, dA$ (simply pressure multiplied by area) and a direction $-\hat{\mathbf{n}}$, because the pressure acts normal to the surface and the direction of the force on the fluid is directed inwards to the control volume.

The other contribution is the sum of all the body forces, which will be usually due to gravity. We let these be denoted by $\mathbf{F}_{\text{body}}$. Equating the rate of change of momentum to the applied forces and taking the pressure force over to the other side we obtain the *integral momentum theorem* for steady flow

$$\int_{\text{CS}} \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA + \int_{\text{CS}} p\, \hat{\mathbf{n}}\, dA = \mathbf{F}_{\text{body}}. \tag{2.8}$$

This form enables us to solve a number of problems yielding the force of fluid on objects and structures. Now the integrals in equation (2.8) will be separated into those over surfaces through which fluid flows and solid surfaces:

$$\sum_{\text{Fluid surfaces}} \left(\int_A \left(\rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}} + p\, \hat{\mathbf{n}}\right)\, dA \right) + \sum_{\text{Solid surfaces}} \left(\int_A \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA + \int_A p\, \hat{\mathbf{n}}\, dA \right) = \mathbf{F}_{\text{body}}.$$

However, as $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ on all solid surfaces, there are no contributions. Also on the solid surfaces, unless we know all details of the flow field, we do not know the pressure p. However the sum of all those contributions is the total force $\mathbf{P}$ of the fluid on the surrounding structure. Hence we have the theorem in a more practical form for calculating the force on objects:

$$\text{Total force on solid surfaces} = \mathbf{P} = -\sum_{\text{Fluid surfaces}} \left(\underbrace{\int_A \left(\rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}} + p\, \hat{\mathbf{n}}\right)\, dA}_{\text{Momentum flux}} \right) + \mathbf{F}_{\text{body}}. \tag{2.9}$$

Note the use of the term *momentum flux* for the integral shown – it includes contributions from the inertial momentum flux and from pressure.

Inertial momentum flux

Here we evaluate the integral describing the transport of momentum by fluid velocity. In many situations *we can choose the control surface such that on each part where the fluid crosses it, the local surface element is planar, and the velocity crosses it at right angles.* We write

$\mathbf{u} = \pm u\hat{\mathbf{n}}$,where u is the fluid speed, whose magnitude might vary over that part of the control surface through which it passes, but whose direction is perpendicular either in the direction of the unit normal or opposite to it. Hence

$$\int_A \rho \mathbf{u}\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dA = \int_A \rho\left(\pm u\hat{\mathbf{n}}\right)\,(\pm u)\;dA = +\hat{\mathbf{n}}\rho\int_A u^2\,dA, \tag{2.10}$$

where, as $(\pm)\times(\pm)$ is always positive, the surprising result has been obtained that the contribution to momentum flux is always in the direction of the outwardly-directed normal, whether the fluid is entering or leaving the control volume. Also we have assumed that the area A is planar such that $\hat{\mathbf{n}}$ is constant, so that we have been able to take the $\hat{\mathbf{n}}$ outside the integral sign.

Approximation of the integral allowing for turbulence and boundary layers

Although it has not been written explicitly, it is understood that equation (2.10) is evaluated in a time mean sense. In equation (2.2) we saw that if a flow is turbulent, then $\overline{u^2} = \bar{u}^2 + \overline{u'^2}$, such that the time mean of the square of the velocity is greater than the square of the mean velocity. In this way, we should include the effects of turbulence in the inertial momentum flux by writing the integral on the right of equation (2.10) as

$$\int_A u^2\,dA = \int_A \left(\bar{u}^2 + \overline{u'^2}\right)\,dA. \tag{2.11}$$

Usually we do not know the nature of the turbulence structure, or even the actual velocity distribution across the flow, so that we approximate this in a simple sense by recognising that the time mean velocity at any point and the magnitude of the turbulent fluctuations are all of the scale of the mean velocity in the flow in a time and spatial mean sense, $\bar{U} = Q/A$, such that we write for the integral in space of the time mean of the squared velocities:

$$\int_A u^2 \, dA = \int_A \left(\bar{u}^2 + \overline{u'^2}\right) \, dA \approx \beta \bar{U}^2 A = \beta \left(\frac{Q}{A}\right)^2 A = \beta \frac{Q^2}{A}. \tag{2.12}$$

The coefficient β is called a Boussinesq coefficient, after the French engineer who introduced it to allow for the spatial variation of velocity. Allowing for the effects of time variation, turbulence, has been a recent addition.

The coefficients β have typical values of 1.05 to something like 1.5 or more in channels of irregular cross-section. Almost all textbooks introduce this quantity for open channel flow (without turbulence) but then assume it is equal to 1. Surprisingly, for pipe flow it seems not to have been used at all. In this course we consider it important and will include it.

2.5 Energy flux and conservation

The energy equation in integral form can be written for a control volume CV bounded by a control surface CS, where there is no heat added or work done on the fluid inside the control volume:

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho e \,\mathrm{d}V}_{\text{Rate of change of energy inside CV}} + \underbrace{\int_{\mathrm{CS}} \rho e\, \mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S}_{\text{Flux of energy across CS}} + \underbrace{\int_{\mathrm{CS}} p\, \mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S}_{\text{Rate of work done by pressure}} = 0, \tag{2.13}$$

where t is time, ρ is density, $\mathrm{d}V$ is an element of volume, ρe is the internal energy per unit volume of fluid, ignoring nuclear, electrical, magnetic, surface tension, and intrinsic energy due to molecular spacing, leaving the sum of the potential and kinetic energies such that the internal energy *per unit mass* is

$$e = gz + \tfrac{1}{2}\left(u^2 + v^2 + w^2\right), \tag{2.14}$$

where the velocity vector $\mathbf{u} = (u, v, w)$ in a cartesian coordinate system, the co-ordinate z is vertically upwards, p is pressure, and dS is the elemental area of the control surface.

Here *steady* flow is considered, at least in a time-mean sense, so that the first term in equation (2.13) is zero. The equation becomes, after dividing by density ρ and taking the long-term time mean, denoted by an overbar:

$$\overline{\int_{\mathrm{CS}} \left(p + \rho g z + \frac{\rho}{2}\left(u^2 + v^2 + w^2\right)\right) \mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S} = 0. \tag{2.15}$$

The energy flux over a section of area A is then $\pm\dot{E}$, depending on whether the flow is leaving/entering the control volume, where

$$\dot{E} = \overline{\int_A \left(p + \rho g z + \frac{\rho}{2}\left(u^2 + v^2 + w^2\right)\right) u\,\mathrm{d}A},$$

Now we consider the individual contributions to this integral.

The pressure distribution in an open channel (river, canal, drain, *etc.*) is usually very close to hydrostatic (streamlines have a very small slope), so that $p/\rho + gz$ is constant over a section. Hence we can take the first two terms of the integral outside the integral sign and use the result that $\int_A u\ dA = Q$ to give

$$\dot{E} = (p + \rho g z)\,Q + \frac{\rho}{2}\overline{\int_A \left(u^2 + v^2 + w^2\right)u\,\mathrm{d}A}, \tag{2.16}$$

where p and z outside the integral are the pressure and elevation at any point on a particular section. *Our treatment is not entirely satisfactory – we have ignored turbulent contributions in the nonlinear term* pQ*, as we almost never know anything about turbulent pressure fluctuations. It really should be written* $\bar{p}\bar{Q} + \overline{p'Q'}$.

Now, in the same spirit as for momentum, when we introduced a coefficient β to allow for a non-constant velocity distribution, we introduce a coefficient α such that it allows for the variation of the kinetic energy term across the section and in time and we write

$$\overline{\int_A \left(u^2+v^2+w^2\right)\, u\,\mathrm{d}A} = \alpha U^3 A = \alpha \frac{Q^3}{A^2}, \tag{2.17}$$

where $U = Q/A$ is the mean velocity. Obviously, α is defined by

$$\alpha = \frac{A^2}{Q^3} \overline{\int_A \left(u^2+v^2+w^2\right)\, u\,\mathrm{d}A} = \frac{1}{U^3 A} \overline{\int_A \left(u^2+v^2+w^2\right)\, u\,\mathrm{d}A} \tag{2.18}$$

which is "the kinetic energy term divided by the value of the term if the flow had a single velocity component constant in time and space". Such a coefficient was introduced by Coriolis, a French military engineer who neglected the other two velocity components and turbulence. Strangely, textbooks even today just use his original definition and take the first component under the integral sign, neglect turbulence, and write

$$\alpha = \frac{A^2}{Q^3} \int_A u^3\,\mathrm{d}A = \frac{1}{U^3 A} \int_A u^3\,\mathrm{d}A, \tag{2.19}$$

where α is the *Coriolis* coefficient, for which a typical value is $\alpha \approx 1.05 – 1.5$. With equation (2.16) and the definition of α from equation (2.18):

$$\text{Rate of energy transport across a section} = \dot{E} = (p + \rho g z)\, Q + \alpha \frac{\rho}{2} \frac{Q^3}{A^2}.$$

This can be written in a factorised form

$$\dot{E} = \underbrace{\rho Q}_{\text{Mass rate of flow}} \times \left(\underbrace{\frac{p}{\rho} + gz + \frac{\alpha}{2}\frac{Q^2}{A^2}}_{\text{Energy per unit mass}} \right).$$

The energy per unit mass has units of Joules per kilogram, $\mathrm{J\,kg^{-1}}$. It is common in civil and environmental engineering problems to factor out gravitational acceleration and to write

$$\dot{E} = \rho g Q \times \left(\underbrace{\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}}_{\text{Mean total head of the flow}} \right), \tag{2.20}$$

where the quantity in the brackets has units of length, corresponding to elevation, and is termed the *Mean total head of the flow* H, the mean energy per unit mass divided by g. This form is more convenient, because often in hydraulics elevations are more important and useful than actual energies. For example, the height of a reservoir surface, or the height of a levee bank on a river might be known and govern design calculations.

Energy conservation equation for a control surface

Now, evaluating the integral energy equation (2.15) using these approximations over each of the parts of the control surface through which fluid flows, numbered 1, 2, 3, … gives an energy conservation equation

$$\pm \dot{E}_1 \pm \dot{E}_2 \pm \dot{E}_3 \pm \ldots = \pm \rho_1 g Q_1 H_1 \pm \rho_2 g Q_2 H_2 \pm \rho_3 g Q_3 H_3 \pm \ldots = 0, \tag{2.21}$$

where the positive/negative sign is taken for fluid leaving/entering the control volume. In almost all hydraulics problems the density can be assumed to be constant, and so dividing through by the common density and gravity, we have

$$\pm Q_1 H_1 \pm Q_2 H_2 \pm Q_3 H_3 \pm \ldots = 0. \tag{2.22}$$

If we consider a control surface enclosing a junction of two pipes or channels, it seems we must include the volume rates of flow as shown.

We will consider a length of pipe or channel in which water enters at only one point 1 and leaves at another 2, so that $-Q_1 H_1 + Q_2 H_2 = 0$, but by volume conservation $Q_1 = Q_2$ so we obtain our familiar result $H_1 = H_2$ or

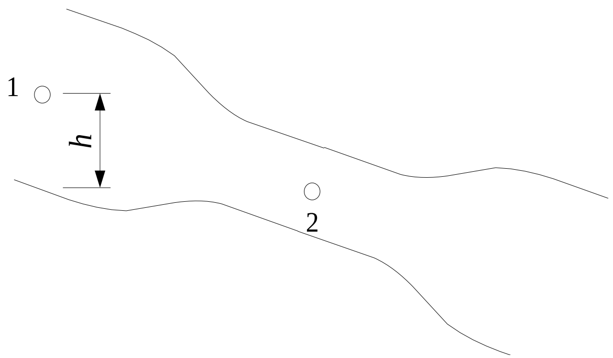
$$\left(\frac{p}{\rho g} + z + \frac{\alpha}{2g} \frac{Q^2}{A^2} \right)_1 = \left(\frac{p}{\rho g} + z + \frac{\alpha}{2g} \frac{Q^2}{A^2} \right)_2 . \tag{2.23}$$

However, to give more accurate practical results, an empirical allowance is usually made for energy losses, and in most applications the equation is used in the form

$$H_1 = H_2 + \Delta H,$$

where ΔH is a head loss. In many situations it is given as an empirical coefficient times the kinetic head.

Example 1 Application of the integral form of the energy equation – The Venturi meter, where we will see that the fact that a coefficient which is necessary to give correct results is not due to head loss, but just due to $\alpha \neq 1$.



Consider the gradual constriction in the pipe shown, where there are pressure tapping points in the horizontal side of the pipe at point 1 before the constriction and point 2 in the constriction. Using a control volume crossing the pipe at 1 and 2 and applying equation (2.23), the integral energy theorem for a single inlet and outlet,

$$\frac{p_1}{\rho g} + z_1 + \frac{\alpha}{2g}\frac{Q^2}{A_1^2} = \frac{p_2}{\rho g} + z_2 + \frac{\alpha}{2g}\frac{Q^2}{A_2^2}.$$

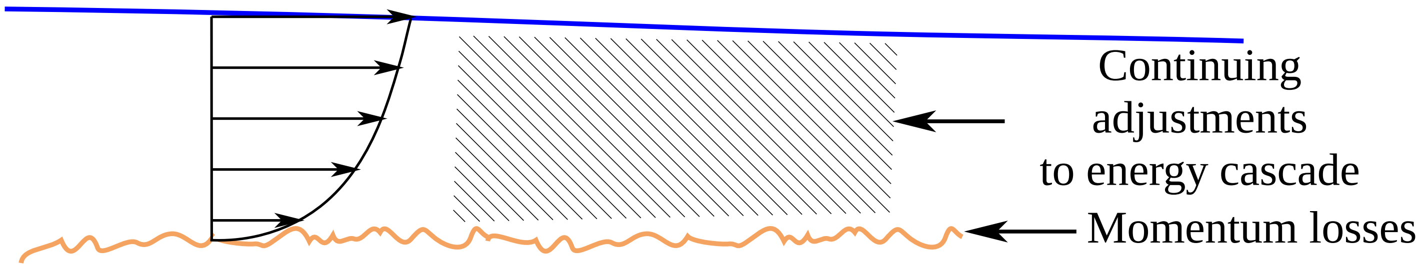
Solving for Q and using $z_1 = z_2 + h$ gives

$$Q = \sqrt{\frac{2}{\alpha}\frac{\frac{p_1 - p_2}{\rho} + gh}{\frac{1}{A_2^2} - \frac{1}{A_1^2}}}.$$

Hence we see that by measuring the pressure change between 1 and 2 we can obtain an approximation to the discharge. It is interesting that other presentations introduce a coefficient C (< 1) in front to agree with experiment. In our presentation, by including the parameter α (> 1) we may have gone some way to quantifying that.

Comments and conclusions from our discussion of energy

- We have considered energy flux and conservation in *integral* form. We have *not* considered Bernoulli's equation, which is only valid along a streamline. It is easier to derive the energy principle in integral form, it is what we use in practice anyway, *and* a little-known fact, Bernoulli's equation is actually based on conservation of momentum. It does not look like it!
- In this course of River Engineering, we will rarely use energy conservation. Momentum conservation is simpler because momentum losses only occur at boundaries, such as the bed of a channel, or at an obstacle. Energy losses due to shear layers, turbulence, viscosity *etc* are diffused through the fluid, which are more complicated than the momentum losses.



- So, why did we consider energy at some length? Knowledge is never wasted; we never know when we might need energy; the treatment of momentum and energy was similar so it was a good time to do it; *and*
- We showed that in two cases, the strangely limited treatment of α and β without including turbulence or other velocity components *and* our demonstration of the nature of the formula for the Venturi meter, the principle: *no-one can be trusted, neither textbooks nor lecturers* ...

3. Resistance in river and other open channel flows

The resistance to the flow of a stream is probably the most important problem in river mechanics.

Page 27: We consider a simple theory based on force balance and some classical fluid mechanics experiments to obtain a flow formula for a wide channel.

Page 29: To obtain the equivalent formula for channels of any section we consider velocity distributions in real streams and develop an approximation giving a general flow formula.

Page 34: We consider an approximation to that formula and find that we have obtained the Gauckler-Manning-Strickler formula, including a theoretical prediction of Strickler's formula for the effect of boundary grain size.

Page 37: Comparison with a series of experiments validates the approach, giving an explicit flow formula for a variety of channel boundaries.

Page 39: The common problem of calculating the water depth for a given flow rate is considered. A computational method is developed and applied.

Page 42: For more general river problems, considering the nature of the bed particles and bed forms, vegetation, meandering, and possibly obstacles, it is better to use a formulation in which forces and the mechanics are clearer: the Chézy-Weisbach flow formula.

Page 44: A large number of stream-gauging results are considered and the values of the Weisbach resistance coefficient, its dependence on grain size, and on the state of the bed are obtained. Empirical formulae are considered.

3.1 A channel flow formula from theory and experiment

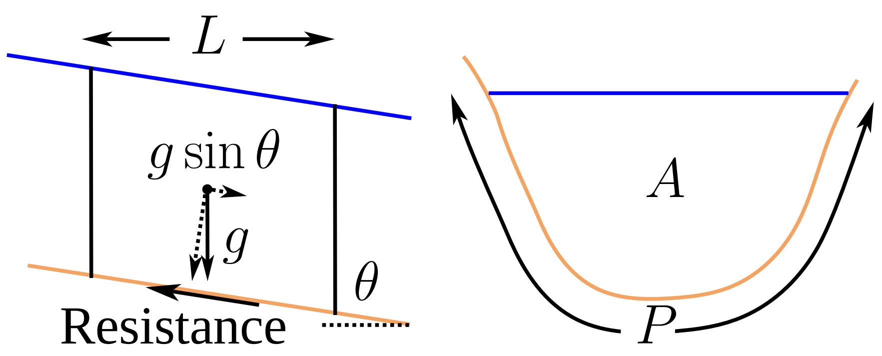


Figure 3.1: Uniform flow in a channel, showing resistance and gravity forces on a finite length, plus cross-section quantities

Here, as an example of the application of rational mechanics, a flow formula for steady uniform flow in channels is developed without using the empirical flow formulae of Gauckler-Manning-Strickler or Chézy-Weisbach.

Consider a horizontal length L of uniform channel flow, inclined at a small angle θ to the horizontal, with cross-sectional area A. The volume of the element is AL, the vertical gravitational force on the water is $\rho g AL$, where ρ is fluid density and g is gravitational acceleration. The component of this along the slope is $\rho g AL \sin\theta$. The resistance force along the slope, of length $L/\cos\theta$ is $\tau PL/\cos\theta$, where τ is the mean resistance shear stress, assumed uniformly distibuted around the wetted perimeter P around which it acts. Equating gravitational and resistance components gives $\tau PL/\cos\theta = \rho g AL \sin\theta$. To high accuracy for small θ, $\cos\theta \approx 1$ and $\sin\theta \approx \tan\theta = S$, the slope, giving

$$\frac{\tau}{\rho} = g\frac{A}{P}S. \tag{3.1}$$

Our problem is now to express shear stress τ in terms of flow quantities.

One of the most famous series of experiments in fluid mechanics was performed by Johann Nikuradse at Göttingen in the 1930s, who studied the flow of fluid over uniformly-rough sand grains. The fluid was actually air, and the sand grains were actually in circular pipes, but the results are still valid enough.

With those results, for a wide channel of depth h with sand grains of size k_{s}, the velocity distribution for fully rough flow (no effects of viscosity), the Prandtl-von Kármán *universal velocity distribution* can be written:

$$u = \frac{u_*}{\kappa} \ln \frac{30\,z}{k_{\mathrm{s}}},$$

in terms of the shear velocity $u_* = \sqrt{\tau/\rho}$, the von Kármán constant $\kappa \approx 0.4$, the vertical co-ordinate z, and where the factor of 30 is for closely-packed uniform sand grains. It varies somewhat with other types of boundary roughness. The mean velocity U is obtained by integrating between 0 and h, such that

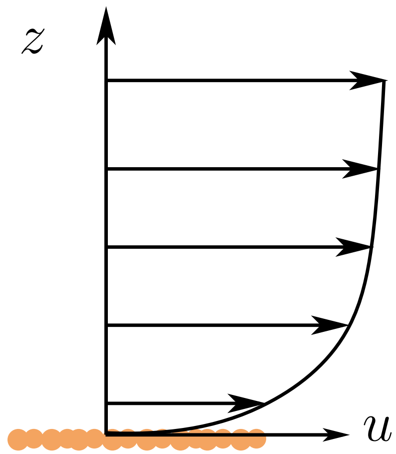


Figure 3.2: Idealised logarithmic velocity profile in turbulent flow over rough bed

$$U = \frac{u_*}{\kappa} \ln \frac{30/\mathrm{e}}{k_{\mathrm{s}}/h}.$$

where e is Euler's number $\exp(1) = 2.718\ldots$, obtaining the result in terms of *relative roughness* k_{s}/h. Now we replace $u_* = \sqrt{\tau/\rho}$ by the expression on the right of equation (3.1) to give

$$U = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln \frac{30/\mathrm{e}}{k_{\mathrm{s}}/h}\right), \tag{3.2}$$

We have obtained something possibly useful – a formula for the mean flow velocity in a wide channel of constant depth h, slope S, and relative roughness k_s/h. We have used simple mechanics plus empirical laboratory results. Surprisingly, the formula is explicit in terms of physical quantities – we have not had to assume a value like the Strickler k_St!

That was for a wide channel with an idealised logarithmic velocity distribution. In nature, for channels of any general cross-section there is the problem that the velocity has a maximum at a somewhere below the surface, and in general the isovels are something like Figure 3.3.

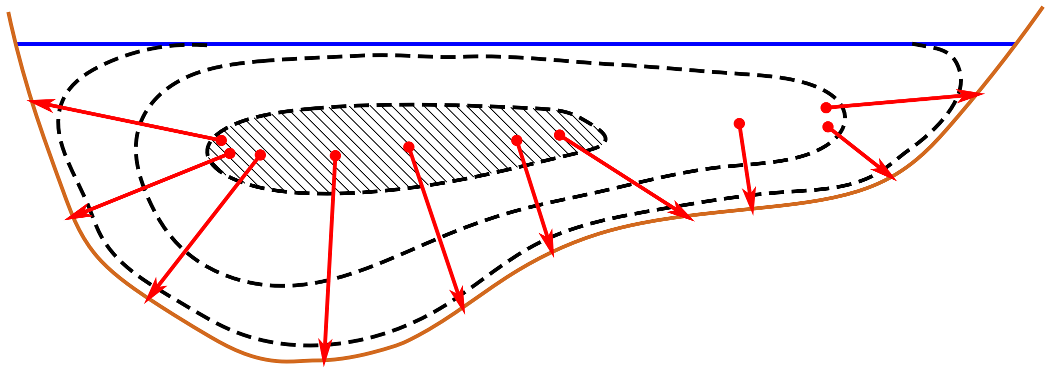


Figure 3.3: Cross-section of flow showing isovels and, for a number of points on the bed, where the fastest-likely fluid comes from and how far it travels, the effective length scale for resistance calculations.

To obtain a flow formula for channels of any cross-section, we hypothesise that the effective depth h for resistance calculations is the typical distance from points with the highest velocity to the nearest point on the bed, as suggested by the red arrows on the figure. If this model is correct, typical length scales as shown by the arrows are somewhat *smaller* than the overall mean depth of flow.

This is a highly approximate model, but at least it is in the spirit of modelling, that it is simple and transparent – and so far has not been obscured by mathematical detail.

Our problem is then, how can we simply approximate that distance?

We consider the experimental data for the vertical position of the locus of velocity maxima in *rectangular* channels from Yang, Tan & Lim (2004). They presented a formula for the height above the bed of the velocity maximum as a function of position across the channel. If the mean value of this is calculated by integration, a formula for the mean elevation of the velocity maximum $z_{\max}/h$ is obtained as a function of aspect ratio (channel width B divided by depth h).

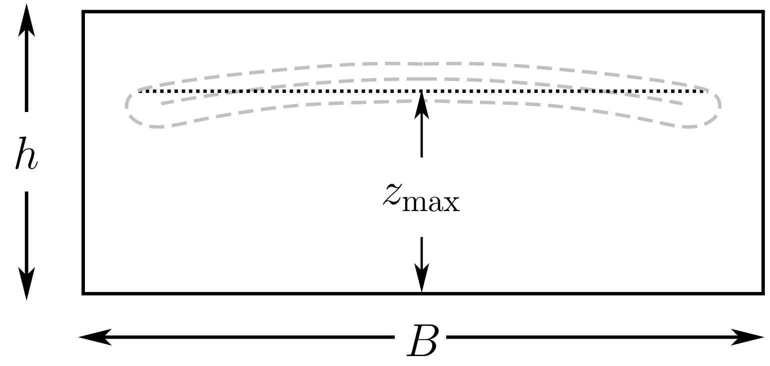


Figure 3.4: Experimental determination of velocity maxima in rectangular channel

The only other length scale we have above in equation (3.2) is the ratio of area to perimeter A/P. As $P > B$ it should be smaller than the mean depth A/B, so we will try it:

$$\frac{A/P}{h} = \frac{Bh}{(B+2h)h} = \frac{B/h}{B/h+2}. \tag{3.3}$$

Both this and the experimental formula for $z_{\max}/h$ are plotted in Figure 3.5. Remarkably and coincidentally, the two coincide closely over a wide range of aspect ratios! We cannot claim that this is a justification as strong as it looks, but in the absence of anything else, instead of h we will use A/P for channels of any cross-section.

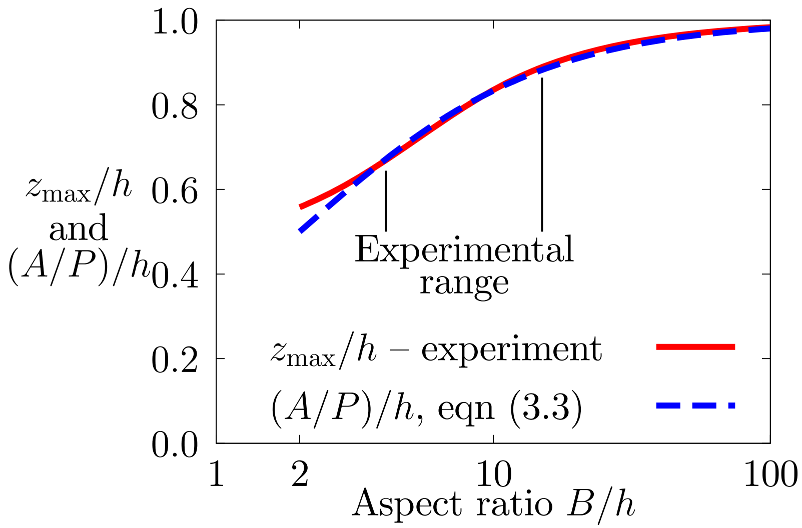


Figure 3.5: Rectangular channels: dimensionless mean elevation of $z_{\max}/h$ and the effective depth $(A/P)/h$

We have seen that A/P appeared naturally in the simple mechanical equilibrium calculation, and here we have found that it mimics the behaviour of $z_{\max}/h$, which is fortunate as it is a rather simpler quantity to calculate in practice, usually with no knowledge of the flow field. We will name A/P the *hydraulic mean depth*, and we will not use the conventional and misleading term “hydraulic radius” (dt. after Strickler – “Profil- oder hydraulischer Radius”). In channels that are wide, which is most, $P \approx B$ and A/P is about the same as the geometric mean depth A/B.

For rectangular channels that are not rectangular we have presented no results. Our suggestion is that A/P will still be a plausible approximation, and it already appears in the Chézy-Weisbach and Gauckler-Manning-Strickler equations (which, officially, we do not yet know in this course!). The use in those equations was justified by Keulegan (1938), however there is much wrong with that work, mathematically correctly integrating logarithmic velocity distributions over various shapes of cross-section but without any attention to real flows in channels.

Our suggested channel flow formula, replacing h by A/P in equation (3.2) is

$$U = \frac{Q}{A} = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{30/\mathrm{e}}{k_\mathrm{s}/\left(A/P\right)}\right), \tag{3.4}$$

where $30/\mathrm{e} \approx 11.0$ is usually written as 12:

$$U = \frac{Q}{A} = \frac{1}{\kappa}\sqrt{g\frac{A}{P}S}\left(\ln\frac{12.}{k_\mathrm{s}/\left(A/P\right)}\right). \tag{3.5}$$

3.2 Generalised notation for flow formulae

We introduce the symbol ε for the relative roughness

$$\varepsilon = \frac{k}{A/P} = \frac{\text{Grain size}}{\text{Hydraulic mean depth}} \approx \frac{\text{Grain size}}{\text{Depth}},$$

also for equivalent uniform sand grain size, $\varepsilon = k_{\mathrm{s}}/(A/P)$. Our flow formula (3.5) is then written in generalised form

$$U = \frac{Q}{A} = \gamma \sqrt{g\frac{A}{P}S}, \tag{3.6}$$

in which we have already obtained the result for the shear velocity

$$u_* = \sqrt{g\frac{A}{P}S}$$

and we introduce the symbol γ for the velocity ratio

$$\frac{U}{u_*} = \gamma = \ln\frac{12.}{\varepsilon},$$

from equation (3.5).

Relative unimportance of grain size

In fact, γ, although all-important for us, is relatively slowly varying with grain size. Consider a small change in the relative roughness $\varepsilon\,(1+\Delta)$. The relative change in the factor γ is

$$\frac{\Delta\gamma}{\gamma} = \frac{\ln\left(12/(\varepsilon\,(1+\Delta))\right)}{\ln\left(12/\varepsilon\right)} - 1 \approx \frac{-\Delta}{\ln\left(12/\varepsilon\right)},$$

having expanded the logarithm as a power series $\ln\left(1+\Delta\right) = \Delta + \ldots$. Now for a value of $\varepsilon = 0.001$ (a $1\,\mathrm{mm}$ grain in $1\,\mathrm{m}$ of water), a relative change of $\Delta = 50\%$ gives a relative change in the factor γ in the equation of only -5%. Even for a much rougher case of $\varepsilon = 0.1$, the same relative change of 50% in grain size changes the left side by just -11%. It does not matter so much if we cannot specify the bed conditions all that accurately.

3.3 The Gauckler-Manning-Strickler formula

We now show that the G-M-S formula is an approximation to the expression we have obtained.

On Figure 3.6 is shown how the dimensionless factor γ varies as a function of relative roughness ε, given by equation (3.5) from experimental fluid mechanics. It is actually possible to approximate that curve closely using a monomial function a/ε^{μ}. The best values of a and μ can be found by performing a least-squares fit over 11 points equally-spaced in $\log\varepsilon$ between $\varepsilon = 0.001$ and 0.1. The result obtained was $\mu = 1/7.001$, which is a surprising coincidence. Now, setting $\mu = 1/7$ and determining just a by optimisation, a value of $a \approx 8.9$ was obtained:

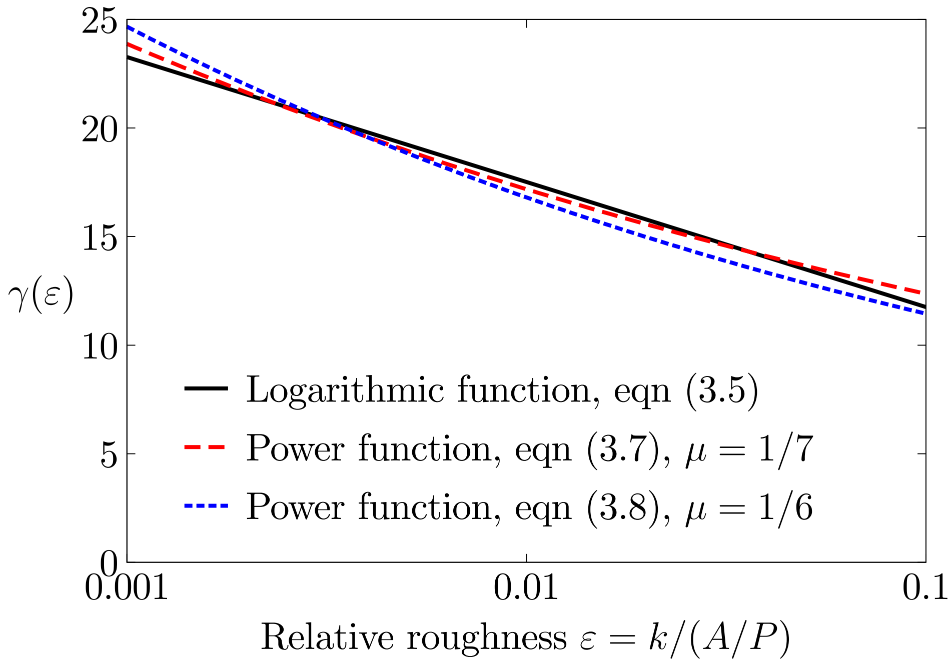


Figure 3.6:

$$\gamma = 8.9 \left(\frac{A/P}{k_{\mathrm{s}}}\right)^{1/7}, \tag{3.7}$$

with close agreement with the expression from the logarithmic velocity distribution shown in the figure. This would give us another flow formula, very similar to the Gauckler-Manning-Strickler

(G-M-S) formula. It is

$$U = \frac{Q}{A} = 8.9 \left(\frac{A/P}{k_{\mathrm{s}}}\right)^{1/7} \sqrt{g\frac{A}{P}S},$$

however unlike G-M-S this is explicit in terms of bed grain size. We do not want to proliferate an already crowded field, so we ignore this. It does, however, suggest the next step. We approximated the result from the logarithmic function again, this time by a function of the form $b/\varepsilon^{1/6}$, where b is a constant. We determined this constant by performing a similar least-squares fit, giving a value of $b \approx 7.8$ such that

$$\gamma = 7.8 \left(\frac{A/P}{k_{\mathrm{s}}}\right)^{1/6}, \tag{3.8}$$

with satisfactory results shown in Figure 3.6, showing that this is also quite a good approximation to our logarithmic function. Substituting into the flow formula, equation (3.6) and re-writing, we obtain

$$U = \frac{Q}{A} = \frac{7.8\sqrt{g}}{k_{\mathrm{s}}^{1/6}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S} = k_{\mathrm{St}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S} = \frac{1}{n} \left(\frac{A}{P}\right)^{2/3} \sqrt{S}, \tag{3.9}$$

which is simply the Gauckler-Manning-Strickler equation, where k_{St} is the Strickler coefficient and $n = 1/k_{\mathrm{St}}$ is the Manning coefficient! Unlike the G-M-S equation, this has given an explicit expression for the Strickler coefficient

$$k_{\mathrm{St}} = \frac{7.8\sqrt{g}}{k_{\mathrm{s}}^{1/6}}. \tag{3.10}$$

A similar result was obtained by Strickler nearly a century ago without optimising software, based entirely on experiment, on boundary roughnesses of equivalent mean diameter from $D = 0.1\,\mathrm{mm}$ to $D = 300\,\mathrm{mm}$,and where that diameter was sometimes calculated from alluvial gravel with relatives lengths of the three axes 1:2:3! For the numerical coefficient he obtained a value of $4.75\sqrt{2} \approx 6.7$, giving his expression

$$k_{\mathrm{St}} = \frac{6.7\sqrt{g}}{D^{1/6}}. \tag{3.11}$$

The expression (3.10) here has been obtained by a quite different route, and the agreement between the two expressions, one based on sand grains glued to the inside of a circular pipe carrying air, is encouraging. Of course, Strickler's result (3.11) is to be preferred.

Sensitivity to boundary particle size

One thing we can do now is, as we did earlier, examine the effect of uncertainty or variability in the size of the boundary particles (and any perceived difference between k_{s} and D), using a power series expansion

$$\frac{\Delta k_{\mathrm{St}}}{k_{\mathrm{St}}} = \left(1 + \frac{\Delta D}{D}\right)^{-1/6} - 1 = -\frac{1}{6}\frac{\Delta D}{D} + \ldots,$$

and so a fractional change in boundary particle size gives a relative change of $1/6$ of that amount in D. Again for this form the exact particle size is actually not so important.

Test of logarithmic and G-M-S formulae

To test the accuracy of the G-M-S formula compared with the logarithmic formula we consider the results of Strickler (1923, Beilage 4), which have been interpreted as the justification for the exponent $2/3$ in the G-M-S formula, and leading to the “S” in that name. Strickler considered results from nine very different channels. For each we calculated the equivalent k_s or D, constant for each channel, by least-squares fitting of the appropriate flow formula to the points, with results shown in the figure. If anything, the Gauckler-Manning-Strickler formula gives better agreement at the lower ends than the logarithmic formula obtained from fluid mechanics experiments.

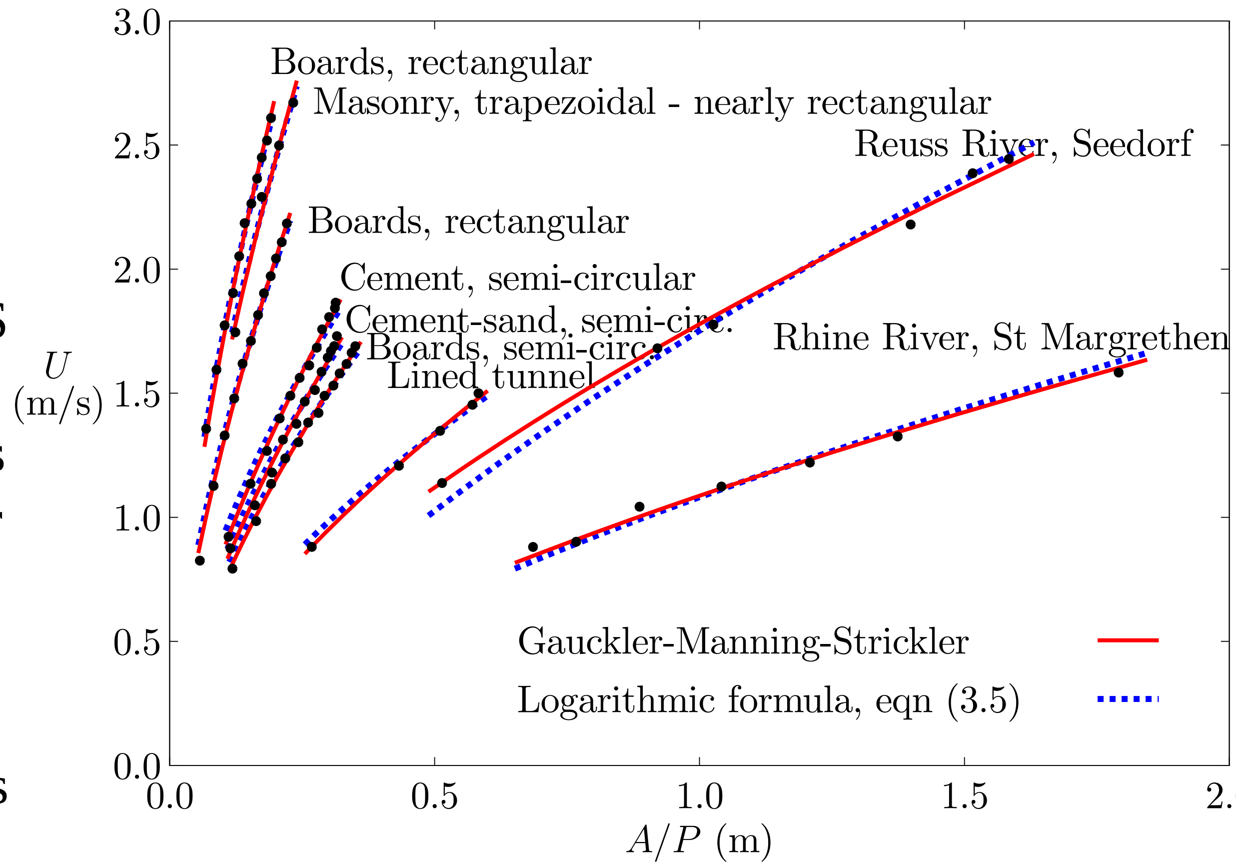


Figure 3.7: Strickler’s results approximated by two flow formulae

3.4 Notation

In the rest of this course, we could write the Gauckler-Manning-Strickler formula in the conventional form

$$U = \frac{Q}{A} = k_{\mathrm{St}} \left(\frac{A}{P}\right)^{2/3} \sqrt{S},$$

however if we use the Strickler expression $k_{\mathrm{St}} = 6.7\sqrt{g}/D^{1/6}$ we prefer to write it in the form which more reflects its nature and our derivation:

$$U = \frac{Q}{A} = \gamma\sqrt{g\frac{A}{P}S} \quad \text{where} \quad \gamma(\varepsilon) = \frac{6.7}{\varepsilon^{1/6}}, \text{ and } \varepsilon = \frac{D}{A/P}. \tag{3.12}$$

- We no longer have the problem of k_{St} with difficult units.
- The characterisation of the resistance has been reduced to that of the dimensionless relative roughness $\varepsilon = D/(A/P)$.
- If asked to estimate the flow at a particular site, we do not have to imagine a value of k_{St} or, like in an Australian water office, ring a friend to see what they used when they worked on a similar stream $20\,\mathrm{km}$ distant. We could estimate the D.
- If the bed material has a size of about $2\,\mathrm{cm}$ (Donau), then we simply use $D = 0.02\,\mathrm{m}$.
- It is much simpler and physically understandable.

3.5 Computation of normal flow

"Normal flow" is the name given to a uniform flow, and the depth is called the normal depth. If the discharge Q, slope S, resistance coefficient k_{St}, and the relationship between area and depth and perimeter and depth are known, the G-M-S formula becomes a transcendental equation for the normal depth h. To solve this is a common problem in river engineering

A numerical method

Any method for the numerical solution of transcendental equations can be used, such as Newton's method. Here we develop a simple method based on *direct iteration*, where we develop a trick, giving us rapid convergence.

In the case of wide channels, (*i.e.* channels rather wider than they are deep, a common case), the wetted perimeter P does not vary much with depth h. Similarly in the expression for the area, the width does not vary much with h. Consider the Gauckler-Manning-Strickler formula in the conventional form, written now

$$Q = k_{\mathrm{St}} \frac{A^{5/3}}{P^{2/3}} \sqrt{S}$$

we divide both sides by $h^{5/3}$, and showing functional dependence of A and P on h:

$$\frac{Q}{h^{5/3}} = k_{\mathrm{St}} \sqrt{S} \frac{(A(h)/h)^{5/3}}{P^{2/3}(h)}.$$

The term $A(h)/h$ is approximately the width of the channel, which for many channels varies little with h, as does the perimeter $P(h)$. So, the right side of the equation varies little with h, so by isolating the $h^{5/3}$ term and taking the $3/5$ power of both sides of the equation, we obtain the equation in a form suitable for direct iteration

$$h = \left(\frac{Q}{k_{\mathrm{St}}\sqrt{S}}\right)^{3/5} \times \frac{P^{2/5}(h)}{A(h)/h}, \qquad (3.13)$$

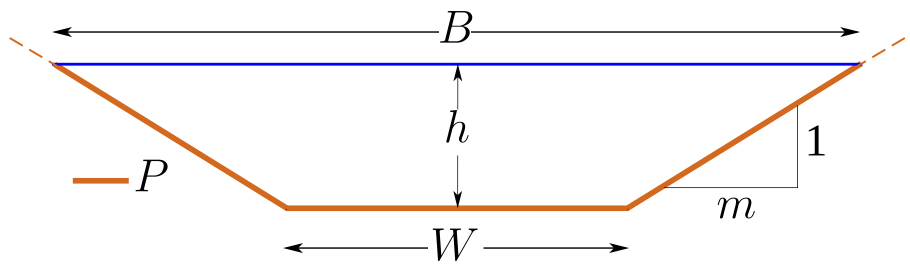
where the first term on the right is a constant for any particular problem, and the second term varies slowly with depth – a primary requirement that the direct iteration scheme be convergent and indeed be quickly convergent.

For an initial estimate we suggest making a rough estimate of the approximate width B_0 and so, making a wide channel approximation, setting $A(h)/h \approx B_0$ and $P(h) \approx B_0$, in the general scheme of (3.13) gives

$$h_0 = \left(\frac{Q}{k_{\mathrm{St}}B_0\sqrt{S}}\right)^{3/5}. \qquad (3.14)$$

Experience with typical trapezoidal sections shows that the method works well and is quickly convergent.

Trapezoidal section



Most canals are excavated to a trapezoidal section, and this is often used as a convenient approximation to river cross-sections too. In many of the problems in this course we will consider the case of trapezoidal sections. Consider the quantities shown in the figure: the bottom width is W, the depth is h, the top width is B, and the *batter slope*, defined to be the ratio of H:V dimensions is m. Geometrically, $B = W + 2mh$, area $A = h\,(W + mh)$, wetted perimeter $P = W + 2\sqrt{1+m^2}h$.

Example 2 Calculate the normal depth in a trapezoidal channel of slope 0.001, $k_{\mathrm{St}} = 25$, bottom width $W = 10\,\mathrm{m}$, with batter slopes $m = 2$, carrying a flow of $20\,\mathrm{m^3s^{-1}}$. We have $A = h\,(10 + 2\,h)$, $P = 10 + 4.472\,h$. For B_0 we use $W = 10\,\mathrm{m}$. Equation (3.14) gives

$$h_0 = \left(\frac{Q}{k_{\mathrm{St}} B_0 \sqrt{S}}\right)^{3/5} = \left(\frac{20}{25 \times 10\sqrt{0.001}}\right)^{3/5} = 1.745\,\mathrm{m}.$$

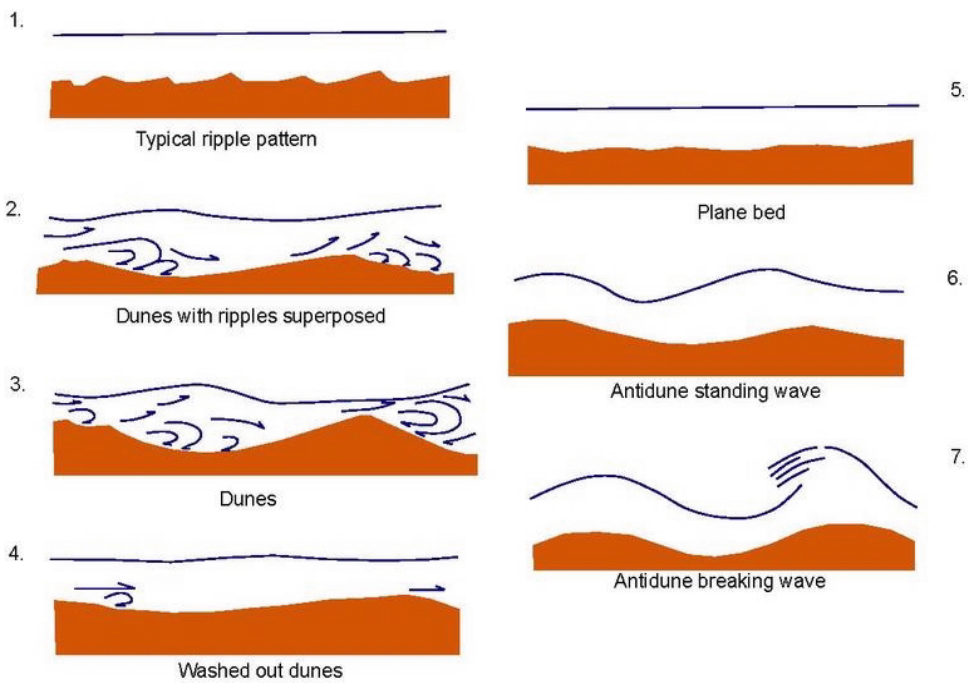
Then, equation (3.13) gives

$$h_{n+1} = \left(\frac{Q}{k_{\mathrm{St}}\sqrt{S_0}}\right)^{3/5} \times \frac{(10 + 4.472\,h_n)^{2/5}}{10 + 2\,h_n} = 6.948 \times \frac{(10 + 4.472\,h_n)^{2/5}}{10 + 2\,h_n}.$$

With $h_0 = 1.745$, $h_1 = 1.629$, $h_2 = 1.639$, $h_3 = 1.638\,\mathrm{m}$, and the method has converged.

3.6 General situations

In many cases the conditions in the river are more complicated than just a layer of uniform regular particles. For example:



- Irregular and variable nature of the bed particle arrangement.
- Bed forms – ripples, dunes, anti-dunes *etc*. Figure from Wikipedia **URL:** *https://en.wikipedia.org/wiki/Bedform*
- Particle movement – if the grains are moving, then the force required to move the grains appears to the water as an additional stress, whether they are moving along the bed, rolling, jumping, or carried suspended in the flow.
- Vegetation – trees standing in the water, grasses, reeds *etc*

Often one is required to adopt a value of resistance coefficient not given by the Strickler formula, but based on experience, knowledge, the Australian telephone method, looking at pictures in books *etc*. None of these are particularly good.

3.7 Chézy-Weisbach flow formula

A problem with the G-M-S form, using a value of k_{St} in more general situations, is that it has little basis in fluid mechanics, the coefficient has no physical significance, and it is in awkward units.

We now relate our results to another well-known standard formulae. Writing shear stress τ in terms of the result obtained from the Darcy-Weisbach formulation of flow resistance in pipes,

$$\frac{\tau}{\rho} = \tfrac{1}{8}\lambda U^2, \tag{3.15}$$

where λ is the Weisbach dimensionless resistance coefficient, expressing the relationship between velocity and stress. From our simple force balance we already have $\tau/\rho = \sqrt{g\,(A/P)\,S}$. Equating gives the Chézy-Weisbach flow formula

$$U = \frac{Q}{A} = \sqrt{\frac{8g}{\lambda}\frac{A}{P}S} = C\sqrt{\frac{A}{P}S}, \tag{3.16}$$

where $C = \sqrt{8g/\lambda}$ is the Chézy coefficient, named after the French military engineer who first wrote down such an open channel flow formula. Comparing our equation (3.5) we see that it is in the same form, such that

$$\frac{\text{Mean velocity}}{\text{Shear velocity}} = \gamma = \sqrt{\frac{8}{\lambda}} \tag{3.17}$$

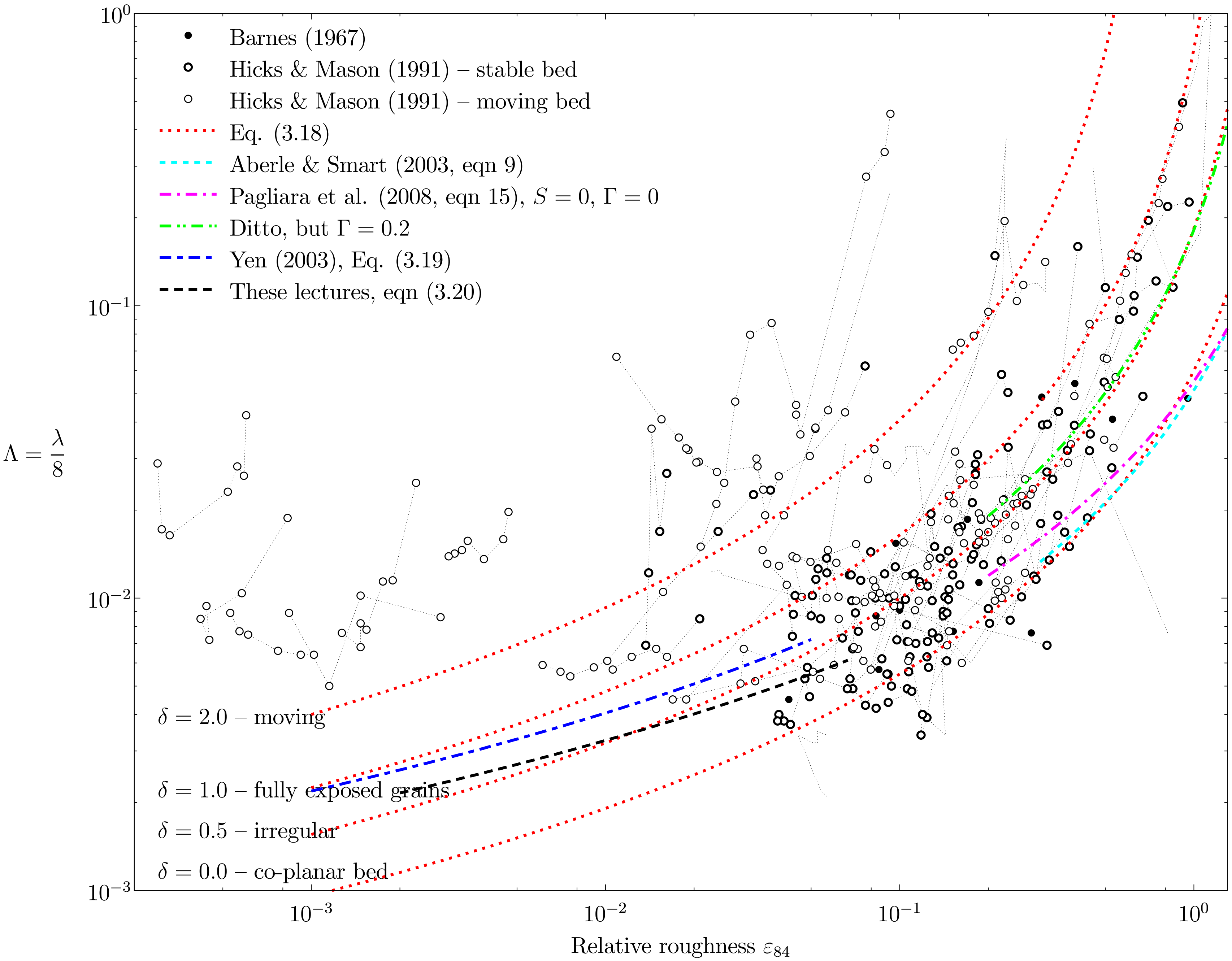
for the flows we have encountered so far in this section, steady, uniform flows in a cross-section which is relatively simple.

Where the resistance to flow is compound – where there are different parts of the stream cross-section with different resistances, where there are obstacles in a stream causing resistance, it is more consistent to use the Chézy-Weisbach formulation, which is more clearly related to forces involved and the mechanics of the problems. This is especially so when we consider the mechanics of flows and flood wave propagation.

Results for resistance coefficients in real rivers

Here we attempt to obtain understanding and a formula for the resistance coefficient using results from a number of field measurements. We considered the results of Hicks & Mason (1991), a catalogue of 558 stream-gaugings from 78 river and canal reaches in New Zealand, of which 55 were sites with grading curves for boundary material, so that particle sizes were known. Neither vegetation nor bed-form resistance can be isolated. Hicks & Mason based their approach on Barnes (1967), who provided values of Manning's resistance coefficient $n = 1/k_{\mathrm{St}}$ for a single flow at each of 50 separate river sites in the United States of America, of which boundary material details were given for 14. We also include those results here.

From both catalogues we took the values of D_{84}, the boundary particle size for which 84% of the material was finer, and from the values of A/P, calculated the relative roughness $\varepsilon_{84} = D_{84}/(A/P)$, and used the measured values of Chézy's C to calculate values of λ. Results are shown on the next page. We have plotted them for a parameter $\Lambda = \lambda/8$, which it will be more convenient for us to use later.



Barnes (1967)
Hicks & Mason (1991) – stable bed
Hicks & Mason (1991) – moving bed
Eq. (3.18)
Aberle & Smart (2003, eqn 9)
Pagliara et al. (2008, eqn 15), $S = 0$, $\Gamma = 0$
Ditto, but $\Gamma = 0.2$
Yen (2003), Eq. (3.19)
These lectures, eqn (3.20)
$\delta = 2.0$ – moving
$\delta = 1.0$ – fully exposed grains
$\delta = 0.5$ – irregular
$\delta = 0.0$ – co-planar bed
$\Lambda = \frac{\lambda}{8}$
Relative roughness ε_{84}
10^{0}
10^{-1}
10^{-2}
10^{-3}

- Many of the results from each study are for large bed material $\varepsilon_{84} > 0.1$, possibly a reflection of the hilly and mountainous nature of New Zealand and Pacific North-West of the United States of America. And Austria.
- There is a wide scatter of results. But not all that very wide if we consider that the streams range from large slow-moving rivers with extremely small grains to mountain torrents with $30\,\mathrm{cm}$ boulders. Most of the results, unless the grains are moving, fall between $\Lambda \approx 0.005$ and 0.02.
- There is, as we have seen, slow variation with relative roughness: am increase in ε by a factor of 10 leads to an increase in Λ of about 2, as we have already seen.
- The points, we believe, have a tendency to group around three of the curves shown and the rest to be bounded below by the fourth (upper) curve shown. The curves have been drawn using the expression, found by trial and error:

$$\Lambda = \frac{0.06 + 0.06\,\delta}{\left(1.0 - 0.6\,\delta - \ln \varepsilon_{84}\right)^2}, \tag{3.18}$$

  with values of $\delta = 0$, 0.5, 1, and 2. The parameter δ is an arbitrary one that we use to identify the state of the particles making up the bed. This will now be explained.
- The first grouping of points comprises those around the bottom curve. We hypothesise that these points, having the lowest resistance, are those forming beds where the particles are relatively co-planar such that the bed is *armoured*. We assigned $\delta = 0$ to this state, and used that in equation (3.18) to plot the curve.
- The next grouping of points is around the second curve from the bottom, which can be seen

to substantially coincide with the a curve corresponding to exposed boulders on top of the bed occupying 0.2 of the surface area. Of course, with a number of these grains thus exposed, the resistance is greater. We assigned a value of $\delta = 0.5$ to this intermediate state.

- Substituting $\delta = 1$ in equation (3.18) gives the third curve on the figure, passing through what we believe is the third grouping of particles. This is probably the state for the maximum resistance for a stable bed corresponding to exposed grains occupying something like 50% of the surface area. Any more such grains will cause shielding of particles, the bed will start to resemble the co-planar case, and resistance will actually be reduced.
- Further evidence supporting our assertions is obtained from the expression proposed by Yen (2002, eqn 19), who considered results from a number of experimental studies using fixed impermeable beds. We used his formula, converted to $\Lambda = \lambda/8$, used an infinite Reynolds number, and converted his equivalent sand roughness $\varepsilon_s = 2\varepsilon_{84}$. It can be seen that the curve passes (left to right) from our curve $\delta = 1$ for small particles, which are unlikely to have the tops levelled so that particles are exposed, to the second curve for larger particles, more likely to be levelled in the laboratory experiments, with $\delta = 0$. Yen obtained the approximation for λ:

$$\lambda = \frac{1}{4.}\left(-\log_{10}\left(\frac{1}{12.}\frac{k_{\mathrm{s}}}{A/P} + \frac{1.95}{\mathsf{R}^{0.9}}\right)\right)^{-2}, \tag{3.19}$$

  where R is the channel Reynolds number $\mathsf{R} = (Q/P)/\nu$, in which ν is the kinematic viscosity.
- The logarithmic formula we obtained quite simply, equation (3.5), leads to, if we use $\varepsilon_s = 2\varepsilon_{84}$,

$$\Lambda = (6 - 2.5\ln(2\varepsilon_{84}))^{-2}, \tag{3.20}$$

quite similar to the results from Yen's formula.

- For points above the third curve almost all experimental points had shear stresses greater than the critical one necessary for movement. If particles move, not only do many particles protrude above others, increasing the stress, but there is the additional force required to maintain the sliding and rolling and jostling of all the particles. Hence, the resistance is greater. And, if there is a need to maintain particles in suspension, that will contribute also to resistance. We have shown the fourth curve as drawn for $\delta = 2$.

Hopefully the figure and approximating curves have given us an idea of the magnitudes and variation of the quantities, and maybe even some results for use in practice.

Non-uniform and unsteady flows

We will be considering flows which are not uniform and those which are neither uniform nor steady. As the length scale of river flows is much longer in space than the cross-sectional dimensions and the time scale of disturbances is much longer than that of local turbulence, we will assume that the boundary stress at each place and at each time is given by the local immediate flow conditions of velocity, in terms of discharge Q and area A. From equation (3.15), and using $\Lambda = \lambda/8$ or from our presentation of the Gauckler-Manning-Strickler equation, we have

$$\frac{\tau}{\rho} = \Lambda U^2 = \Lambda \left(\frac{Q(x,t)}{A(x,t)} \right)^2 = \frac{1}{\gamma^2(\varepsilon)} \left(\frac{Q(x,t)}{A(x,t)} \right)^2, \tag{3.21}$$

where we could use a value of Λ from the figure on page 45 or from the formulae given therein, or we could use our result for the Gauckler-Manning-Strickler formula where $\gamma(\varepsilon) = 6.7/\varepsilon^{1/6}$.

4. Froude number

William Froude (1810-1879, pronounced as in "food") was a naval architect who proposed similarity rules for free-surface flows. A Froude number is a dimensionless number from a velocity scale U and a length scale L, $\mathsf{F} = U/\sqrt{gL}$. In the original definition, of a ship in deep water, the only length scale was L, the length of the ship. In river engineering it is not obvious what the length scale is. Might it be the wetted perimeter P, might it be the geometric mean depth A/B, where A is cross-sectional area and B is surface width?

In fact, the answer will usually depend on the problem. If we consider the *mean total head* of a channel flow at a section, where U is mean velocity and η is surface elevation:

$$H = \eta + \alpha\frac{U^2}{2g},$$

it does *not* appear explicitly. Neither does it appear in the momentum flux M at a section

$$M = \rho g \left(A\bar{h} + \beta U^2 A\right).$$

Here, we are going simply to define it in terms of a vertical length scale, the depth scale A/B:

$$\mathsf{F}^2 = \frac{U^2}{gA/B} = \frac{Q^2 B}{gA^3}, \tag{4.1}$$

where $Q = UA$ is discharge. For many years the lecturer was more specific, using terms of kinetic and potential energy, now all he says is that F^2 is a measure of dynamic effects relative to gravitational, the latter measured by mean depth.

Even if we were to consider rather more complicated problems such as the unsteady propagation of waves and floods, and to non-dimensionalise the equations, we would find that the Froude number F itself never appears in the equations, but always as $\alpha\mathsf{F}^2$ or $\beta\mathsf{F}^2$, depending on whether energy or momentum considerations are being used.

Flows which are fast and shallow have large Froude numbers, and those which are slow and deep have small Froude numbers. Generally F^2 is an expression of the wave-making ability of a flow, and in conversation we usually use "high/ low Froude number" as an expression of how fast a flow is. For example, consider a river or canal which is $2\,\mathrm{m}$ deep flowing at $0.5\,\mathrm{ms}^{-1}$ (make some effort to imagine it - we can well believe that it would be able to flow with little surface disturbance!). We have

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 2}} = 0.11 \quad \text{and} \quad F^2 = 0.012\,,$$

and we can imagine that the wavemaking effects are small. Now consider flow in a street gutter after rain. The velocity might also be $0.5\,\mathrm{ms}^{-1}$, while the depth might be as little as $2\,\mathrm{cm}$. The Froude number is

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 0.02}} = 1.1 \quad \text{and} \quad F^2 = 1.2\,,$$

and we can easily imagine it to have many waves and disturbances on it due to irregularities in the gutter.

Near-constancy of Froude number in a stream

It is interesting to calculate the Froude number F of a steady uniform flow given by the Chézy-Weisbach formula for discharge:

$$Q = \sqrt{\frac{gA^3}{\Lambda P} S}\ .$$

Immediately this gives

$$\mathsf{F}^2 = \frac{Q^2 B}{gA^3} = \frac{S}{\Lambda}\frac{B}{P}\ ,$$

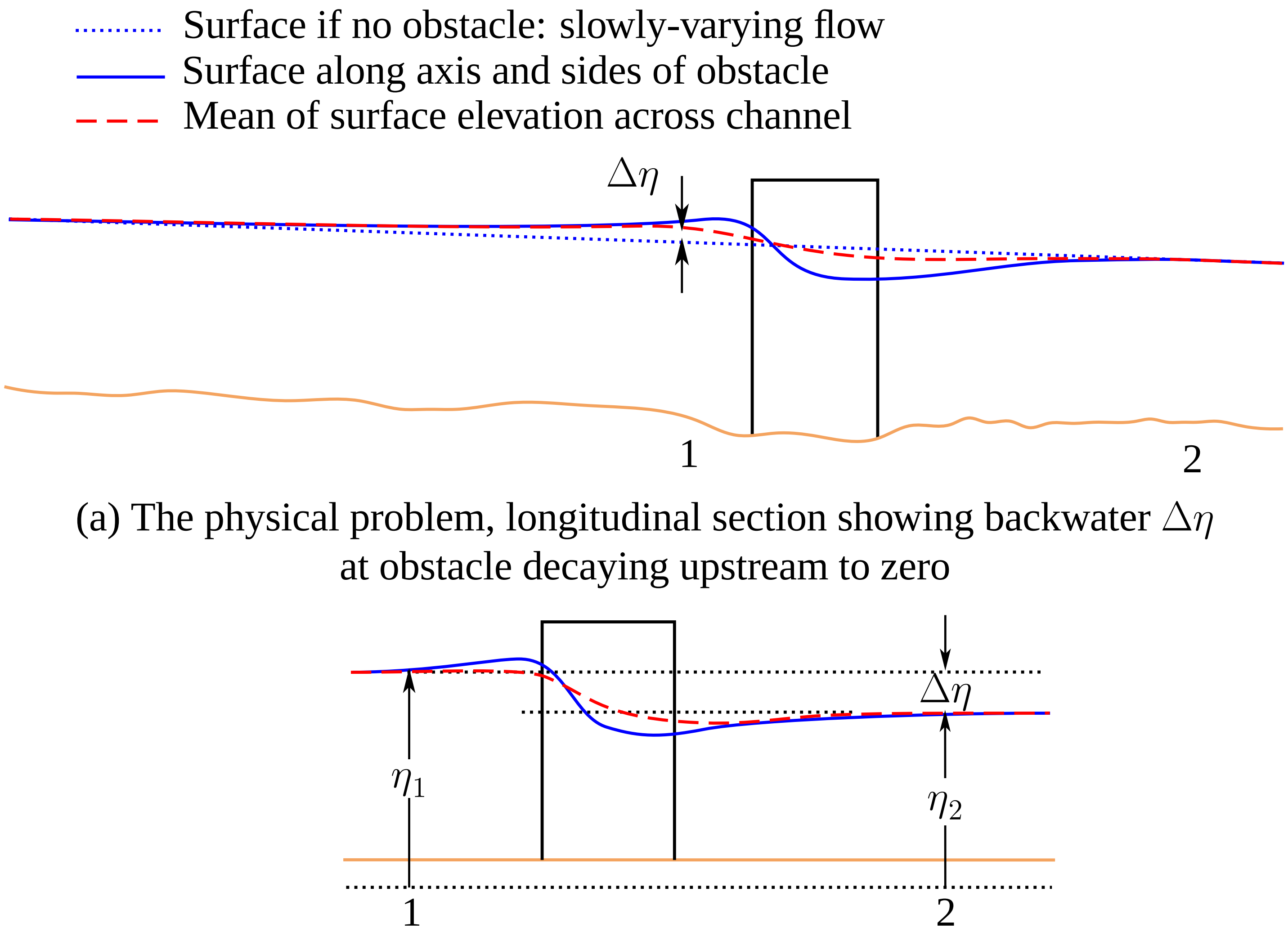
and as $B \approx P$ for wide channels, we see that the square of the Froude number is approximately equal to the ratio of bed slope S to resistance coefficient Λ, giving some significance and physical feeling for Λ. This means that for a particular reach of river, where slope S is effectively independent of flow, where B/P also does not vary much with the flow and Λ often does not vary much, the Froude number F does not change much with flow. While a flood might look more dramatic than a more-common low flow, because it is faster and higher, the Froude number is roughly the same for both.

5. The effect of obstructions on streams – an approximate method

River Traun, Bad Ischl, Oberösterreich

Structures such as weirs can almost completely block a river, but there are also other types of obstacles that are only a partial blockage, such as the piers of a bridge, blocks on the bed, Iowa vanes, *etc.* or possibly more importantly, the effects of trees placed in rivers (”Large Woody Debris”), used in their environmental rehabilitation. It might be important to know what the forces on the obstacles are, or, more importantly for us, what effects the obstacles have on the river.

The physical problem and its idealisation



(a) The physical problem, longitudinal section showing backwater $\Delta\eta$ at obstacle decaying upstream to zero

(b) The idealised problem, uniform channel with no friction or slope

Figure 5.1: A typical physical problem of flow past a bridge pier, and its idealisation for hydraulic purposes

Momentum flux in a channel

The momentum flux across a section is defined to be the sum of the pressure force, plus the mass rate of transport multiplied by the velocity. For a vertical section, the mass rate of transport is $\rho u \, \mathrm{d}A$, so the momentum flux is

$$M = \int_A \left(p + \rho u^2\right) \, \mathrm{d}A.$$

Substituting the hydrostatic pressure distribution, $p = \rho g \left(\eta - z\right)$, where η is the free surface elevation, we obtain

$$M = \rho \int_A \left(g\left(\eta - z\right) + u^2\right) \, dA.$$

- The integral $\int_A (\eta - z)\, dA$ is simply the first moment of area about a transverse horizontal axis *at the surface*, we can write it as

  $$\int_A (\eta - z)\, dA = A\bar{h}$$

  where $\bar{h}$ is the *depth* of the centroid of the section *below the surface*.
- The velocity contribution we have already evaluated as $\int_A u^2 \, dA \approx \beta U^2 A = \beta Q^2/A$.

Collecting contributions, we have the expression for the momentum flux at a section

$$M = \rho \left(gA\bar{h} + \beta \frac{Q^2}{A}\right). \tag{5.1}$$

Momentum conservation

Consider the momentum conservation equation if a force P is applied in a negative direction to a flow between two sections 1 and 2:

$$P = \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \rho\left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2, \tag{5.2}$$

Usually one wants to calculate the effect of the obstacle on water levels. The effects of drag can be estimated by knowing the area of the object measured transverse to the flow, a, the drag coefficient C_{D}, and u, the mean fluid speed past the object:

$$P = \frac{1}{2}\rho C_{\mathrm{D}} u^2 a, \tag{5.3}$$

and so, substituting into equation (5.2) gives, after dividing by density,

$$\frac{1}{2}C_{\mathrm{D}} u^2 a = \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A}\right)_2. \tag{5.4}$$

We consider the velocity u on the obstacle as being proportional to the upstream velocity, such that we write

$$u^2 = \Gamma\left(\frac{Q}{A_1}\right)^2, \tag{5.5}$$

where Γ is a coefficient which recognises that the velocity which impinges on the object is generally not equal to the *mean* velocity in the flow. For a small object near the bed, Γ could be quite small; for an object near the surface it will be slightly greater than 1; for objects of a vertical scale that of the whole depth, $\Gamma \approx 1$. Equation (5.4) becomes

$$\frac{1}{2}\Gamma\, C_{\mathrm{D}} \frac{Q^2}{A_1^2} a = \left(gA\bar{h} + \beta\frac{Q^2}{A} \right)_1 - \left(gA\bar{h} + \beta\frac{Q^2}{A} \right)_2 \tag{5.6}$$

A typical problem is where the downstream water level is given (sub-critical flow, so that the control is downstream), and we want to know by how much the water level will be raised upstream if an obstacle is installed. As both A_1 and $\overline{h}_1$ are functions of h_1, so that we would need to know in detail the geometry of the stream, and then to solve the transcendental equation for h_1. However, by *linearising* the problem, solving it approximately, we obtain a simple explicit solution that tells us rather more.

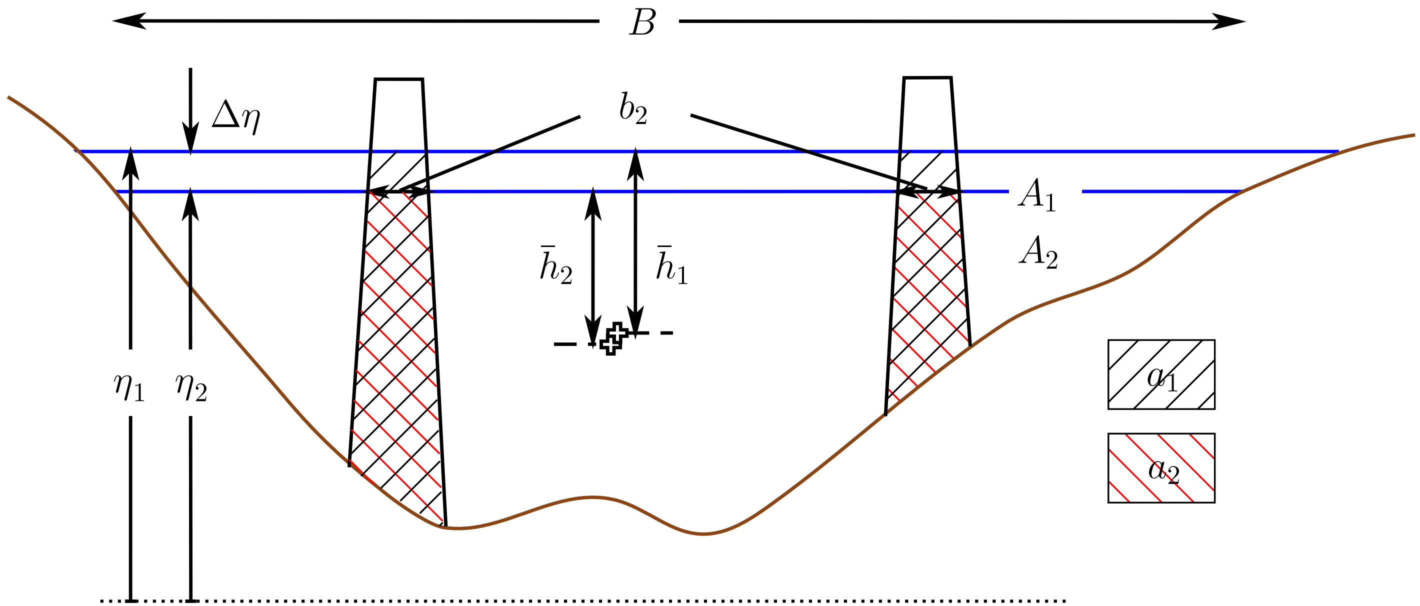


Figure 5.2: Cross-section showing dimensions for water levels at 1 and 2

Consider the stream cross-section shown in Fig. 5.2, with a small change in water level $\eta_1 = \eta_2 + \Delta\eta$. We now use geometry to obtain approximate expressions for quantities at 1 in terms of those at 2. It is easily shown that

$$A_1 = A_2 + B_2\,\Delta\eta + O\left((\Delta\eta)^2\right), \text{ and } \left(A\bar{h}\right)_1 = \left(A\bar{h}\right)_2 + A_2\,\Delta\eta + O\left((\Delta\eta)^2\right) \; ,$$

and similarly we write for the blockage area $a_1 = a_2 + b_2\,\Delta\eta + O\left((\Delta\eta)^2\right)$, where b_2 is the surface width of the obstacle (which for a submerged obstacle would be zero). We have actually been using Taylor series expansions, but the physical interpretations seem simpler than the mathematical!

The momentum equation (5.6) gives us

$$\frac{1}{2}\Gamma\, C_{\mathrm{D}} \frac{Q^2}{A_1^2} a_2 = gA_2\,\Delta\eta + \beta\frac{Q^2}{A_2 + B_2\,\Delta\eta} - \beta\frac{Q^2}{gA_2}.$$

Now we use a power series expansion in $\Delta\eta$ to simplify the term in the denominator:

$$\frac{1}{A_2 + B_2\,\Delta\eta} = \frac{1}{A_2\left(1 + B_2\,\Delta\eta/A_2\right)} = \frac{1}{A_2}\left(1 + B_2\,\Delta\eta/A_2\right)^{-1} \approx \frac{1}{A_2}\left(1 - B_2\,\Delta\eta/A_2\right),$$

neglecting terms like $(\Delta\eta)^2$ (see equation A-1). The momentum equation becomes

$$\frac{1}{2}\Gamma\, C_{\mathrm{D}} \frac{Q^2}{gA_1^2} a_2 \approx \Delta\eta\, A_2\left(1 - \beta\frac{Q^2 B_2}{gA_2^3}\right).$$

As $A_1 = A_2 + O(\Delta\eta)$ we replace A_1 by A_2 and introduce $\mathsf{F}_2^2 = Q^2 B_2/gA_2^3$, the square of the Froude number of the downstream flow. The equation is easily solved to give an explicit approximation for the dimensionless drop across the obstacle $\Delta\eta/\left(A_2/B_2\right)$, where A_2/B_2 is the mean downstream depth:

$$\frac{\Delta\eta}{A_2/B_2} = \frac{\frac{1}{2}\,\Gamma\,C_{\mathrm{D}}\,\mathsf{F}_2^2}{1-\beta\mathsf{F}_2^2}\frac{a_2}{A_2}. \tag{5.7}$$

This explicit approximate solution has revealed the important quantities of the problem to us and how they affect the result: downstream Froude number $\mathsf{F}_2^2 = Q^2B_2/gA_2^3$ and the relative blockage area a_2/A_2. For subcritical flow $\beta\mathsf{F}_2^2 < 1$ the denominator in (3.13) is positive, and so is $\Delta\eta$, so that the surface drops from 1 to 2, as we expect. If the flow is supercritical, $\beta\mathsf{F}_2^2 > 1$, we find $\Delta\eta$ negative, and the surface rises between 1 and 2. If the flow is near critical $\beta\mathsf{F}_2^2 \approx 1$, the change in depth will be large, which is made explicit, and the theory will not be valid.

We could immediately estimate how important this is. We see that, for small Froude number $\mathsf{F}_2^2 \ll 1$, such that $1-\beta\mathsf{F}_2^2 \approx 1$, the relative change of depth is equal to $\frac{1}{2}$ times $\Gamma \approx 1$ (for a body extending the whole depth), times $C_{\mathrm{D}} \approx 1$ for cylinders *etc*., multiplied by F_2^2, usually small, multiplied by the blockage ratio a_2/A_2, which is also probably small. So, the relative result is usually small. However, the absolute value might still be finite compared with resistance losses, as will be seen below.

Another benefit of the approximate analytical solution is that it shows that such an obstacle forms a control in the channel, so that the finite sudden change in surface elevation $\Delta\eta$ is a function of Q^2, or Q a function of $\sqrt{\Delta\eta}$, in a manner analogous to a weir. In numerical river models it should ideally be included as an internal boundary condition between different reaches as if it were a type of fixed control.

The mathematical step of linearising has revealed much to us about the nature of the problem that the original momentum equation did not.

Example 3 It is proposed to build a bridge, where the bridge piers occupy about 10% of the "wetted area" of a river with Froude number 0.5 (which is quite large). How much effect will this have on the river level upstream?

As the bridge piers occupy all the depth, we have $\Gamma = 1$. A typical drag coefficient is $C_\mathrm{D} \approx 1$. We will use $\beta = 1$ (this is an estimate!). So we find, using equation (3.13):

$$
\begin{aligned}
\frac{\Delta\eta}{A_2/B_2} &= \frac{\frac{1}{2}\,\Gamma\, C_\mathrm{D}\, \mathsf{F}_2^2}{1-\beta \mathsf{F}_2^2}\frac{a_2}{A_2} \\
&\approx \tfrac{1}{2} \times 1 \times 1 \times 0.1 \times \frac{0.5^2}{1-0.5^2} \\
&= 0.017,
\end{aligned}
$$

about 2% of the mean depth. This seems small, but if the river were $2\ \mathrm{m}$ deep, there is a $4\ \mathrm{cm}$ drop across the bridge. If the slope of the river were $S = 10^{-4}$, this would correspond to the surface level change in a length of $400\ \mathrm{m}$, which can hardly be neglected.

6. Reservoir routing

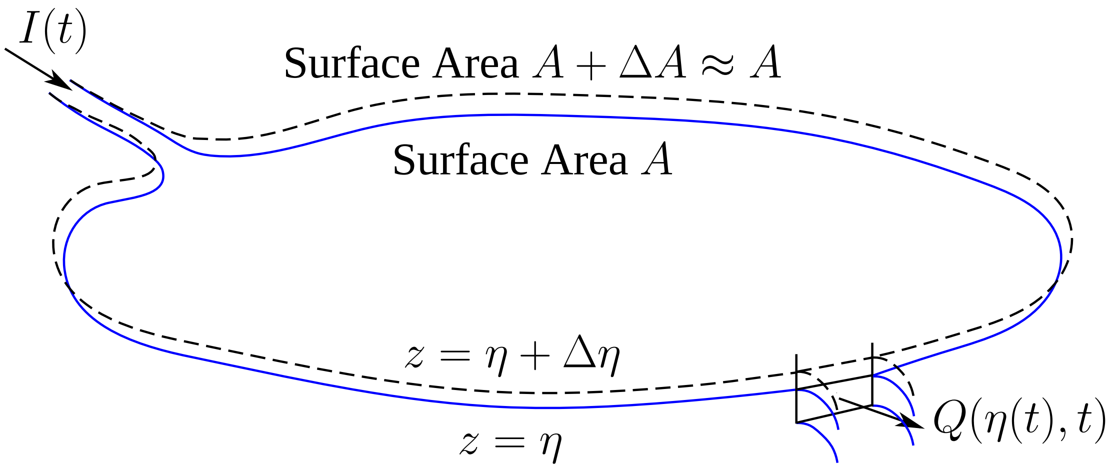


Figure 6.1: Reservoir or tank, showing surface level varying with inflow, determining the rate of outflow

Consider the problem shown in figure 6.1, where a generally unsteady inflow rate $I(t)$ enters a reservoir or a storage tank, and we have to calculate what the outflow rate $Q(t)$ is, as a function of time t. The action of the reservoir is usually to store water, and to release it more slowly, so that the outflow is delayed and the maximum value is less than the maximum inflow. Some reservoirs, notably in urban areas, are installed just for this purpose, and are called *detention* reservoirs or storages. The procedure of solving the problem is also called *Level-pool Routing*.

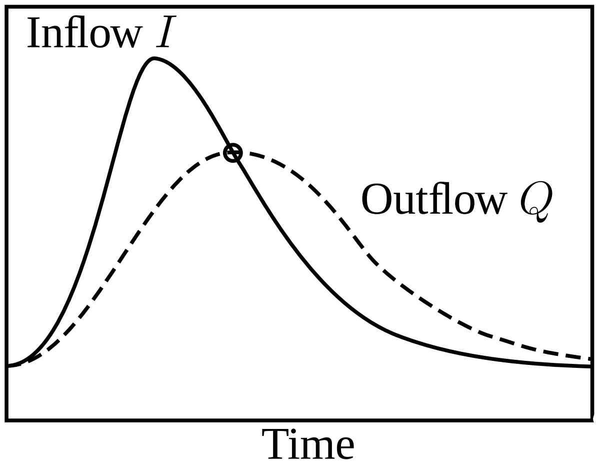


Figure 6.2:

The process is shown in figure 6.2. When a flood comes down the river, inflow increases, the water level rises in the reservoir until at the point O when the outflow over the spillway now balances the inflow. At this point, outflow and surface elevation in the reservoir have a maximum. After this, the inflow might reduce quickly, but it still takes some time for the extra volume of water to leave the reservoir.

Detention reservoir in a public park in Melbourne, Australia

It is simple and obvious to write down the relationship stating that the rate of surface rise $\mathrm{d}\eta/\mathrm{d}t$ is equal to the net rate of volume increase divided by surface area:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{I(t) - Q(\eta, t)}{A(\eta)}\ , \tag{6.1}$$

where η is the free surface elevation, and $A(\eta)$ is the surface area, possibly given from planimetric information from contour maps, and $Q(\eta, t)$ is the volume rate of outflow, which is usually a simple function of the surface elevation η, from a weir or gate formula, usually involving terms like $(\eta - z_{\text{outlet}})^{1/2}$ and/or $(\eta - z_{\text{crest}})^{3/2}$, where z_{outlet} is the elevation of the pipe or tailrace outlet to atmosphere and z_{crest} is the elevation of the spillway crest. There might be extra dependence on time t is if the outflow device is opened or closed. This is a differential equation for the surface elevation itself. The procedure of solving it is called *Level-pool Routing*.

The traditional method of solving the problem, described in almost all books on hydrology, is to use an unnecessarily complicated method called the “Modified Puls” method of routing, which solves a transcendental equation for a single unknown quantity, the volume in the reservoir, at each time step. It is simpler and more fundamental to treat the problem as a differential equation (Fenton 1992). :-)

Numerical solution of the differential equation by Euler's method

Euler's method is the simplest (but least-accurate) of all methods, being of first-order accuracy only. For river engineering purposes it is usually quite good enough. However there is a good method for making it more accurate, which we will use. Euler's method is to approximate the derivative in a differential equation at a time step i by a forward difference expression in terms of a time step Δ, here applying it to equation (6.1):

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}t}\right|_i \approx \frac{\eta_{i+1}-\eta_i}{\Delta} = \frac{I(t_i)-Q(\eta_i,t_i)}{A(\eta_i)} ,$$

giving the scheme to calculate the value of η at t_{i+1} as

$$\eta_{i+1} = \eta_i + \Delta\frac{I(t_i)-Q(\eta_i,t_i)}{A(\eta_i)} + O\left(\Delta^2\right) , \tag{6.2}$$

where we use the notation η_i for the solution at time step i. We have shown that the error of this approximation is proportional to Δ^2. It is necessary to take small enough Δ that this is small.

Accurate results with simple methods – Richardson extrapolation

We introduce a clever device for obtaining more accurate solutions from Euler's method and others.

Consider the numerical value of any part of a computational solution for some physical quantity ϕ obtained using a time or space step Δ, such that we write $\phi(\Delta)$. Let the computational scheme be of known nth order such that the *global* error of the scheme at any point or time is proportional to

Δ^n, then if $\phi(0)$ is the exact solution, we can write the expression in terms of the error at order n:

$$\phi(\Delta) = \phi(0) + b\Delta^n + \dots \ , \tag{6.3}$$

where $\phi(0)$ is the solution for a vanishingly small time step, so that it should be exact. The b is an unknown coefficient; the neglected terms vary like Δ^{n+1}. If we have two numerical simulations or approximations with two different Δ_1 and Δ_2 giving numerical values $\phi_1 = \phi(\Delta_1)$ and $\phi_2 = \phi(\Delta_2)$ then we write (6.3) for each:

$$\begin{aligned} \phi_1 &= \phi(0) + b\Delta_1^n + \dots \ , \\ \phi_2 &= \phi(0) + b\Delta_2^n + \dots \ . \end{aligned}$$

These are two linear equations in the two unknowns $\phi(0)$ and b. Eliminating b, which is not important, between the two equations and neglecting the terms omitted, we can solve for $\phi(0)$, an approximation to the exact solution:

$$\phi(0) = \frac{\phi_2 - r^n \phi_1}{1 - r^n} + O\left(\Delta_1^{n+1}, \Delta_2^{n+1}\right) \ , \tag{6.4}$$

where $r = \Delta_2/\Delta_1$. The errors are now proportional to step size to the power $n+1$, so that we have gained a higher-order scheme without having to implement any more sophisticated numerical methods, just with a simple numerical calculation. This procedure, where n is known, is called *Richardson extrapolation to the limit.*

1. For simple Euler time-stepping solutions of ordinary differential equations, $n = 1$, and if we

perform two simulations, one with a time step Δ and then one with $\Delta/2$, we have

$$\phi(t,0) = 2\,\phi(t,\Delta/2) - \phi(t,\Delta) + O\left(\Delta^2\right), \tag{6.5}$$

where the numerical solution at time t has been shown as a function of the step. This is very simply implemented.

2. For the evaluation of an integral by the trapezoidal rule, $n = 2$.

Example 4 Consider a small detention reservoir, square in plan, with dimensions 100m by 100m, with water level at the crest of a sharp-crested weir of length of $b = 4\,\mathrm{m}$, where the outflow over the sharp-crested weir can be taken to be

$$Q\left(\eta\right) = 0.6\sqrt{g}b\eta^{3/2}, \tag{6.6}$$

where $g = 9.8\,\mathrm{ms}^{-2}$. The surrounding land has a slope (V:H) of about 1:2, so that the length of a reservoir side is $100 + 2 \times 2 \times \eta$, where η is the surface elevation relative to the weir crest, and

$$A(\eta) = \left(100 + 4\eta\right)^2.$$

The inflow hydrograph is:

$$I(t) = Q_{\text{min}} + \left(Q_{\text{max}} - Q_{\text{min}}\right)\left(\frac{t}{T_{\text{max}}}e^{1-t/T_{\text{max}}}\right)^5, \tag{6.7}$$

where the event starts at $t = 0$ with Q_{min} and has a maximum Q_{max} at $t = T_{\text{max}}$. This general form of inflow hydrograph mimics a typical storm, with a sudden rise and slower fall, and will be used in other places in this course. In the present example we consider a typical sudden local storm event, with $Q_{\text{min}} = 1\ \text{m}^3\text{s}^{-1}$ at $T_{\text{max}} = 1800\,\text{s}$.

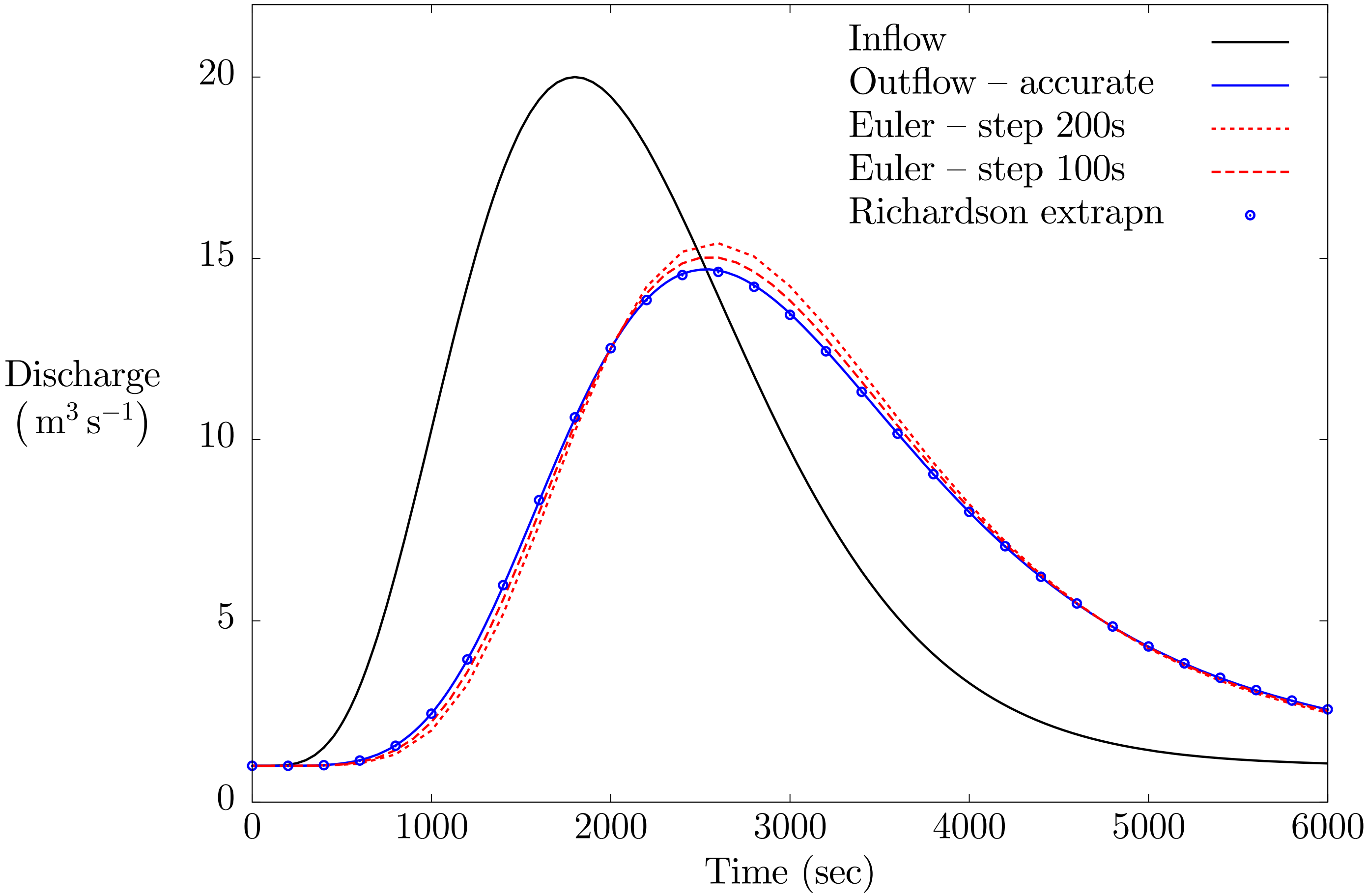


Figure 6.3: Computational results for the routing of a sudden storm through a small detention reservoir

The problem was solved with an accurate 4th-order Runge-Kutta scheme, and the results are shown as a solid blue line on figure 6.3, to provide a basis for comparison. Next, Euler's method (equation 6.2) was used with 30 steps of $200\,\text{s}$, with results that are barely acceptable. Halving the time step to $100\,\text{s}$ and taking 60 steps gave the slightly better results shown. It seems, as expected from knowledge of the behaviour of the global error of the Euler method, that it has been halved at each point. Next, applying Richardson extrapolation, equation (6.5), gave the results shown by the solid points. They almost coincide with the accurate solution, and cross the inflow hydrograph with an apparent horizontal gradient, as required, whereas the less-accurate results do not. Overall, it seems that the simplest Euler method can be used, but is better together with Richardson extrapolation. In fact, there was nothing in this example that required large time steps – a simpler approach might have been just to take rather smaller steps.

The role of the detention reservoir in reducing the maximum flow from $20\,\mathrm{m^3s^{-1}}$ to $14.7\,\mathrm{m^3s^{-1}}$ is clear. If one wanted a larger reduction, it would require a larger spillway. It is possible in practice that this problem might have been solved in an inverse sense, to determine the spillway length for a given maximum outflow.

7. The one-dimensional equations of river hydraulics

These are the fundamental equations that are used to describe the propagation of floods and disturbances in rivers. They are called the *long wave equations*, the *shallow water equations*, or the *Saint-Venant equations*, and are mass and momentum conservation equations for water.

The equations are a pair of partial differential equations in the independent variables x (distance along the stream) and t (time). A typical flood routing problem is for large extra values of discharge Q to be introduced at the upstream initial point, and then for a number of time steps, to solve the equations along the channel to obtain the progress of the flood at each time.

We will also consider a mass conservation equation for soil. In their steady form, the equations describe how water level and velocities vary along a stream, and what effects boundary changes such as sand removal might have on flooding.

We make the traditional approximation that all rivers are straight. Later we will see that it is quite accurate, even for meandering streams.

The model is one-dimensional. We do not consider details of motion in the plane across the stream – all quantities are averaged across it. This does not mean that we assume they are constant. This approach requires surprisingly few approximations – the model is a good simple model of complicated reality.

It is easier to use cartesian co-ordinates, for which we use x the horizontal distance along the stream, y the horizontal transverse co-ordinate, and z the vertical, relative to some arbitrary origin.

7.1 Mass conservation of water and soil

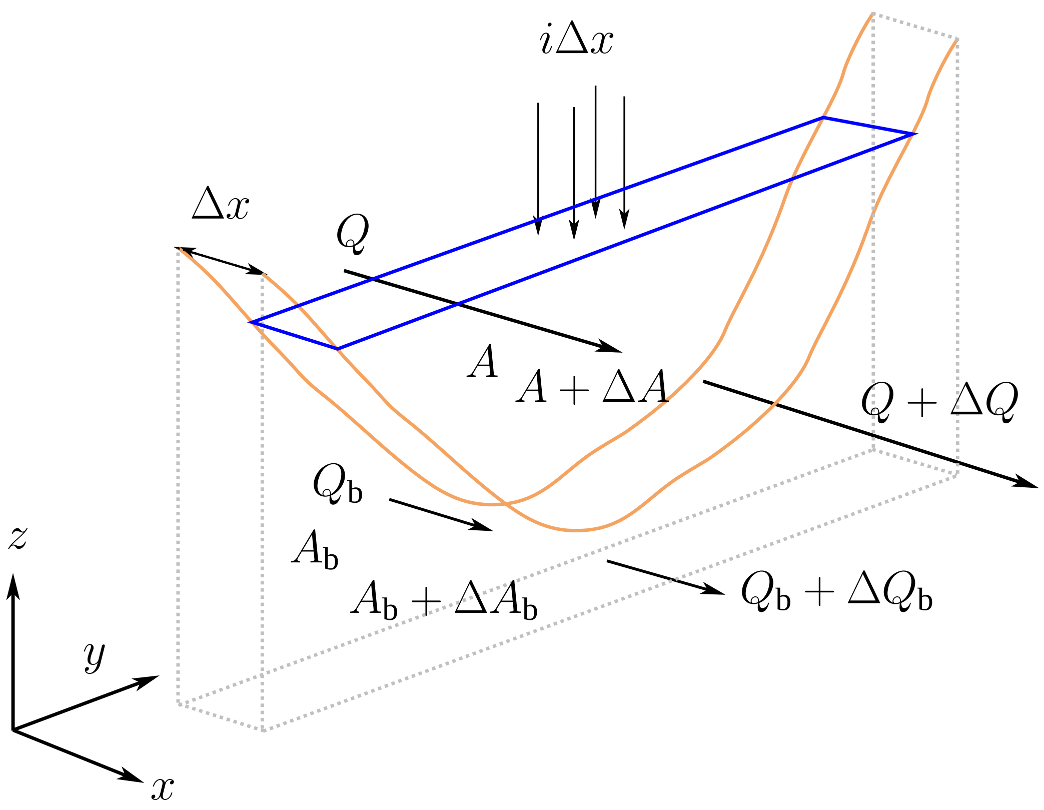


Figure 7.1: Elemental length of channel showing control volumes

Consider Figure 7.1, showing an elemental slice of channel of length Δx with two stationary vertical faces across the flow. It includes two different control volumes. The free surface and the interface between them may move. The surface shown by solid lines contains water and possibly suspended soil grains. The surface shown by dotted lines contains the soil moving as bed load and extends down into the soil such that there is no motion at its far boundaries. Each is modelled separately. We assume that the density of the fluid (water plus suspended soil particles) is constant, so that we can consider *volume* conservation.

On the upstream vertical face at any instant, there is a volume flux (rate of volume flow) Q, and that on the downstream face is $Q + \Delta Q$, so that

$$\text{Net volume flow rate of fluid } \textit{leaving} \text{ across vertical faces} = \Delta Q = \frac{\partial Q}{\partial x}\Delta x + \text{terms in } (\Delta x)^2 .$$

If rainfall, seepage, or tributaries contribute an inflow volume rate i per unit length of stream, the volume flow rate of this other fluid *entering* the control volume is $i\,\Delta x$. If the sum of the two contributions is not zero, then volume of fluid is changing inside the elemental slice, so that the water level will change in time. The rate of change with time t of fluid volume is $\partial A/\partial t \times \Delta x$. For volume to be conserved (mass, but we assume the water is incompressible) this is equal to the net rate of fluid entering the control volume, dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{7.1}$$

This is the *mass conservation equation*. Remarkably for hydraulics, it is almost exact. The only approximation has been that the channel is straight. It is also linear in the two dependent variables A and Q.

The composite bulk density ρ_{b} of the bed load composed of larger soil particles and is assumed to be constant. The bed has a cross-sectional area A_{b}, the bulk volumetric flow rate is Q_{b}, and there is an inflow of mass rate $\dot{m}_i$ per unit length, possibly due to deposition or erosion. Mass conservation is calculated following the same reasoning as for the channel, giving Exner's[1] equation:

$$\frac{\partial A_{\mathrm{b}}}{\partial t} + \frac{\partial Q_{\mathrm{b}}}{\partial x} = \frac{\dot{m}_i}{\rho_{\mathrm{b}}}. \tag{7.2}$$

The volume transport rate used here is the bulk flow rate; it is related to the volume flow rate of solid matter Q_{s} used in transport formulae, by $Q_{\mathrm{s}} = Q_{\mathrm{b}}\,(1-\varphi)$, where φ is the porosity.

[1] `https://de.wikipedia.org/wiki/Felix_Maria_von_Exner-Ewarten` – Austrian – Director of the Zentral Anstalt für Meteorologie und Geodynamik

Upstream Volume

The mass conservation equation (7.1) suggests the introduction of a function $V(x, t)$ which is the volume upstream of point x at time t, such that for the channel flow

$$\frac{\partial V}{\partial x} = A \quad \text{and} \quad \frac{\partial V}{\partial t} = \int^{x} i\left(x'\right)\, \mathrm{d}x' - Q. \tag{7.3}$$

The derivative of volume with respect to distance x gives the area, as shown, while the time rate of change of volume upstream is given by the rate at which the volume is increasing due to inflow, minus the rate at which volume is passing the point and therefore no longer upstream. Substituting for A and Q into equation (7.1):

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial x}\left(-\frac{\partial Q}{\partial t}\right) + \frac{\partial}{\partial x}\int^{x} i\left(x'\right)\, \mathrm{d}x' = i\ ,$$

which is identically satisfied. By introducing V we automatically satisfy one of the two conservation equations and reduce the number of unknowns from two (A and Q) to one (V). Sometimes this can be very helpful.

In the case of the bed load, a similar quantity $V_{\mathrm{b}}(x, t)$ can be introduced such that the mass conservation equation (7.2) is identically satisfied.

Use of surface elevation instead of cross-sectional area

We usually work in terms of water surface elevation (“stage”) η, which is easily measurable and which is practically more important. We make a significant assumption here, but one that is usually accurate: the water surface is horizontal across the stream. Now, if the surface changes by an amount $\delta\eta$ in an increment of time δt, then the area changes by an amount $\delta A = B\,\delta\eta$, where B is the width of the stream surface. Taking the usual limit of small variations in calculus, we obtain $\partial A/\partial t = B\,\partial\eta/\partial t$, and the mass conservation equation can be written

$$B\frac{\partial\eta}{\partial t} + \frac{\partial Q}{\partial x} = i. \tag{7.4}$$

The discharge Q could be written as $Q = UA$, where U is the mean streamwise velocity over a section, and substituted into this. However, the discharge is more practical and fundamental than the velocity, and that will not be done here.

7.2 Momentum conservation equation for channel flow

The equation

The conservation of momentum principle is now applied to the mixture of water and suspended solids in the main channel for a moving and deformable control volume (White 2003, §§3.2 & 3.4). The x-component is

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{CV}} \rho u \,\mathrm{d}V}_{\text{Unsteady term}} + \underbrace{\int_{\mathrm{CS}} \rho u\, \mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}}\,\mathrm{d}S}_{\text{Fluid inertia term}} = P_x, \tag{7.5}$$

where $\mathbf{u}$ is the fluid velocity with x-component u, $\mathrm{d}V$ is an element of volume, $\mathbf{u}_\mathrm{r}$ is the velocity vector of the fluid relative to the local element of the control surface, which is possibly moving itself, $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards, and $\mathrm{d}S$ is an elemental area of the surface. The quantity $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}}$ is the component of relative velocity normal to the surface at any point. It is this velocity that is responsible for the transport of any quantity across the surface, momentum here. P_x is the force exerted on the fluid in the control volume by both body forces, which act on all fluid particles, and surface forces which act only on the control surface.

Hydraulic approximations

1. **Unsteady term**

   The element of volume is $\mathrm{d}V = \Delta x\, \mathrm{d}A$, and the term contribution can be written

$$\rho\, \Delta x\, \frac{\mathrm{d}}{\mathrm{d}t} \int_A u\, \mathrm{d}A = \rho\, \Delta x\, \frac{\partial Q}{\partial t}, \tag{7.6}$$

   where the integral $\int_A u\, \mathrm{d}A$ has a simple and practical significance – it is just the discharge Q, so that the contribution of the term can be written simply as shown, but where again it has been necessary to use the partial differentiation symbol, as Q is a function of x as well. No additional approximation has been made in obtaining this term. It can be seen that the discharge Q plays a simple role in the momentum of the flow.

2. **Fluid inertia term**

   The second term on the left of equation (7.5) is $\int_{\mathrm{CS}} \rho u\, \mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}}\, \mathrm{d}S$, has its most important contributions from the stationary vertical faces perpendicular to the main flow.

   a. **Top and bottom, possibly moving surfaces:** we have chosen our control surface to coincide with these boundaries so that no fluid crosses them, $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}} = 0$ and there is no contribution.

b. **Stationary vertical faces:** on the upstream face, $\mathbf{u}_\mathrm{r}\cdot\hat{\mathbf{n}} = -u$, giving the contribution to the term of $-\rho\int_A u^2\,\mathrm{d}A$. The downstream face at $x+\Delta x$ has a contribution of a similar nature, but positive, and where all quantities have increased over the distance Δx. The net contribution, the difference between the two, after neglecting terms like $(\Delta x)^2$, can be written

$$\rho\,\Delta x\,\frac{\partial}{\partial x}\int\limits_A u^2\,\mathrm{d}A.$$

In equation (2.12), much earlier, we approximated the integral over the cross section and with a mean in time, in terms of a Boussinesq coefficient β such that the contribution to the equation is then simply but approximately.

$$\rho\,\Delta x\,\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right). \tag{7.7}$$

It is useful to retain the β, unlike many presentations that implicitly assume it to be 1.0, as it is a signal and reminder to us that we have introduced an approximation.

c. **Lateral momentum contributions:** If there is also fluid entering or leaving from rainfall, tributaries, or seepage, there are contributions over the other faces. Their contribution to momentum exchange is small and uncertain and we will ignore them.

3. **Contributions to force** P_x

a. **Body force**: for the straight channel considered, the only body force acting is gravity; we will consider only the x-component of the momentum equation, which have chosen to be horizontal, as gravity only has a component in the $-z$ direction, there will be no contribution from gravity to our equation! The manner in which gravity acts is to cause pressure gradients in the fluid, giving rise to the following term, due to pressure variations around the control surface.

b. **Pressure forces**: these act normally to the control surface. The direction of the pressure force on the fluid at the control surface is $-\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward-directed normal; its local magnitude is $p\,\mathrm{d}S$, where p is the pressure and dS an elemental area of the control surface. Hence, the total pressure force is the integral $-\int_{\text{CS}} p\hat{\mathbf{n}}\,\mathrm{d}S$. This is difficult to evaluate for arbitrary control surfaces, as the pressure and the non-constant unit vector have to be integrated over all the faces. A much simpler derivation is obtained if the term is evaluated using Gauss' Divergence Theorem:

$$-\int_{\text{CS}} p\hat{\mathbf{n}}\,\mathrm{d}S = -\int_{\text{CV}} \nabla p\,\mathrm{d}V,$$

where $\nabla p = (\partial p/\partial x, \partial p/\partial y, \partial p/\partial z)$, the vector gradient of pressure. This has turned a complicated surface integral into a volume integral of a simpler quantity.

We only need the x component $-\int_{\text{CV}} \partial p/\partial x\,\mathrm{d}V$, the volume integral of the streamwise pressure gradient. The hydraulic approximation now has a problem, because we have not attempted to calculate the detailed pressure distribution throughout the flow. However, in most places in most channel flows the length of disturbances is much greater than the depth, so that streamlines in the flow are only very gently sloping and gently curved, and the pressure in the fluid is accurately given by the equivalent hydrostatic pressure, that due to a stationary column of water of the same depth. Hence we write for a point of elevation z, our equation (2.5) gives

$$p = \rho g \times \text{Depth of water above point} = \rho g(\eta - z),$$

where η is the elevation of the free surface above that point. The quantity that we need is the horizontal pressure gradient $\partial p/\partial x = \rho g\, \partial\eta/\partial x$, and so the streamwise pressure gradient is entirely due to the slope of the free surface. The contribution is

$$-\int\limits_A \frac{\partial p}{\partial x}\,\mathrm{d}V \approx -\rho\,\Delta x\, g \int\limits_A \frac{\partial \eta}{\partial x}\,\mathrm{d}A \approx -\rho\,\Delta x\, gA\frac{\partial \eta}{\partial x}, \tag{7.8}$$

where any variation with y has been ignored, as the surface elevation usually varies little across the channel, and so $\partial\eta/\partial x$ is constant over the section and has been able to be taken outside the integral, which is then simply evaluated.

c. **Resistance due to shear:** there is little that we can say that is exact about the shear forces. We have already considered resistance in some detail, however, and in equation (3.21) we we have

$$\frac{\tau}{\rho} = \Lambda U^2 = \Lambda\left(\frac{Q\,(x,t)}{A\,(x,t)}\right)^2 = \frac{1}{\gamma^2(\varepsilon)}\left(\frac{Q\,(x,t)}{A\,(x,t)}\right)^2,$$

where we could use a value of Λ from the figure on page 45 or from the formulae given therein, or we could use our result for the Gauckler-Manning-Strickler formula where $\gamma(\varepsilon) = 6.7/\varepsilon^{1/6}$. The value of τ is the mean around the solid boundary, so to obtain the force we multiply by the wetted perimeter P and length of the element Δx and instead of Q^2 we write $-Q\,|Q|$, to allow for possible negative Q in estuaries, as resistance always opposes the motion:

$$\text{Total horizontal shear force on control surface} = -\rho\,\Delta x\,\Lambda \frac{Q\,|Q|}{A^2} P. \tag{7.9}$$

Collection of terms and discussion

Now all contributions from the hydraulic approximations to terms in equation (7.5) are collected, using equations (7.6), (7.7), (7.8), and (7.9), and bringing all derivatives to the left and others to the right, all divided by $\rho\Delta x$, gives the momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\Lambda P \frac{Q\,|Q|}{A^2}. \tag{7.10}$$

It is convenient to generalise the resistance term so as to be able to incorporate Gauckler-Manning-Strickler resistance as well as situations where a *Rating Curve* is known from river measurements, giving a relationship between measured discharge, supposed steady and uniform, and local cross-sectional area, $Q_{\mathrm{r}}(A)$. We write the equation as

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -\Omega Q\,|Q|\,, \tag{7.11}$$

where the coefficient Ω, of dimensions L^{-3}, is a function of resistance coefficient, cross-sectional area, and wetted perimeter.

$$\Omega = \begin{cases} gA\tilde{S}/Q_{\mathrm{r}}^2(A), & \text{in terms of rated discharge } Q_{\mathrm{r}}(A); \\ \Lambda P/A^2, & \text{Chézy-Weisbach, where } \Lambda = \lambda/8 = g/C^2; \\ gP^{4/3}/k_{\mathrm{St}}^2 A^{7/3}, & \text{Gauckler-Manning-Strickler.} \end{cases} \tag{7.12}$$

Example 5 Verify the use of the three resistance forms for steady uniform flow, on a uniform slope $\tilde{S} = S$.

In this case, the flow is steady so the first term in equation (7.11) is zero, and uniform so that the second is zero. The surface slope $\partial\eta/\partial x = -S$, and as Q is positive, $Q\,|Q| = Q^2$ and the momentum equation (7.11) gives $-gAS = -\Omega Q^2$, so that

$$Q = \sqrt{\frac{gAS}{\Omega}} = \begin{cases} \sqrt{\frac{gAS}{gAS}Q_{\mathrm{r}}^2} = Q_{\mathrm{r}}, & \text{in terms of rated discharge } Q_{\mathrm{r}}(A); \\ \sqrt{\frac{gA^3S}{\Lambda P}} = A\sqrt{\frac{g}{\Lambda}\frac{A}{P}S}, & \text{Chézy-Weisbach;} \\ \sqrt{\frac{gAS}{gP^{4/3}}k_{\mathrm{St}}^2 A^{7/3}} = Ak_{\mathrm{St}}\left(\frac{A}{P}\right)^{2/3}\sqrt{S}, & \text{Gauckler-Manning-Strickler.} \end{cases} \tag{7.13}$$

At this stage the non-trivial assumptions in the derivation are stated, roughly in decreasing order of importance (they are actually not very restrictive at all!):

1. Resistance to flow is modelled empirically. The Navier-Stokes equations are not being used.
2. All surface variation is sufficiently long and of small slope that the pressure throughout the flow

is given by the hydrostatic pressure corresponding to the depth of water above each point.

3. Effects of curvature of the stream course are ignored.
4. In the momentum flux term the effects of both non-uniformity of velocity over a section and turbulent fluctuations are approximated by a momentum or Boussinesq coefficient.
5. Surface elevation η across the stream is constant.

Relating surface slope $\partial\eta/\partial x$ and $\partial A/\partial x$

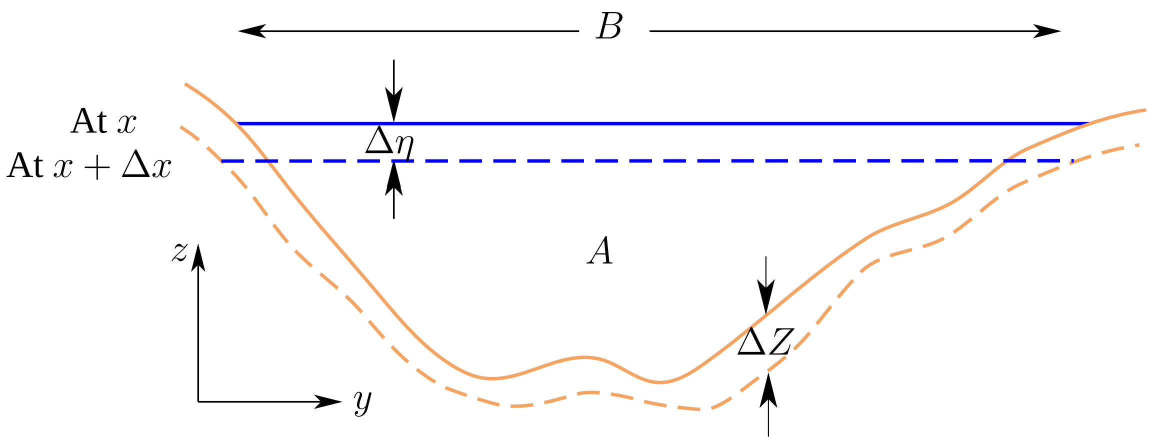


Figure 7.2: Two channel cross-sections separated by Δx

In the momentum equation (7.10) when expanded, the dependent variables are discharge Q and a mixture of derivatives of area $\partial A/\partial x$ and surface elevation $\partial\eta/\partial x$. We must relate the two, and now consider the bottom geometry in greater detail, although in practice the precise details of the bed are often not known. This will help us know when to make approximations.

The cross-section of a river in Figure 7.2 shows how ambiguous and possibly non-unique the concept of the "bottom" of the stream may be. In a distance Δx the surface elevation may change by an amount $\Delta\eta$ as shown, so that the contribution to the change in cross-section area ΔA is $B \times \Delta\eta$, where $\Delta\eta$ is usually negative as the surface drops downstream. The change in the bed is

ΔZ, which in general varies across the section, with contribution to ΔA of $-\int_B \Delta Z\, dy$, the area between the solid and dotted lines on the figure corresponding to the bed at the two locations. The minus sign is because, if the bed drops away and ΔZ is negative, as usual, the contribution to area increase is positive. Combining the two terms,

$$\Delta A = B\,\Delta\eta - \int\limits_B \Delta Z\,\mathrm{d}y \tag{7.14}$$

For the second contribution, the integral of the change in bed elevation across the stream, we introduce the symbol $\tilde{S}$ for the mean downstream bed slope across the section such that

$$\tilde{S} = -\frac{1}{B}\int\limits_B \frac{\partial Z}{\partial x}\,\mathrm{d}y, \tag{7.15}$$

where the negative sign has been introduced such that in the usual case when Z decreases with x, this mean downstream bed slope at a section is positive. In an important problem where bed details might be known, this can be evaluated. In the usual case where bed topography is poorly known, a reasonable local approximation or assumption is made. Using equations (7.14) and (7.15) we can write

$$\Delta A = B\,\Delta\eta + B\tilde{S}\,\Delta x,$$

where in a distance Δx the *mean* bed level across the channel then changes by $-\tilde{S}\times\Delta x$ under the water. In the rare case where the sides of the stream are vertical diverging or converging walls, an extra term would have to be included. Taking the usual calculus limit, we obtain

$$\frac{\partial A}{\partial x} = B \left(\frac{\partial \eta}{\partial x} + \tilde{S} \right), \tag{7.16}$$

which might have been able to have been written down without the mathematical details.

7.3 Forms of the governing equations

We use equation (7.16) to eliminate first $\partial\eta/\partial x$ and then $\partial A/\partial x$ to give two alternative forms of the momentum equation governing flows and long waves in waterways. In both cases, we restate the corresponding mass conservation equation, using (7.1) and (7.4), to give the pairs of equations:

Formulation 1 – Long wave equations in terms of area A and discharge Q

Eliminating $\partial\eta/\partial x$ gives the equations in terms of A and Q:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i, \tag{7.17a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) + \frac{gA}{B}\frac{\partial A}{\partial x} = gA\tilde{S} - \Omega Q\,|Q|\,. \tag{7.17b}$$

Formulation 2 – Long wave equations in terms of stage η and discharge Q

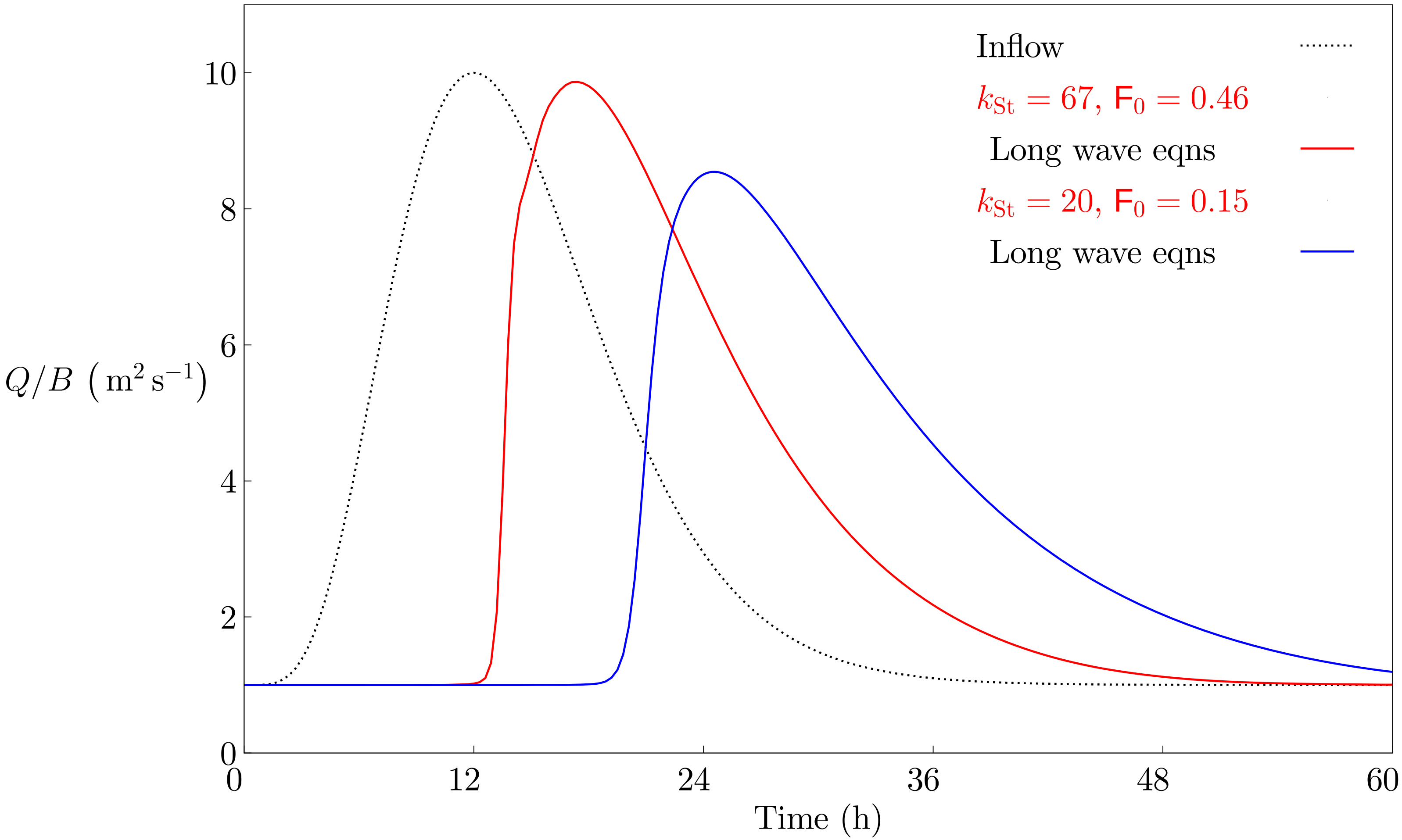
Now eliminating $\partial A/\partial x$, but retaining A in all coefficients, as it can be calculated in terms of η:

$$\frac{\partial \eta}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = \frac{i}{B}, \tag{7.18a}$$

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2 B}{A^2}\right)\frac{\partial \eta}{\partial x} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Omega Q\,|Q|\,. \tag{7.18b}$$

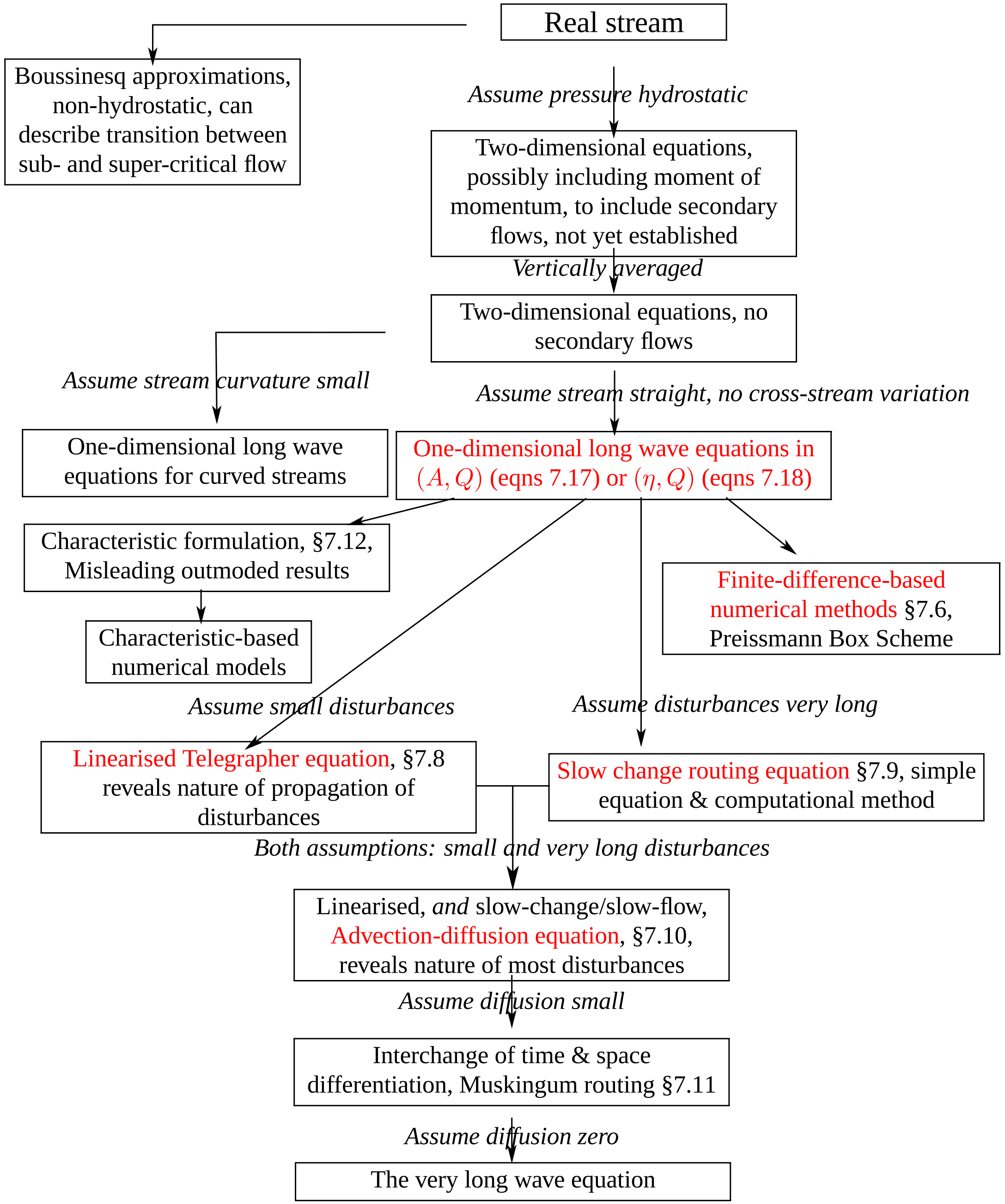
These equations are the basis of computational hydraulics and flood routing. There is much commercial software written to solve them. They are actually quite simple in the form here!

7.4 Examples of flood propagation



As an example we consider an infinitely-wide (no side friction) channel with a channel slope $S = 0.0005$ and length 50 km. Two different boundary resistances were considered, Strickler $k_{St} = 67$ for a smooth boundary to give a large Froude number and $k_{St} = 20$ for a natural boundary.

7.5 Hierarchy of one-dimensional open channel theories and approximations



7.6 Numerical solution of the long wave equations – FTQS scheme

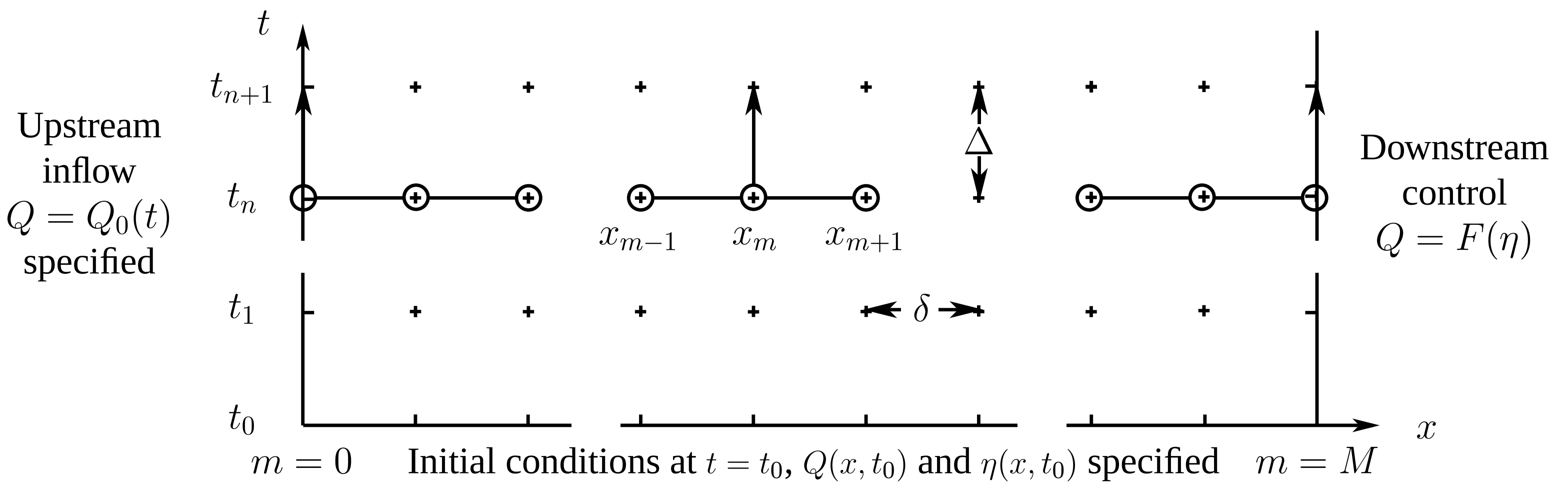


Figure 7.3: (x, t) axes showing computational grid, initial and boundary conditions, and three computational modules

We use a scheme where time derivatives are approximated using forward differences, and where x-derivatives are approximated using quadratic approximation, fitting a quadratic to three points, giving the **F**orward-**T**ime-**Q**uadratic-**S**pace scheme. The x-derivatives are

$$\left.\frac{\partial f}{\partial x}\right|_0 = \frac{-3f_0 + 4f_1 - f_2}{2\delta} + O(\delta)\ , \tag{7.19a}$$

$$\left.\frac{\partial f}{\partial x}\right|_m = \frac{f_{m+1} - f_{m-1}}{2\delta} + O(\delta)\,, \quad \text{for } m = 1, \ldots, M-1\ , \tag{7.19b}$$

$$\left.\frac{\partial f}{\partial x}\right|_M = \frac{f_{M-2} - 4f_{M-1} + 3f_M}{2\delta} + O(\delta)\ . \tag{7.19c}$$

Using the obvious forward difference expressions for the time derivatives, the scheme applied to the (A, Q) formulation, equations (7.17), becomes

$$\frac{A_{m,n+1} - A_{m,n}}{\Delta} = i - \left.\frac{\partial Q}{\partial x}\right|_{m,n}, \tag{7.20a}$$

$$\frac{Q_{m,n+1} - Q_{m,n}}{\Delta} = -\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right) - \frac{gA}{B}\frac{\partial A}{\partial x} + gA\tilde{S} - \Omega Q\left|Q\right|\Big|_{m,n}. \tag{7.20b}$$

Both expressions are easily re-arranged to give explicit formulae for the terms in red, the values of A and Q at point m at the next time level $n+1$, each in terms of three values of A and three values of Q at the computational points at the previous time t_n, using one of the equations (7.19).

Liggett & Cunge (1975) claimed that the above scheme, the simplest and most obvious, was unconditionally unstable. This had some important implications, for it meant that the world was forced into using complicated schemes such as the Preissmann Box scheme, which form the basis of all commercial software. The lecturer (Fenton 2014) has discovered that their analysis is wrong, and that the scheme has a quite acceptable stability limitation, and it opens up the possibility for simpler computations of floods and flows in open channels. The Preissmann Box Scheme allows much larger time steps, but it is very complicated to apply.

7.7 Initial and boundary conditions

Initial conditions

Usually there is some initial flow in the channel which is constant if there is no inflow, $Q(x, t_0) = Q_0$. The next step is to determine the initial distribution of surface elevation η. The conventional method is to solve the Gradually-varied flow equation, using the equations and methods described in §8, as well as the downstream boundary condition, which is about to be described. A simpler method is to use the unsteady equations and computation scheme that will be used later anyway – simply start with an approximate solution for $\eta(x, t_0)$ (a straight line?) and let the unsteady dynamics take over, allowing disturbances to propagate downstream and out of the computational domain until the solution is steady. Then, for example, the main computation can be started.

Boundary conditions

Upstream

It is usually the upstream boundary condition that drives the whole model, where a flood or wave enters, via the specification of the time variation of $Q = Q(x_0, t)$ at the boundary. The surface elevation there is obtained as part of the computations. A common model inflow hydrograph is:

$$Q(x_0, t) = Q_{\text{min}} + (Q_{\text{max}} - Q_{\text{min}}) \left(\frac{t}{T_{\text{max}}} e^{1 - t/T_{\text{max}}} \right)^5,$$

where the event starts at $t = 0$ with Q_{min} and has a maximum Q_{max} at $t = T_{\text{max}}$.

We have two variables, however, Q and either A or η. To obtain this we just use the mass conservation FTQS expression (7.20a) to obtain the updated value of A or η at $m = 0$ at t_{n+1}. The equation of course applies up to and including the boundary point.

Downstream boundary - known stage-discharge relationship

- Where there is a downstream control structure such as a spillway, weir, gate, or flume, the stage-discharge relationship $Q(x_M, t) = F(\eta(x_M, t))$ must be known. For example, a weir might have a flow formula such as

$$Q = 0.6\sqrt{g}b\,(\eta - z_\mathrm{c})^{3/2}$$

  where b is the crest length and z_c is the elevation of the crest.
- We assume that the $Q = F(\eta)$ relationship is not affected by unsteadiness and non-uniformity, which probably holds for relatively short control structures mentioned
- A potential difficulty – we have one equation too many: we have the FTQS finite difference formulae based on mass conservation for $\eta(x_M, t_{n+1})$ and momentum conservation for $Q(x_M, t_{n+1})$ *and* the relation between Q and η
- However, a sudden change in section where a typical spillway, weir, gate, or flume is placed actually violates a fundamental assumption of the long wave momentum equation, that variation in the channel is long. We can easily ignore that equation near such a sudden change
- Fortunately, the mass conservation equation, is still valid near a sudden change – it requires only the assumption that water surface is horizontal across the channel.
- The procedure is: obtain the updated value $\eta(x_M, t_{n+1})$ from the FTQS finite difference formula for the *mass conservation* equation (7.20a), using values of Q at x_{M-2}, x_{M-1}, x_M and t_n and then use the *stage-discharge relationship* to calculate $Q(x_M, t_{n+1}) = F(\eta(x_M, t_{n+1}))$

Open downstream boundary

- A common boundary is where the computational domain is artificially truncated at some point in the stream. This is sometimes called *Normal Depth* boundary and standard practice is that the computational domain be artificially extended and this boundary condition be used far enough downstream from the study area that it does not affect the results there.
- The lecturer prefers a different approach, and this is simply to treat the boundary as if it were just any other part of the river (which it is!) and to use both long wave equations to update both η and Q there, calculating the necessary derivatives $\partial\eta/\partial x$ and $\partial Q/\partial x$ from upstream finite difference formulae and simply treating the end point as if it were an ordinary point in the stream and using *both* FTQS formulae for $\eta\left(x_M, t_{n+1}\right)$ and $Q\left(x_M, t_{n+1}\right)$ there, with the three-point leftwards approximations for the last point x_M in terms of values at x_{M-2}, x_{M-1}, and x_M.
- This works very well in practice.

7.8 Nature of the equations and solutions

The Telegraph equation as a model for long wave propagation

We recall the long wave equations in terms of area, equations (7.17):

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=i,$$
$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+\frac{gA}{B}\frac{\partial A}{\partial x}=gA\tilde{S}\left(1-\frac{Q^2}{Q_{\mathrm{r}}^2(A)}\right),$$

where we have now written the resistance term in terms of the function of area $Q_{\mathrm{r}}(A)$ which is the Rating Curve relationship, or that given by, say, the Gauckler-Manning-Strickler formula. For steady uniform flow such that all derivatives on the left are zero, the equation becomes $Q=Q_{\mathrm{r}}(A)$, which we require. We also recall the relationships (7.3) for the upstream volume:

$$\frac{\partial V}{\partial x}=A \quad \text{and} \quad \frac{\partial V}{\partial t}=\int^{x} i\left(x'\right)\,\mathrm{d}x'-Q. \tag{7.21}$$

Substituting these into the mass conservation equation, as we did in §7.1.1, we find that it is identically satisfied (we expect volume to satisfy a volume conservation equation). The momentum conservation equation becomes

$$\frac{\partial^2 V}{\partial t^2}+2\beta\frac{Q}{A}\frac{\partial^2 V}{\partial x\partial t}+\left(\beta\frac{Q^2}{A^2}-\frac{gA}{B(A)}\right)\frac{\partial^2 V}{\partial x^2}+gA\tilde{S}\left(1-\left(-\frac{\partial V/\partial t}{Q_{\mathrm{r}}(A)}\right)^2\right)=0$$

where symbols Q and A have been retained in coefficients of second derivatives.

- The momentum equation has become a second-order partial differential equation in terms of the single variable V.
- And it is unusable in this ugly form. It is more useful in theoretical works and where approximations can be made, as we now do.
- We linearise the equation by considering relatively small disturbances about a uniform flow with area A_0 and discharge Q_0. Substituting the series

$$V = A_0 x - Q_0 t + \varepsilon v,\ A = A_0 + \varepsilon v_x,\ Q = Q_0 - \varepsilon v_t,\ \text{and } Q_{\mathrm{r}}(A) = Q_0 + Q_0'\,\varepsilon v_x,$$

where εv is a small quantity, a deviation of upstream volume from that of uniform flow, $v_t = \partial v/\partial t$, $v_x = \partial v/\partial x$, and $Q_0' = \mathrm{d}Q_{\mathrm{r}}/\mathrm{d}A|_0$.

Performing power series operations, the second order derivatives can simply be written down with constant coefficients. The gravity and resistance terms become, making use of the power series expansion to first order, $(1+\varepsilon a)^n = 1 + \varepsilon n a + \ldots$:

$$\begin{aligned} g\left(A_0 + \varepsilon v_x\right) S_0 \times \left(1 - \left(\frac{Q_0 - \varepsilon v_t}{Q_0 + Q_0'\,\varepsilon v_x}\right)^2\right) &= g A_0 S_0 \times \left(1 - \left(\frac{1 - \varepsilon v_t/Q_0}{1 + Q_0'/Q_0\,\varepsilon v_x}\right)^2\right) \\ &= g A_0 S_0 \times \left(1 - \left(1 - 2\varepsilon v_t/Q_0\right)\left(1 - 2Q_0'/Q_0\,\varepsilon v_x\right)\right) \\ &= g A_0 S_0 \times \left(1 - \left(1 - 2\varepsilon v_t/Q_0 - 2Q_0'/Q_0\,\varepsilon v_x\right)\right) \\ &= \varepsilon \frac{2 g A_0 S_0}{Q_0}\left(\frac{\partial v}{\partial t} + Q_0' \frac{\partial v}{\partial x}\right). \end{aligned}$$

We obtain the linearised momentum equation as the *Telegraph equation*:

$$\sigma_0 \left(\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} \right) + \frac{\partial^2 v}{\partial t^2} + 2\beta U_0 \frac{\partial^2 v}{\partial x \partial t} - (C_0^2 - \beta^2\, U_0^2) \frac{\partial^2 v}{\partial x^2} = 0. \tag{7.22}$$

- σ_0 – **resistance parameter / inverse time scale**: this is actually an important channel parameter, determining the nature of wave behaviour and computational solution properties

$$\sigma_0 = \frac{2gA_0S_0}{Q_0} = 2\frac{gS_0}{U_0} = \frac{\partial}{\partial Q} \left(gAS\frac{Q^2}{Q_\mathrm{r}^2} \right) \bigg|_0$$

  It is the derivative with respect to Q of the resistance term in the momentum equation. We could argue by a rough electrical analogy that the resistance term in the momentum equation is equivalent to potential difference or voltage, while discharge Q is equivalent to current. As the derivative of voltage with respect to current gives electrical resistance, σ_0 can be thought of as a *resistance parameter* in our nonlinear case.

- c_0 – **wave speed**: This will be shown to be the speed of very long period waves, which means for us the propagation speed of flood waves:

$$c_0 = \frac{\mathrm{d}Q_\mathrm{r}}{\mathrm{d}A} \bigg|_0 .$$

  For the Gauckler-Manning-Strickler equation, $Q_\mathrm{r} = k_\mathrm{St} A^{5/3}/P^{2/3}\sqrt{S}$, for a wide stream, ignoring change of P with A, this gives $c_0 \approx \frac{5}{3}U_0$, so that a good estimate of the speed of propagation of a flood wave is to multiply the stream velocity by $5/3$. This velocity is important, and will be studied more practically later.

- $U_0 = Q_0/A_0$ – **mean fluid velocity**: used for simplicity.
- C_0 – **the speed of not-so-long waves**:

$$C_0 = \sqrt{gA_0/B_0 + \left(\beta^2 - \beta\right) U_0^2},$$

  In most textbooks this is written, not unreasonably, implicitly with $\beta = 1$ such that $C_0 = \sqrt{gA_0/B_0}$, which is usually said to be the "celerity" or "long wave speed" or "dynamic wave speed". Below it will be shown that it is actually the speed of waves only in the limit of shorter waves, but still long enough that the hydrostatic approximation holds. We call these "not-so-long" waves. They occur when waves are due to rapid gate movements. This velocity is less-important than is generally believed.

We now obtain some simple solutions to the Telegraph equation in two limits.

Very long waves – the longest flood waves

- For disturbances with a long period, such that $\partial^2/\partial t^2 \ll \sigma_0 \partial/\partial t$, "very long waves", the last three terms in the equation can be neglected, and it becomes the advection equation

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} = 0, \qquad \text{(Very long wave equation)}$$

$$\text{Solution: } v = f_1\left(x - c_0 t\right), \qquad (7.23)$$

where $f_1\left(.\right)$ is an arbitrary function given by the upstream conditions. To show this consider a moving variable $X = x - c_0 t$, and $v = f_1(X)$. By the *chain rule* for partial differentiation,

$$\frac{\partial v}{\partial t} = \frac{\partial f_1(X)}{\partial t} = \frac{df_1(X)}{dX}\frac{\partial X}{\partial t} = -c_0 \frac{df_1(X)}{dX}, \quad \text{and}$$

$$\frac{\partial v}{\partial x} = \frac{\partial f_1(X)}{\partial x} = \frac{df_1(X)}{dX}\frac{\partial X}{\partial x} = 1 \times \frac{df_1(X)}{dX},$$

and the equation is satisfied for *any* $f_1(X)$, whatever the upstream conditions determine.

- This solution is a wave propagating downstream at speed c_0 with no change or diffusion.
- The equation has been known as the "kinematic wave equation" and c_0 the "kinematic wave speed", because the approximation has previously been believed to be such that dynamic terms of order F^2 in the momentum equation have been neglected.
- Here we have shown that the only approximation has been that the wave period is long. No approximation has been made by neglecting dynamical terms. A better name is the *Very Long Wave Equation*, VLWE.

Not-so-long waves – in the shorter limit of waves from the long wave equations

- In the other limit, for disturbances which are shorter, such that $\partial^2/\partial t^2 \gg \sigma_0 \partial/\partial t$, for which we use the term "not-so-long" waves, the Telegraph equation becomes
$$\frac{\partial^2 v}{\partial t^2} + 2\beta U_0 \frac{\partial^2 v}{\partial x \partial t} - (C_0^2 - \beta^2 U_0^2)\frac{\partial^2 v}{\partial x^2} = 0,$$
which is a second-order wave equation with solutions
$$v = f_{21}\left(x - \left(\beta U_0 + C_0\right) t\right) + f_{22}\left(x - \left(\beta U_0 - C_0\right) t\right)$$
where $f_{21}\left(.\right)$ and $f_{22}\left(.\right)$ are arbitrary functions determined by boundary conditions both upstream and downstream.
- In this case the solutions are waves propagating upstream and downstream at velocities of $\beta U_0 \pm C_0$, such that in the usual terminology C_0 is the "long wave speed", and the waves travel relative to an advection velocity βU_0, where the presence of β is slightly surprising.
- We have shown here that C_0 is the speed of waves that are actually not so long, apparently paradoxically – they are long enough that the pressure distribution in the fluid is still hydrostatic, but they are short in terms of time scales given by the resistance characteristics.

Intermediate period waves

- In the general case, solutions of the long wave equations show wave propagation characteristics, velocity and rate of decay, that depend on the period of the waves, so that the waves are actually
  - diffusive – different period components decay at different rates, and
  - dispersive – different components travel at different speeds
- One can obtain solutions for the propagation behaviour in terms of wave period, but the operations are complicated, and they are not included here.
- The widespread belief, printed in all textbooks, is wrong, that all waves obeying the long wave equations travel at a speed $C \approx \sqrt{gA/B} \approx \sqrt{g \times \text{Depth}}$. The behaviour is very much more complicated. There is no such thing as "**the** long wave speed".

Solving the long wave equations numerically overcomes all such problems, but it is nice to know what physical processes are at work.

7.9 Slow change routing equation

The Telegraph equation (equation 7.22 on page 93) is

$$\sigma_0\left(\frac{\partial v}{\partial t}+c_0\frac{\partial v}{\partial x}\right)+\frac{\partial^2 v}{\partial t^2}+2\beta U_0\frac{\partial^2 v}{\partial x\partial t}+\beta^2U_0^2\frac{\partial^2 v}{\partial x^2}-C_0^2\frac{\partial^2 v}{\partial x^2}=0.$$

Previously for very long waves, we included just the first σ_0 terms. We now make a rather better approximation, including the last term, ignoring just the terms shown light blue. The corresponding full momentum conservation equation is

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+\frac{gA}{B}\frac{\partial A}{\partial x}=gA\tilde{S}\left(1-\frac{Q^2}{Q_{\mathrm{r}}^2}\right).$$

It can be shown that neglecting those first two terms of the momentum equation, the time derivative and the fluid momentum term, one is making, surprisingly, not a low-Froude number "kinematic" approximation as has been believed, but a *Very Long Wave* or *Slow Change* approximation. The criterion is in terms of the dimensionless time scale $\sigma_0 T$ where $\sigma_0 = 2gS/U_0$, in which U_0 is the mean flow velocity, and T is the time scale.For the approximation to be accurate, waves should be sufficiently long that $\sigma_0 T \gtrsim 20$, where here T is wave period. The equation can be re-arranged to give the momentum equation in the form of a simple expression for Q for the period limitation shown:

$$Q(A)=\underbrace{Q_{\mathrm{r}}(A)}_{\text{Steady, uniform}}\times\sqrt{1-\frac{1}{B(A)S}\frac{\partial A}{\partial x}},\quad\text{for}\quad gST/U_0\gtrsim 10\,. \tag{7.24}$$

It is interesting that just using the rating expression $Q = Q_\mathrm{r}(A)$ is already an approximation to the momentum equation.

This approximate simple momentum equation is surprisingly accurate, and can be used for long wave propagation problems in streams. It is possible to substitute it into the mass conservation equation to give a single partial differential equation in area $A\,(x,t)$. However, that is complicated, and inflow boundary conditions are often expressed in terms of a discharge hydrograph, when the area A formulation is not so convenient.

A more general approach in terms of *Upstream Volume* can be developed. Substituting for $A = \partial V/\partial x$ and $Q = -\partial V/\partial t$ gives the single equation in the single dependent variable V:

$$\frac{\partial V}{\partial t} + Q_\mathrm{r}\,(\partial V/\partial x)\,\sqrt{1 - \frac{1}{S\,B(\partial V/\partial x)}\frac{\partial^2 V}{\partial x^2}} = 0\ , \tag{7.25}$$

where both breadth B and Q_r have been written as functions of area $A = \partial V/\partial x$. We call this the *Slow change routing equation*. It is a single equation in a single unknown. The only approximation relative to the long wave equations has been that the variation with time is slow, such that $gST/U \gtrsim 10$. Boundary conditions involving discharge Q or stage η can be incorporated using equations (7.21) and the geometrical relationship between A and surface elevation η at a point.

The equation can be used for simulations, for which it is necessary to use the quadratic approximation to the second derivative, similar to those for the first derivative in equation (7.19) on page 85:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_m = \frac{f_{m-1} - 2f_m + f_{m+1}}{\delta^2},$$

and as the second derivative of a quadratic function is constant, this is the value used also at $m-1$ and $m+1$ at boundaries if necessary. For most flood routing problems the slow change routing equation gives results as good as the long wave equations. For short waves it is less accurate. The only approximation relative to the long wave equations has been that the variation with time is slow, such as for flood waves.

7.10 Advection-diffusion model equation

In fact, the *slow change routing equation* is more use to us here for purposes of understanding – to show us the *nature* of solutions and how waves in rivers behave.

It is a fully-nonlinear (no assumption of smallness) advection-diffusion equation, which is made clearer if we write $Q_\mathrm{r} = U_\mathrm{r} A$, where U_r is the mean flow velocity as given by Gauckler-Manning-Strickler *etc.*:

$$\frac{\partial V}{\partial t} + U_\mathrm{r}\left(\partial V/\partial x\right)\frac{\partial V}{\partial x}\sqrt{1 - \frac{1}{S\,B(\partial V/\partial x)}\frac{\partial^2 V}{\partial x^2}} = 0\ , \tag{7.26}$$

so that in this form with little approximation, the advective term with the single derivative $\partial V/\partial x$ is multiplied by a term containing a diffusive correction with $\partial^2 V/\partial x^2$.

Advection term

To the lowest level of approximation, when waves are so long that diffusion plays no role, equation (7.26) becomes the nonlinear advection equation

$$\frac{\partial V}{\partial t} + U_{\mathrm{r}}(V_x)\frac{\partial V}{\partial x} = 0\ . \tag{7.27}$$

This generally nonlinear equation has some interesting properties. Superficially it seems that volume V is advected at a velocity of U_{r}, the mean velocity of flow in the stream, as one might expect. However nonlinearity of the equation causes us a surprise. We consider a general unsteady flow superimposed on a steady uniform flow, of area A_0 and discharge $Q_0 = U_0 A_0$, such that, as previously to obtain the Telegraph equation, we write $V = A_0 x - U_0 A_0 t + v\,(x,t)$, where $v\,(x,t)$ is the unsteady or non-uniform contribution. Substituting into equation (7.27)

$$-U_0 A_0 + \frac{\partial v}{\partial t} + U_{\mathrm{r}}\left(A_0 + \frac{\partial v}{\partial x}\right) \times \left(A_0 + \frac{\partial v}{\partial x}\right) = 0\ ,$$

and now assuming that the unsteady/non-uniform terms $\partial v/\partial t$ and $\partial v/\partial x$ are small compared with the underlying flow, and taking a Taylor expansion of U_{r} about A_0, we obtain the advection equation

$$\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} + \left(U_0 + A_0\left.\frac{dU_{\mathrm{r}}}{dA}\right|_0\right)\frac{\partial v}{\partial x} = 0. \tag{7.28}$$

We have obtained, rather more simply this time, the very long wave equation (Very long wave equation). It is now clearer that, even if the underlying flow is carried at a velocity U_0, any variation in the flow is advected at the very long wave speed c_0 given by the Kleitz-Seddon formula:

$$c_0(A) = U_0 + A_0 \left.\frac{dU_\mathrm{r}}{dA}\right|_0 = \left.\frac{dQ_\mathrm{r}}{dA}\right|_0 . \tag{7.29}$$

This nonlinear mathematical artefact is not so obvious, physically! Evaluating c_0 for some cases, we find

$$c_0 = \begin{cases} \mathrm{d}Q_\mathrm{r}/\mathrm{d}A|_0 \,, & \text{General expression} \\ \frac{3}{2}U_0\left(1 - \frac{1}{3}A_0 P_0'/P_0\right) , & \text{Chézy-Weisbach} \\ \frac{5}{3}U_0\left(1 - \frac{2}{5}A_0 P_0'/P_0\right) , & \text{Gauckler-Manning-Strickler} \end{cases} ,$$

where $P_0' = \mathrm{d}P/\mathrm{d}A|_0$.

Example 6 Estimate the effect of side resistance on flood wave speed for a river of bottom width $20\ \mathrm{m}$, side slopes 2:1 (H:V) and a depth of $2\ \mathrm{m}$.

Using our relationships $B = W + 2mh$, $A = h\,(W + mh)$, and $P = W + 2\sqrt{1+m^2}h$ we have $\mathrm{d}P/\mathrm{d}h = 2\sqrt{1+m^2}$ and $\mathrm{d}A/\mathrm{d}h = B$

$$\begin{aligned} -\frac{2}{5}\frac{A_0}{P_0}\left.\frac{\mathrm{d}P}{\mathrm{d}A}\right|_0 &= -\frac{2}{5}\frac{A_0}{P_0}\left.\frac{\mathrm{d}P/\mathrm{d}h}{\mathrm{d}A/dh}\right|_0 = -\frac{2}{5}\frac{A_0}{P_0}\left.\frac{\mathrm{d}P/\mathrm{d}h}{B}\right|_0 = -\frac{2}{5}\frac{h\,(W+mh)}{W+2\sqrt{1+m^2}h}\frac{2\sqrt{1+m^2}}{W+2mh} \\ &= -\frac{2}{5}\frac{2\,(10+2\times 2)}{10+2\times 2\sqrt{1+2^2}}\frac{2\sqrt{1+2^2}}{10+2\times 2\times 2} = -15\% \end{aligned}$$

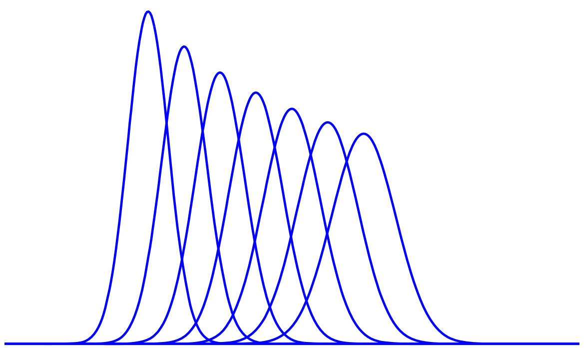
Diffusion term

Now we consider the effects of the "diffusion term", with the second derivative.

In physics, the process of diffusion occurs because of a continuous process of random particle movements, where any irregularities in concentration ϕ of a substance are smoothed out. In a stationary medium, the governing diffusion equation is

$$\frac{\partial \phi}{\partial t} = \kappa \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right), \qquad \text{(Diffusion equation)}$$

where κ is the coefficient of diffusion. The significance of the equation is that any regions of "curvature" (where there is not a linear variation of ϕ) will be smoothed out. For example, near a local maximum in concentration, even in just one dimension, $\partial^2\phi/\partial x^2$ is negative, and this means that $\partial\phi/\partial t$ there is negative, and the concentration is reduced. The reverse applies near a minimum. Quantities that show diffusive behaviour are temperature, electric charge, and pollution concentration.



Linearising the slow change routing equation, now including the second derivative term, gives the *Advection-diffusion equation*:

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} - \frac{Q_0}{2B_0 S} \frac{\partial^2 v}{\partial x^2} = 0, \qquad \text{(Advection-diffusion equation)}$$

and we see that the diffusion coefficient is given by $Q_0/2B_0S$, where B_0 is the width of the undisturbed stream. Typical solutions of the equation are shown in the figure. This is a good simple approximate model of flood propagation. It tells us most importantly, how fast the flood wave moves.

As a first estimate, we might assume that diffusion is unimportant, and that the flood peak might be the same downstream as it was upstream (as implied by the very long wave advection equation). For practical problems, however, the problem then arises as how to estimate the importance of diffusion. We can go back to the momentum equation (7.24) and consider the term which leads to diffusion

$$\left(1 - \frac{1}{B(A)S}\frac{\partial A}{\partial x}\right)^{1/2}.$$

This is not a particularly helpful criterion, as we do not know what the x-derivative is until we solve a problem. However, we can use our knowledge that, *to first approximation*, the flood wave propagates unchanged, satisfying the very long wave equation

$$\frac{\partial A}{\partial t} + c_0\frac{\partial A}{\partial x} = 0.$$

The term containing diffusion, using a power series expansion to first order becomes

$$\begin{aligned} 1 - \tfrac{1}{2}\frac{1}{B(A)S}\frac{\partial A}{\partial x} &= 1 + \tfrac{1}{2}\frac{1}{c_0 B(A)S}\frac{\partial A}{\partial t} \\ &= 1 + \tfrac{1}{2}\frac{1}{c_0 S}\frac{\partial \eta}{\partial t}, \end{aligned}$$

where η is the surface elevation, such that $\partial A/\partial t = B\partial\eta/\partial t$. We have the result

$$\text{Relative importance of diffusion} = \frac{1}{2c_0 S}\frac{\partial \eta}{\partial t}. \tag{7.30}$$

This has an interesting simple physical significance: we can consider it to be

$$\text{Relative importance of diffusion} = \frac{1}{2}\frac{\text{Vertical water surface velocity / Horizontal water surface velocity}}{\text{Vertical fall of bed / Horizontal distance}}.$$

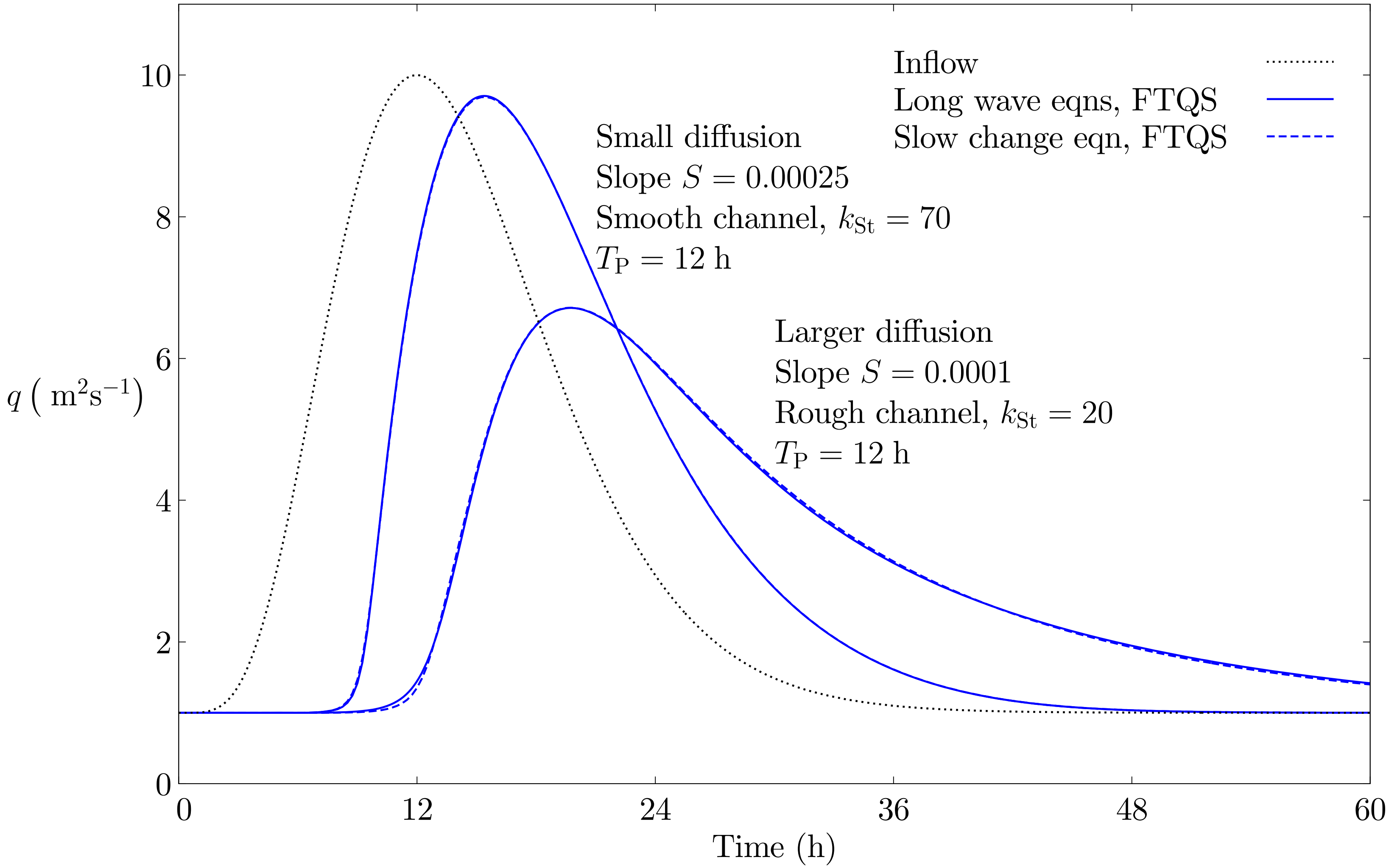
At the beginning of any problem one does not know $\partial\eta/\partial t$. However, it is not difficult to estimate it to give the relative importance of diffusion. Historical records should give us an approximate idea of the maximum (**P**eak) flood level η_{P} to be expected, as well as the time at which it occurs, or which value is given for a particular event. In this case we have

$$\text{Relative importance of diffusion} = \frac{1}{2c_0 S}\frac{\eta_{\mathrm{P}} - \eta_0}{t_{\mathrm{P}} - t_0}, \tag{7.31}$$

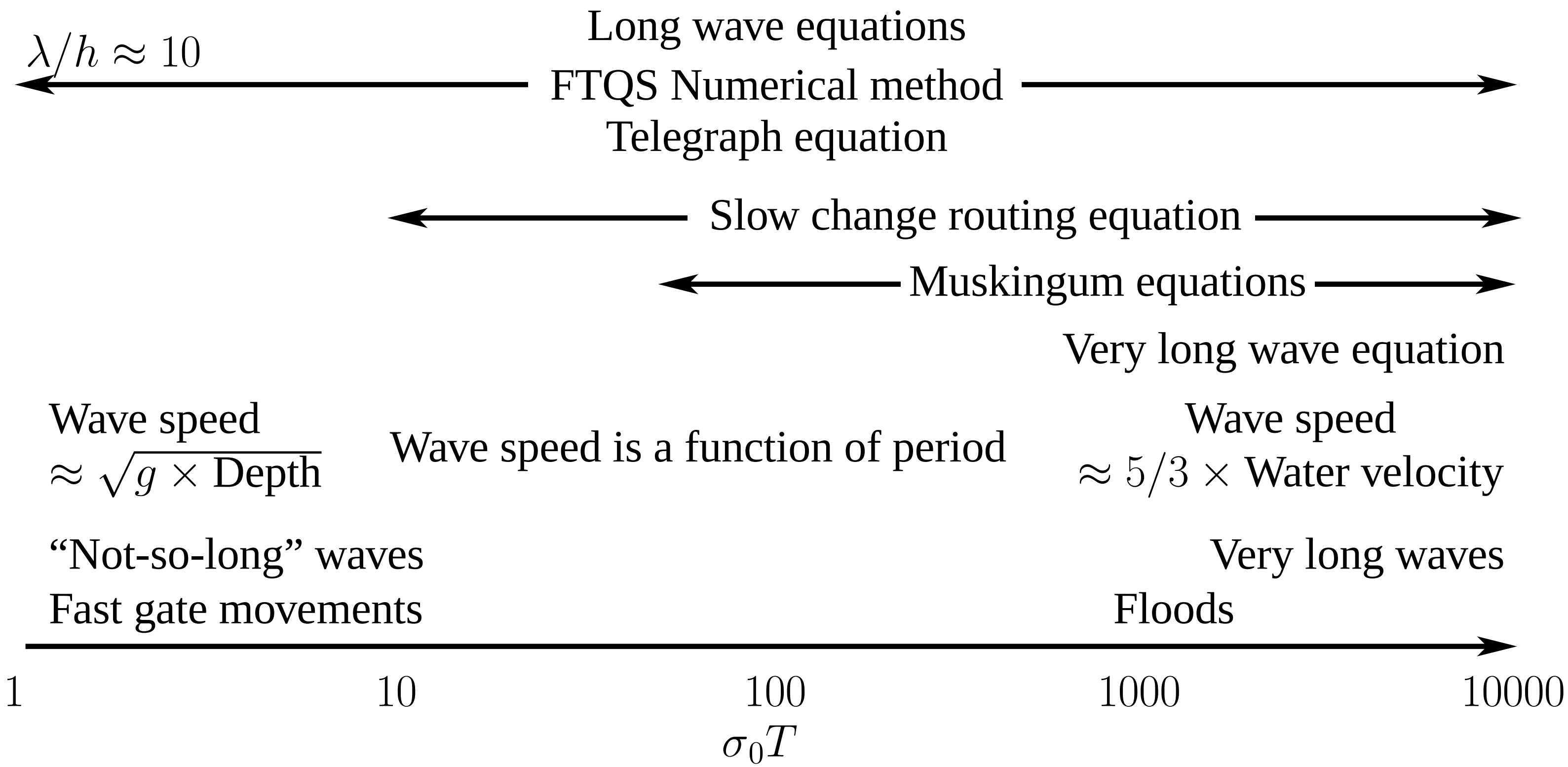
and this is enough for an estimate. Our expressions might also be useful to give us a general idea of when diffusion is important. As c_0 is proportional to U_0 and that is proportional to $\sqrt{S}$, whether Gauckler-Manning-Strickler or Chézy-Weisbach resistance formula are used, we obtain the important result that the relative importance of diffusion is proportional to $S^{-3/2}$, which is a very strong result showing that diffusion is more important for small slopes.

Two examples, showing smaller and greater diffusion effects

The figure shows results from two computations, for a $50\,\mathrm{km}$ length of river, first with a very smooth boundary and steeper slope so that diffusion is not so large, the second for a rougher boundary and smaller slope.



Summary of theories, names, and ranges of application



7.11 Muskingum methods

The advection-diffusion equation, re-written with the symbol D_0 for the coefficient of diffusion, $D_0 = Q/2B_0S$, is

$$\frac{\partial v}{\partial t} + c_0\frac{\partial v}{\partial x} - D_0\frac{\partial^2 v}{\partial x^2} = 0. \qquad \text{(Advection-diffusion equation)}$$

It is a good simple approximate model of flood propagation – not as good as the fully nonlinear *slow change routing equation*. Both have, however, a finite stability criterion – strangely, the numerical simulation with the second derivative diffusion term makes the computation less stable! However, numerical solution is not a problem – one simply takes smaller steps until it works. In the last 40 years, however, there have been a large number of papers published using *Muskingum methods*, named after a river in the USA where such a method was first applied. They mimic the advection-diffusion equation, are supposed to be simple and plausibly seem so, and have been widely used. People have obtained the methods, sometimes from a simple reservoir routing approach, sometimes from the long wave equations, using long, complicated and arbitrary methods. The problem is to obtain a single finite difference equation in a single variable. (Using upstream volume V solves that problem rather better!). A typical Muskingum scheme is written, where c_m^n is the very long wave speed $dQ_\mathrm{r}/dA|_m^n$, and D_m^n is the coefficient of diffusivity $D_m^n = Q_m^n/2B_m^nS$, and so on:

$$(-\delta - \Delta c_m^n + 2D_m^n/c_m^n)\, Q_m^n + (-\delta + \Delta c_{m+1}^n - 2D_{m+1}^n/c_{m+1}^n)\, Q_{m+1}^n$$
$$+ \left(\delta - \Delta c_m^{n+1} - 2D_m^{n+1}/c_m^{n+1}\right) Q_m^{n+1} + \left(\delta + \Delta c_{m+1}^{n+1} + 2D_{m+1}^{n+1}/c_{m+1}^{n+1}\right) Q_{m+1}^{n+1} = 0\ .$$

This can be re-arranged to give an explicit expression for Q_{m+1}^{n+1}, the top right point shown in the blue computational module in Figure 7.4 in terms of the known two values of the discharge at time n, Q_m^n and Q_{m+1}^n and the known value at the previous space point Q_m^{n+1}.

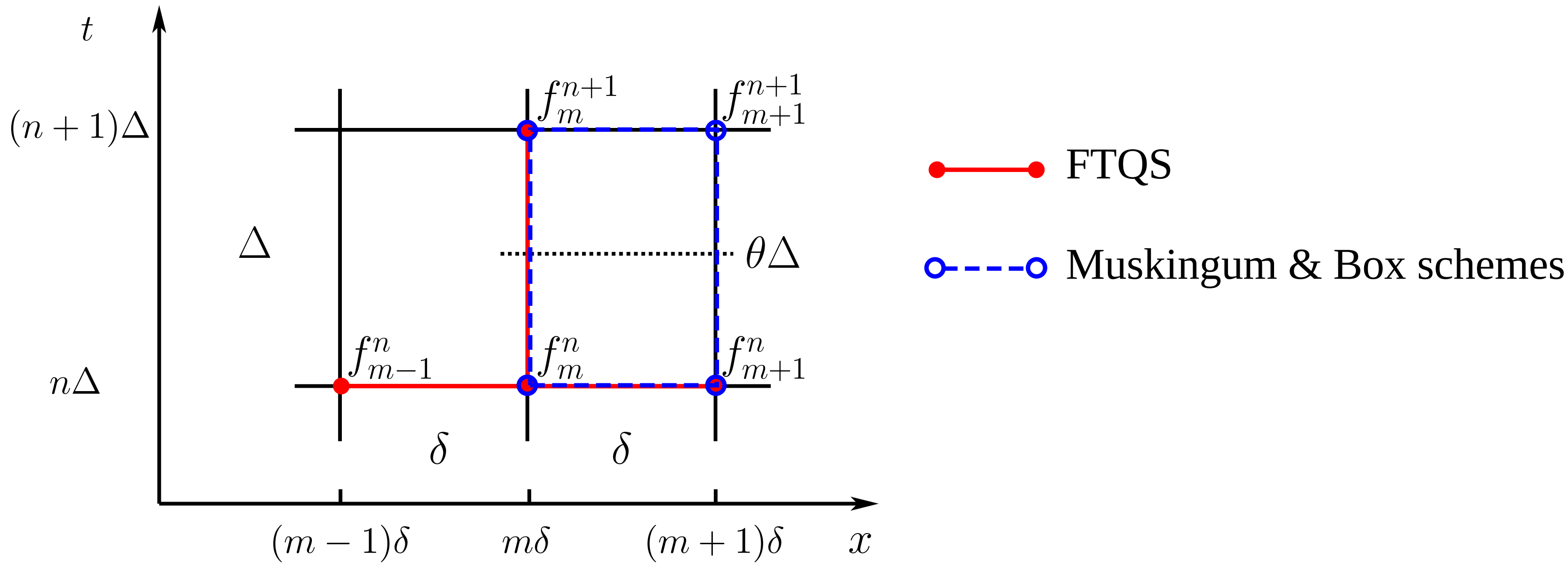


Figure 7.4: Computational stencils

However, if one performs a *Consistency Analysis*, and writes the point values Q_m^n *etc* as bi-dimensional Taylor series, one finds that the differential equation that the Muskingum formula actually satisfies is

$$\frac{\partial Q}{\partial t} + c_0 \frac{\partial Q}{\partial x} + \frac{D_0}{c_0} \frac{\partial^2 Q}{\partial x \partial t} = 0,$$

and *not* the desired Advection-diffusion equation

$$\frac{\partial Q}{\partial t} + c_0 \frac{\partial Q}{\partial x} - D_0 \frac{\partial^2 Q}{\partial x^2} = 0.$$

One can use the first two terms in the Muskingum equation to write $\partial Q/\partial t \approx -c_0 \partial Q/\partial x$ and substitute this into the mixed derivative $\partial^2 Q/\partial x \partial t$ to give the advection-diffusion equation, but that is accurate only for small diffusion.

Muskingum methods work surprisingly well for small-diffusion problems, but in general, they solve the wrong equation, are numerically diffusive, and are to be avoided.

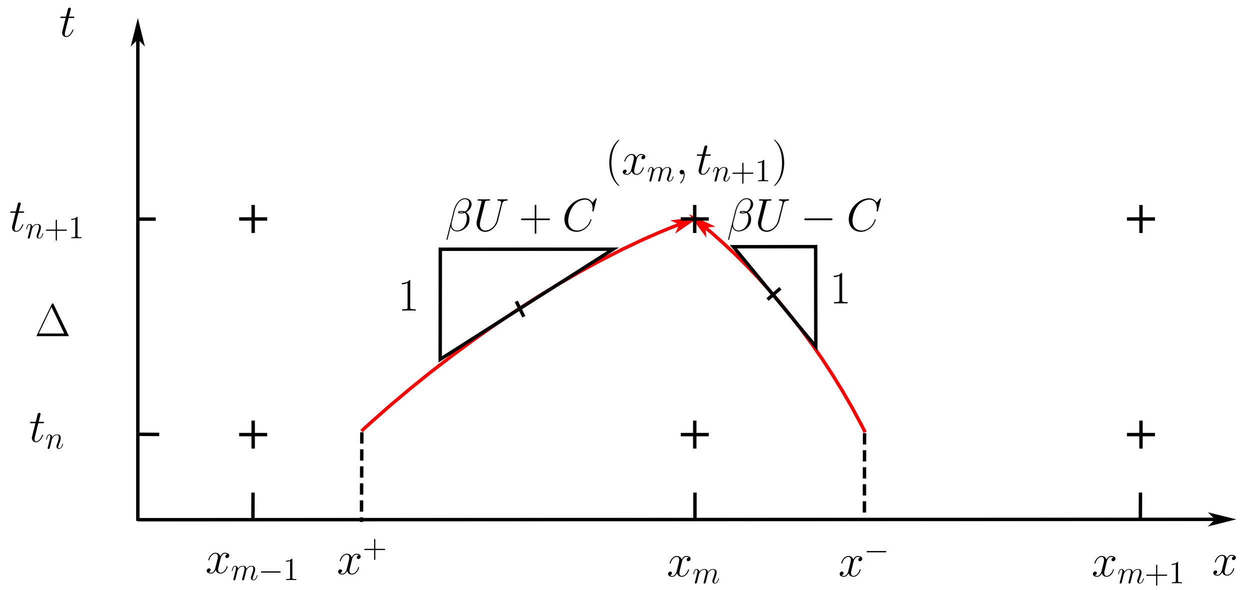


Figure 7.5: (x, t) axes, showing computational module with characteristics

This method is described in many books. The lecturer believes that it is something of an accident of history, and that the deductions that emerge from it are misleading and have caused several important misunderstandings about the nature of wave propagation in open channels.

Each of the pairs of long wave equations (7.17) and (7.18), which are partial differential equations, can be expressed as four ordinary differential equations. Two of the differential equations are for paths for $x(t)$, a path known as a *characteristic*:

$$\frac{dx}{dt} = \beta U \pm C, \tag{7.32}$$

where $U = Q/A$ is the mean fluid velocity in the waterway at that section and the velocity C is

$$C = \sqrt{\frac{gA}{B} + U^2\left(\beta^2 - \beta\right)},$$

often incorrectly described as the "long wave speed". It is, as equation (7.32) shows, the speed of the characteristics relative to the flowing water. The two contributions $\pm C$ correspond to downstream and upstream propagation of information. Two characteristics that meet at a point are shown on Figure 7.5. The "downstream" or "+" characteristic has a velocity at any point of $\beta U + C$. In the usual case where U is positive, both parts are positive and the term is large. As shown on the diagram, the "upstream" or "-" characteristic has a velocity $\beta U - C$, which is usually negative and smaller in magnitude than the other. Not surprisingly, upstream-propagating disturbances travel more slowly. The characteristics are curved, as all quantities determining them are not constant, but functions of the variable A, B, and Q.

The other two differential equations for η and Q can be established from the long wave equations:

$$B\left(-\beta\frac{Q}{A} \pm C\right)\frac{d\eta}{dt} + \frac{dQ}{dt} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Lambda P\frac{Q\left|Q\right|}{A^2}, \tag{7.33}$$

On each of the two characteristics given by the two alternatives of equation (7.32), each of these two equations holds, taking the corresponding plus or minus signs in each case. To advance the solution numerically means that the four differential equations (7.32) and (7.33) have to be solved over time, usually using a finite time step Δ. Figure 7.5 shows the nature of the process on a plot of x against t.

The usual computational problem is, for a time $t_{n+1} = t_n + \Delta$, and for each of the discrete points x_m, to determine the values of x^+ and x^- at which the characteristics cross the previous time level t_n. From the information about η and Q at each of the computational points at that previous time level, the corresponding values of η^+, η^-, Q^+, and Q^- are calculated and then used as initial values in the two differential equations (7.33) which are then solved numerically to give the updated values $\eta\left(x_m, t_{n+1}\right)$ and $Q\left(x_m, t_{n+1}\right)$, and so on for all the points at t_{n+1}.

In textbooks and research papers, characteristics seem wrongly to be believed to have an almost supernatural property that the partial differential equations do not. An advantage of characteristics has been believed to be that numerical schemes are relatively stable. The lecturer is not convinced that they are any more stable then finite difference approximations to the original partial differential equations, but this remains to be proved conclusively.

In fact, the use of characteristics has led to a widespread misconception in hydraulics where C is understood to be the speed of propagation of all waves. It is not – it is the speed of *characteristics*. If surface elevation were constant on a characteristic there would be some justification in using the term ”wave speed” for the quantity C, as disturbances travelling at that speed could be observed. However as equation (7.33) holds, in general neither η (surface elevation – the quantity that we see), nor Q, is constant on the characteristics and one does not have observable disturbances, something that we would call a wave, travelling at C relative to the water. While C may be the speed of propagation of information in the waterway relative to the water, it cannot properly be termed the wave speed as it would usually be understood. In this course we have already examined at length the real nature of the propagation speed of waves.

7.13 Implicit methods – the Preissmann Box scheme

The most popular commercial numerical method for solving the long wave equations in time are Implicit Box (Preissmann) models, where the derivatives are replaced by finite-difference equivalents based on the rectangular blue module in Figure 7.4 on page 109:

$$\begin{aligned}
\frac{\partial f}{\partial x}(m,n) &\approx \frac{1}{\delta}\left[\theta\left(f_{m+1}^{n+1}-f_{m}^{n+1}\right)+(1-\theta)\left(f_{m+1}^{n}-f_{m}^{n}\right)\right], \\
\frac{\partial f}{\partial t}(m,n) &\approx \frac{1}{2\Delta}\left[\left(f_{m+1}^{n+1}-f_{m+1}^{n}\right)+\left(f_{m}^{n+1}-f_{m}^{n}\right)\right], \\
\bar{f}(m,n) &\approx \frac{1}{2}\left[\theta\left(f_{m+1}^{n+1}+f_{m}^{n+1}\right)+(1-\theta)\left(f_{m+1}^{n}+f_{m}^{n}\right)\right],
\end{aligned}$$

where θ is a coefficient that determines how much weight is attached to values at time $n+1$ (unknown, shown red) and how much to those at n (known, shown blue). Now, in the long wave equations (7.17) or (7.18) we use these expressions for all derivatives and also the averaged quantities $\bar{f}$ for those that occur algebraically. Considering all the modules at a certain time level, we have a set of $2M$ simultaneous complicated nonlinear algebraic equations in the values of Q and η at all points along the channel. The method is very complicated, but it is robust and stable, and large time steps can be taken. It is neutrally stable if $\theta=\frac{1}{2}$. In practice, one uses a larger value, such as $\theta=0.6$, and the scheme is stable because it is computationally-diffusive. Several well-known commercial programs are available. For human purposes, it is simpler and better to use an explicit finite difference FTQS scheme.

Evolution of flood wave – large diffusion case

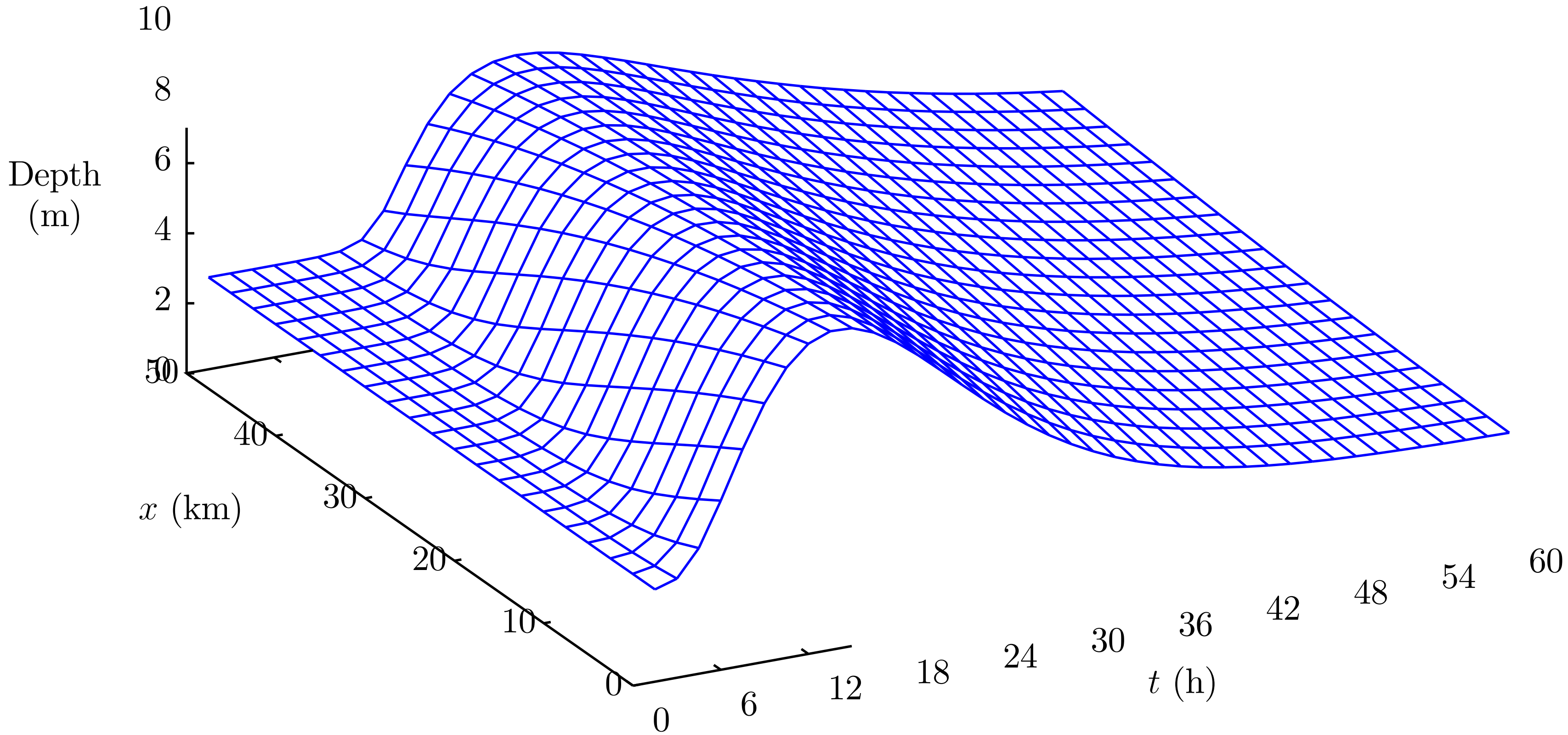
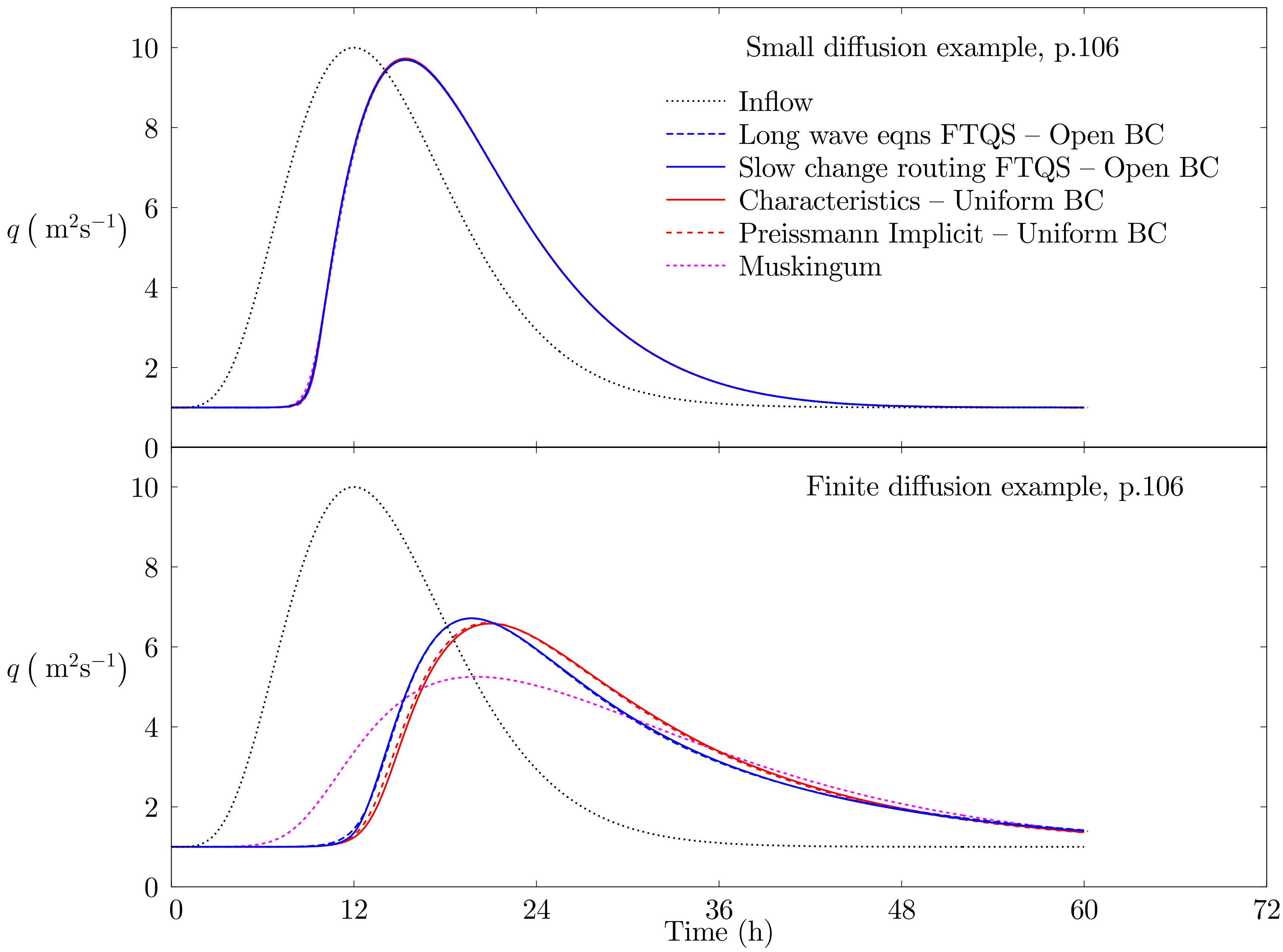


Figure 7.6:

Comparison of different methods



Conclusions

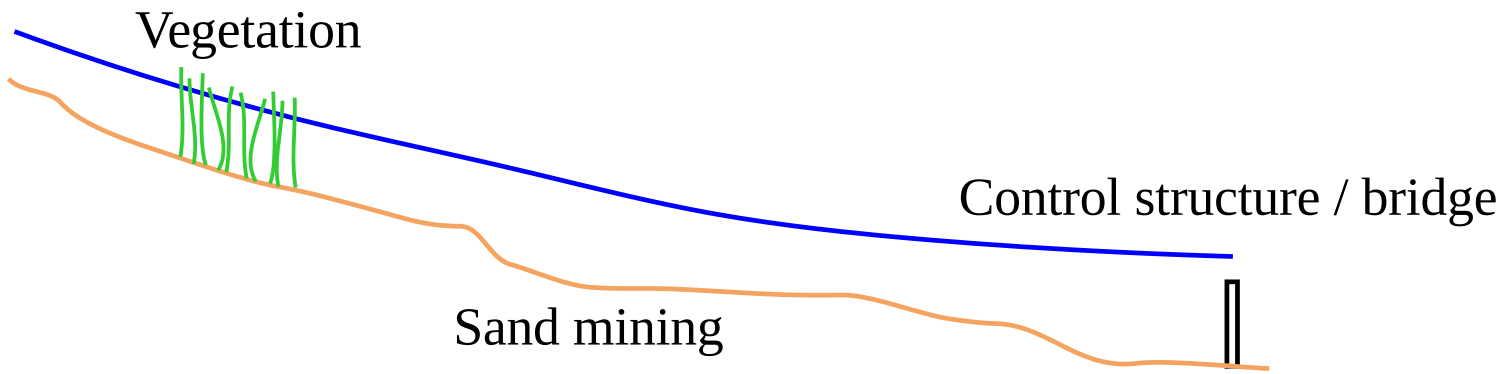
Small diffusion case:

- All methods performed well. Muskingum was accurate; the incorrect uniform flow downstream boundary condition did not matter (all motion is like simple advection – there is little diffusion for downstream effects to be felt upstream).

Finite diffusion case:

- Muskingum showed far too much diffusion when that was important. Such methods have been known to be problematical for small slopes, but nobody has called them out.
- The uniform flow downstream boundary condition: the Preissmann Implicit Box scheme and the Method of Characteristics agreed quite well with each other, but there were finite differences with those of the open downstream boundary condition.
- Both the FTQS finite difference schemes (solving the long wave equations and the slow change routing equation) agreed closely with each other using the more correct open boundary condition. They are the simplest methods and the best. One has to take relatively small time steps ($30\,\mathrm{s}$ used in the examples, compared with $900\,\mathrm{s}$ for the implicit and Muskingum methods), but their simplicity means that computational time is short.
- One can devise an example with a more rapidly-rising flood wave where the slow change routing equation no longer agrees so well. It is simple, and is in terms of a single variable, so that we used it to show the nature of approximations. *In general, however, solving the long wave equations themselves using our explicit FTQS scheme is the best of all.*

8. Steady flow



- A common task in river engineering is to calculate the free surface elevation along a steadily flowing stream.
- Simply the solution of a first-order differential equation – often obscured in writings.
- Flow is usually sub-critical, so the control / boundary condition is at the downstream end and one computes upstream.
- Alternative approach suggested here, using cross-sectional area as the dependent variable, requiring little knowledge of the details of the underwater topography.
- Traditional textbook methods are unsatisfactory: the "Standard Step" method is unnecessarily complicated and the "Direct Step" method is incorrect.
- Application of simple explicit numerical methods is described.
- If A is not used, for non-prismatic streams all methods require much data. Often that is not available. An approximate linearised model of flow in a river is made. This gives us insight into the nature of the problem, as well as simple approximate answers.

Use of area A and application to streams of unknown bathymetry

For steady flow where Q is constant so that $\partial Q/\partial t$ and $\partial Q/\partial x$ are zero, the long wave momentum equation (7.17b) on page 82 in terms of cross-sectional area A, gives one version of the GVFE in terms of area A:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = B\frac{\tilde{S} - Q^2/K^2}{1 - \beta \mathsf{F}^2} = B\frac{\tilde{S} - Q^2 P^{4/3}/k_{\mathrm{St}}^2 A^{10/3}}{1 - \beta Q^2 B/gA^3} \,. \tag{8.1}$$

In the resistance term we are using the conveyance K, which is a function of section properties and the Strickler coefficient k_{St}

$$K = k_{\mathrm{St}}\frac{A^{5/3}}{P^{2/3}} \,, \tag{8.2}$$

such that for uniform flow, $\tilde{S} = S =$ constant, $Q_{\mathrm{r}} = K\sqrt{S}$.

The ordinary differential equation (8.1) is valid also for non-prismatic channels. The mean bed slope at a section $\tilde{S}$, can be variable but is, usually poorly known and is often just estimated, like the other parameters of the problem; β might be something like 1.1. The coefficient k_{St} is also often poorly known.

In the differential equation there are strongly-varying functions of the dependent variable itself, A^3 and possibly $A^{10/3}$, plus the usually slowly-varying functions $B(A)$ and $P(A)$. This suggests that using the GVFE in terms of A has an important advantage: one needs few details of the under-water topography. It is not necessary to know the precise details of the underwater bathymetry other than those weakly-varying functions $B(A)$ and $P(A)$. The obvious approximation could be made that they are constant and equal; river width often does not vary much.

To start numerical solution, one would need to know the area at a control where surface elevation might be known. The solution in terms of area might be enough, to give an idea of how far upstream the effects of a structure or channel changes extend. It is surprising that we can do so much with so little information. However, if one needed a value of surface elevation η at a certain value of x, one would then need cross-sectional details there to go from the computed A to η.

Customary use of a quantity h called the "water depth"

The long wave momentum equation (7.18b) in terms of surface elevation η, for Q constant so that $\partial Q/\partial t$ and $\partial Q/\partial x$ are zero gives another version of the GVFE:

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{\tilde{S}\beta\mathsf{F}^2 - Q^2/K^2}{1-\beta\mathsf{F}^2} \left(\approx -Q^2/K^2 \text{ for } \mathsf{F}^2 \text{ small, the common case}\right)$$

The tradition is not to use η, but instead a depth-like quantity $h = \eta - Z_0$, where Z_0 is the elevation

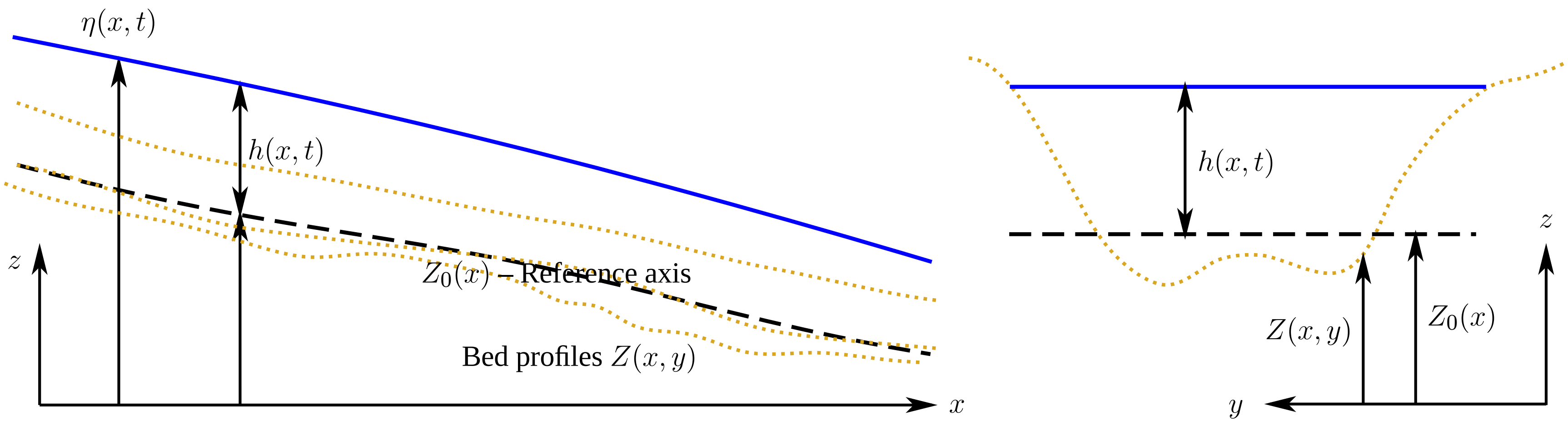


Figure 8.1: Complicated reality

of a longitudinal axis, almost always the supposed bed of the channel. The GVFE becomes

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 + \beta\left(\tilde{S} - S_0\right)\mathsf{F}^2 - Q^2/K^2}{1 - \beta\mathsf{F}^2},$$

where $S_0 = -\mathrm{d}Z_0/\mathrm{d}x$, the slope of the reference axis, positive downwards. We almost never know the details of $\tilde{S}$ so here we assume that $\tilde{S} = S_0$, which we now write as S, giving

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S - Q^2/K^2}{1 - \beta\mathsf{F}^2}$$

where in general both K and F are functions of both x and h, while in a prismatic channel, functions just of h.

Because of our use of h, we pretend that we know the bed in great detail, or, that our channel looks like this:

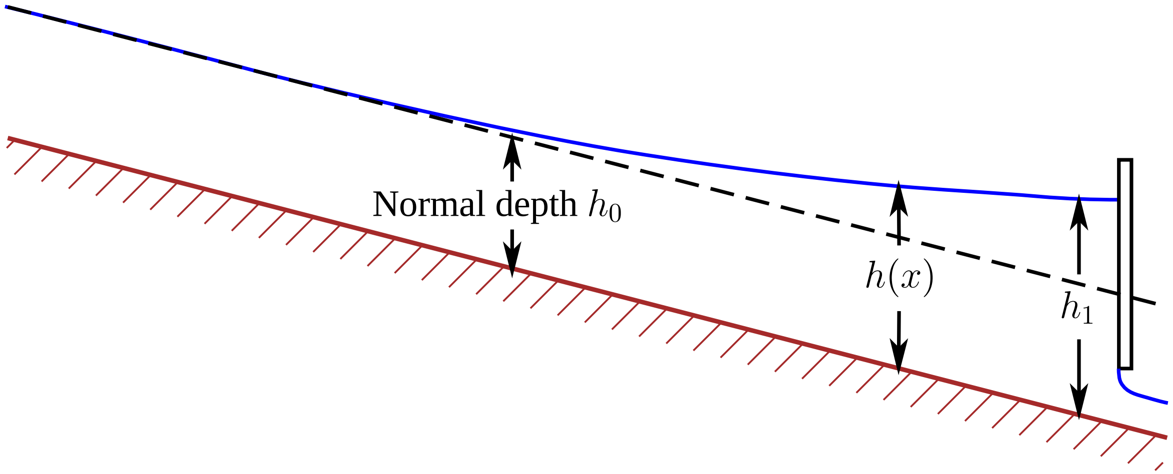


Figure 8.2: Our simpler model

This shows a typical subcritical flow retarded by a structure, showing the free surface disturbance decaying upstream, and if the channel is prismatic, to constant normal depth.

8.2 Traditional textbook methods – some old, complicated, and wrong

The "Standard" step method

The almost trivial energy derivation, ignoring non-prismatic effects, is that the rate of change of total head H is given by the empirical expression for the energy gradient

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -Q^2/K^2\,(x,h) \quad \text{where} \quad H = Z_0(x) + h + \alpha\frac{Q^2}{2gA^2(x,h)}.$$

The computational approximation scheme is

$$\frac{H_{i+1}\,(h_{i+1}) - H_i\,(h_i)}{x_{i+1} - x_i} = -\tfrac{1}{2}Q^2\left(\frac{1}{K^2\,(x_i, h_i)} + \frac{1}{K^2\,(x_{i+1}, h_{i+1})}\right)$$

J. Fenton, Australia 1966;
H. Honsowitz, Austria, 1970?
solving transcendental equations

- The method advocated by Chow (1959) in a pre-computer era and still suggested by textbooks.
- $H(h)$ and $K(x,h)$ are both complicated geometrical functions of h, the unknown h_{i+1} is deep inside left and right sides.
- Requires numerical solution of a transcendental equation at each time step.

The “Direct” step method – distance calculated from depth

- Applied by taking steps in the water depth and calculating the corresponding step in x.
- It has some advantages: iterative methods are not necessary (“Direct”).
- Practical disadvantages are:
  - It is applicable only to prismatic sections
  - Results are not obtained at specified points in x
  - As uniform flow is approached the steps become infinitely large
  - AND, it is wrong, as we now show

Consider the “specific head”, the head relative to the local channel bottom, denoted here by H_0:

$$H_0(h) = H(h) - Z = h + \alpha \frac{Q^2}{2gA^2(h)}.$$

The differential equation becomes, after inverting each side

$$\frac{\mathrm{d}x}{\mathrm{d}H_0(h)} = \frac{1}{S - Q^2/K^2}.$$

A mistake and a correction

- The differential equation is now approximated, the left side by a finite difference expression $(x_i - x_{i+1}) \,/\, (H_{0,i} - H_{0,i+1})$.

- For the right side the numerical method as set out in textbooks is to take the mean of just the *denominator* at beginning and end points, and so to write

$$x_{i+1} = x_i + \frac{H_{0,i+1} - H_{0,i}}{\frac{1}{2}\left(S_i - Q^2/K_i^2 + S_{i+1} - Q^2/K_{i+1}^2\right)}$$

where the red shows the quantity that is a supposed mean value.

- While this is a plausible approximation, it is not mathematically consistent. What should be done is to use the mean value at beginning and end points of the *whole* right side of the differential equation, to give a trapezoidal approximation of the right side, which leads to

$$x_{i+1} = x_i + \left(H_{0,i+1} - H_{0,i}\right) \frac{1}{2}\left(\frac{1}{S_i - Q^2/K_i^2} + \frac{1}{S_{i+1} - Q^2/K_{i+1}^2}\right).$$

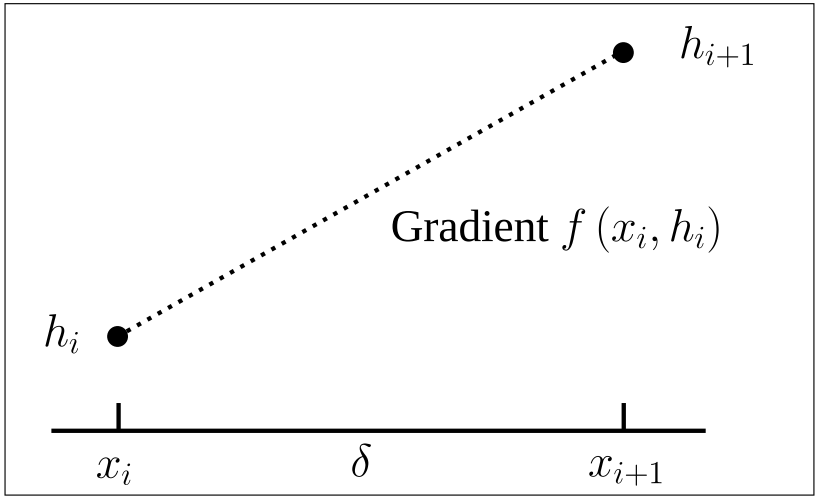
8.3 Standard simple numerical methods for differential equations

We write the differential equation as

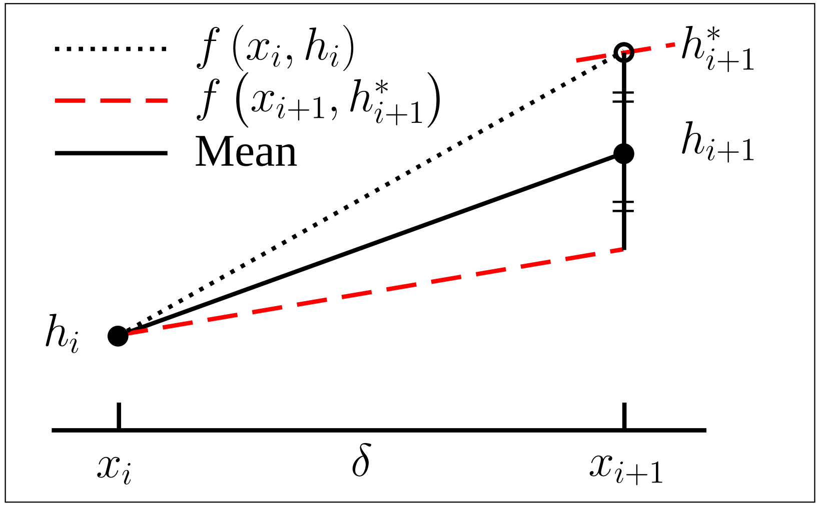
$$\frac{\mathrm{d}h}{\mathrm{d}x} = f(x,h) = \frac{S(x) - Q^2/K^2(x,h)}{1 - \beta \mathsf{F}^2(x,h)}$$

The two simplest numerical methods are:

Euler



Heun



$$h_{i+1} \approx h_i + \delta f(x_i, h_i) + O(\delta^2)$$

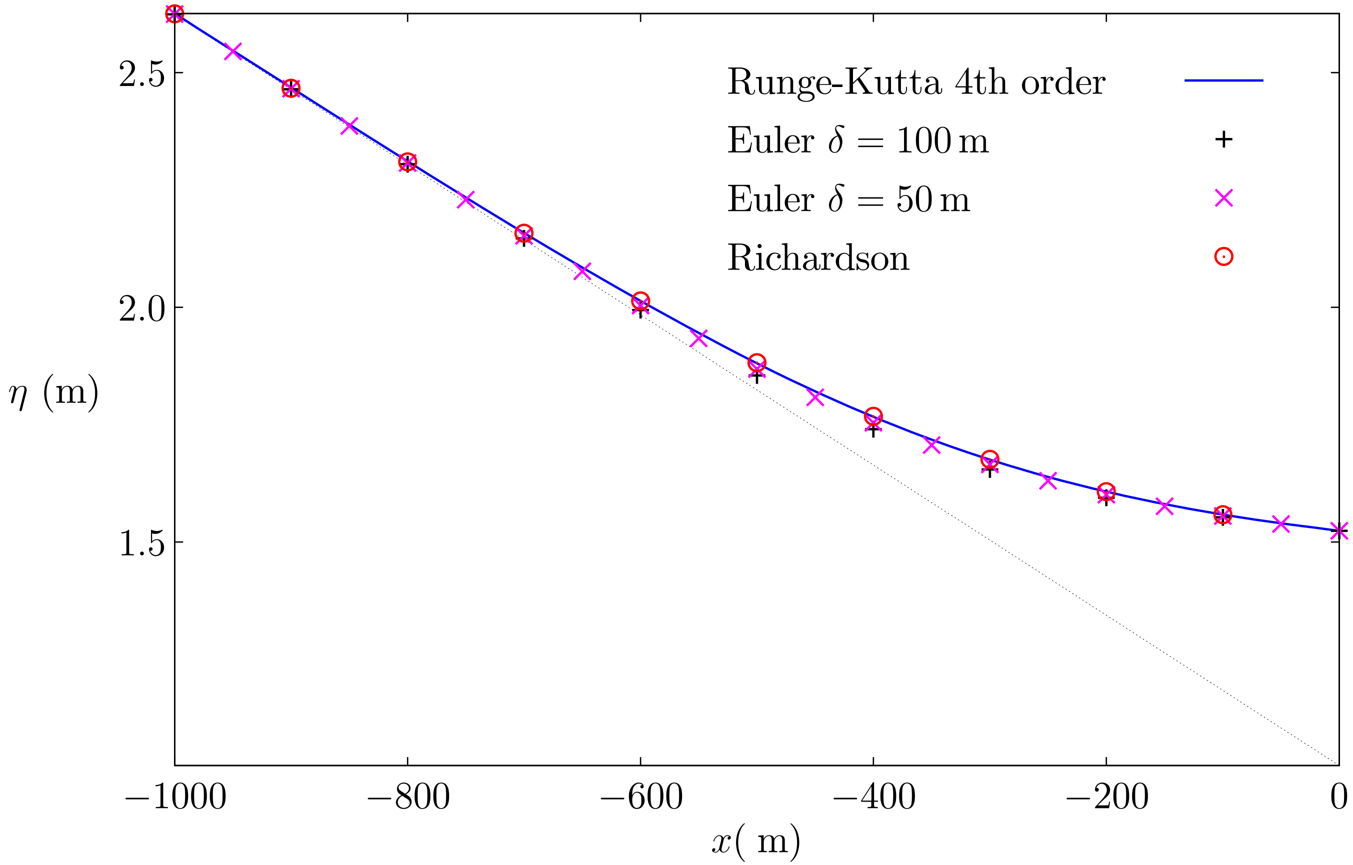
$$h^*_{i+1} \approx h_i + \delta f(x_i, h_i),$$

$$h_{i+1} \approx h_i + \frac{\delta}{2}\left(f(x_i, h_i) + f\left(x_{i+1}, h^*_{i+1}\right)\right) + O(\delta^3)$$

- Euler's method is the simplest but least accurate – yet it might be appropriate for open channel problems where quantities may only be known approximately
- One can use simple modifications such as Heun's method to gain better accuracy, or use Richardson extrapolation, or even more simply, just take smaller steps δ
- For greater accuracy one can use the Trapezoidal method, simply repeating the second Heun step several times, setting $h^*_{i+1} = h_{i+1}$ each time
- Often these two methods are not presented in hydraulics textbooks as alternatives, yet they are simple and flexible, and reveal the nature of what we are doing
- The step δ can be varied at will, to suit possible irregularly spaced cross-sectional data
- In many situations, where $\mathsf{F}^2 \ll 1$, we can ignore the $\beta\mathsf{F}^2$ term in the denominators, giving a notationally simpler scheme

Comparison of schemes

Example 7 A flow of $11.33\ \mathrm{m^3\,s^{-1}}$ passes down a trapezoidal channel of gradient $S = 0.0016$, bed width $6.10\ \mathrm{m}$ and channel side slopes $H : V = 2$, $\alpha = \beta = 1.1$, and $k_{\mathrm{St}} = 40$. At $x = 0$ the flow is backed up to a depth of $1.524\ \mathrm{m}$. Compute the backwater curve for $1000\ \mathrm{m}$ in 10 steps and then 20, then perform Richardson extrapolation for a more accurate estimate.



Convergence of numerical schemes

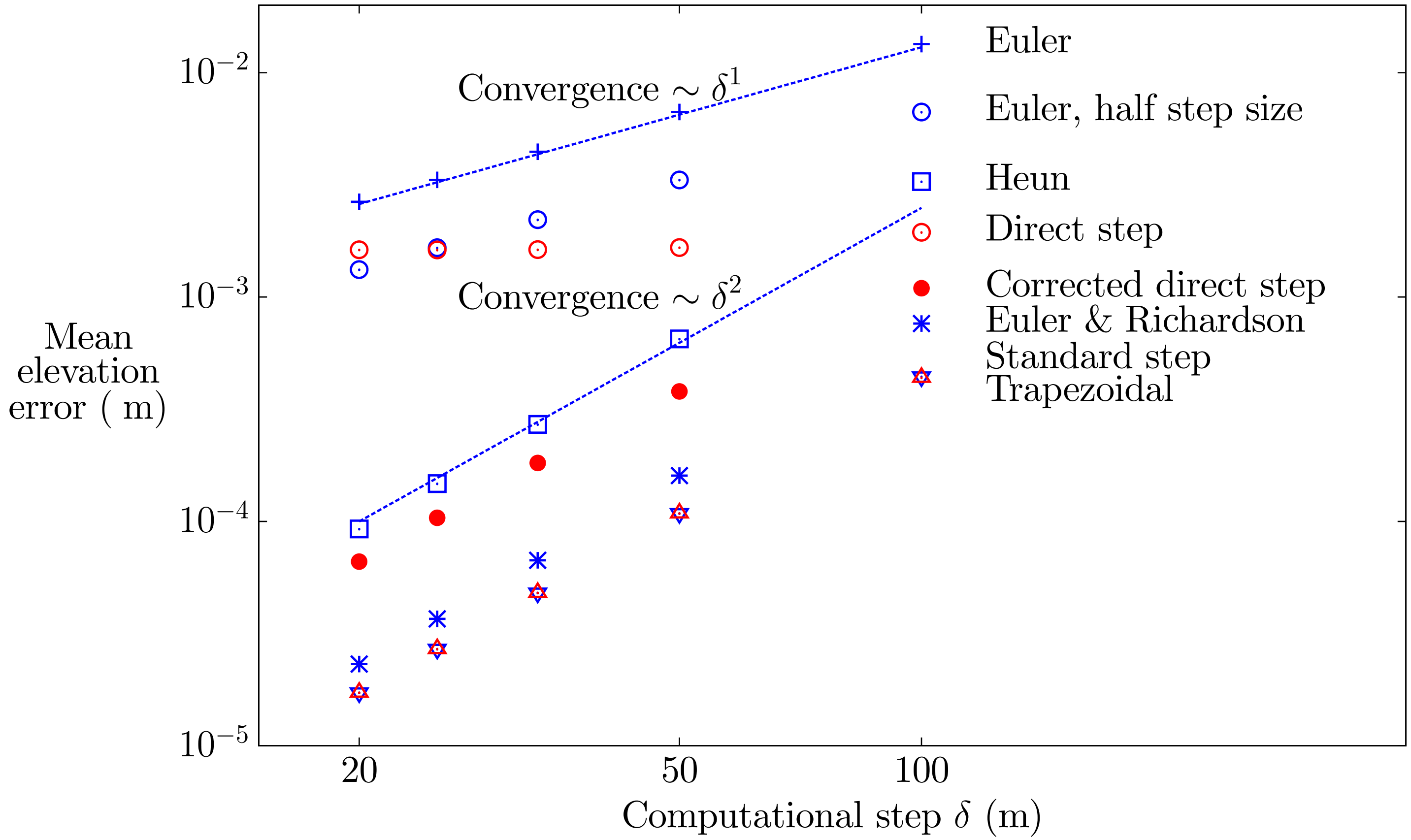


Figure 8.3: Comparison of accuracy - logarithmic scales

- Using Euler, then applying Richardson extrapolation, gave the third most accurate of all the methods, more than enough for practical purposes
- The most accurate were the Standard step method and the Trapezoidal method
- There *is* something wrong with the conventional Direct step method as we have suggested, while the corrected scheme is highly accurate

8.4 A simple model of steady flow in a river

- Often the precise details of a stream are not known, and it is quite legitimate to make approximations
- These might give us more insight and understanding of the problem
- Now a model is made where the GVFE is linearised and a general solution obtained
- Simple deductions as to the length of backwater effects can be made
- One can calculate an approximate solution for a whole stream if the variation in the resistance coefficient and geometry are known or can be estimated
- There is more of a balance between what we know (usually little) and the (un)sophistication of the model

The GVFE is

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S - Q^2/K^2(x,h)}{1 - \beta \mathsf{F}^2(x,h)}$$

We linearise the problem (similar to obtaining the Telegraph equation) and consider small perturbations about an underlying uniform flow of slope S_0 and depth h_0, such that we write

$$h = h_0 + \varepsilon h_1(x) + \ldots,$$

where ε is a small quantity expressing the magnitude of deviations from uniform.

Similarly we also let the possible non-constant slope be

$$S = S_0 + \varepsilon S_1(x) + \ldots .$$

In a real stream varying along its length, both K and F are functions of x and h. We write the series:

$$K = K_0 + \varepsilon K_1(x) + \varepsilon h_1(x) K_{h0} + O\left(\varepsilon^2\right) ,$$

where K_1 is a change caused by a change in the channel properties in x, whether the resistance coefficient or the cross-section, and $K_{h0} = \mathrm{d}K/\mathrm{d}h|_0$ expresses the change of conveyance with water depth. We also write

$$\mathsf{F}^2 = \mathsf{F}_0^2 + O\left(\varepsilon\right) + \ldots ,$$

in which we will find that terms in ε are not necessary.

Multiplying through by $1 - \beta \mathsf{F}^2$, setting dh_0/dx to zero for uniform flow and neglecting terms in ε^2:

$$\varepsilon \left(1 - \beta \mathsf{F}_0^2\right) \frac{\mathrm{d}h_1(x)}{\mathrm{d}x} = S_0 + \varepsilon S_1(x) - \frac{Q^2}{K_0^2}\left(1 - 2\varepsilon \frac{K_1(x)}{K_0} - 2\varepsilon h_1(x) \frac{K_{h0}}{K_0}\right) .$$

At zeroeth order ε^0 we obtain

$$S_0 - Q^2/K_0^2 ,$$

an expression of whichever flow formula is being used, and is identically satisfied.

At ε^1, we obtain the linear differential equation

$$\frac{\mathrm{d}h_1}{\mathrm{d}x} - \gamma h_1 = \phi(x)$$

where γ is a constant:

$$\gamma = 2\frac{S_0 K_{h0}/K_0}{1-\beta \mathsf{F}_0^2} = \frac{S_0}{1-\beta \mathsf{F}_0^2} \times \begin{cases} 2\dfrac{\mathrm{d}K/\mathrm{d}h|_0}{K_0}, & \text{General expression;} \\ 3\dfrac{B_0}{A_0} - \dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Chézy-Weisbach;} \\ \frac{10}{3}\dfrac{B_0}{A_0} - \frac{4}{3}\dfrac{\mathrm{d}P/\mathrm{d}h|_0}{P_0}, & \text{Gauckler-Manning;} \end{cases} \qquad (8.3)$$

and the forcing term on the right is

$$\phi(x) = \frac{S_0}{1-\beta \mathsf{F}_0^2}\left(\frac{S_1(x)}{S_0} + \frac{2K_1(x)}{K_0}\right), \qquad (8.4)$$

showing the effects of fractional changes in slope and conveyance K.

Solving the differential equation

The differential equation is in *integrating factor* form, and can be solved by multiplying both sides by $e^{-\gamma x}$ and writing the result

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\gamma x} h_1\right) = e^{-\gamma x}\phi\left(x\right),$$

which can be integrated to give

$$h_1 = e^{\gamma x} \left(\int^x e^{-\gamma x'} \phi(x') \, dx' + \text{Constant} \right),$$

where x' is a dummy variable. Returning to physical variables, $h = h_0 + \varepsilon h_1$ gives the solution

$$h = h_0 + He^{\gamma x} + \int^x e^{\gamma(x-x')} \phi(x') \, dx'$$

The part of the solution $He^{\gamma x}$ is that obtained by Samuels (1989), giving the solution for backwater level in a uniform channel by evaluating the constant of integration using a downstream boundary condition $h = H$ at $x = 0$. The solution shows how the surface decays upstream at a rate $e^{\gamma x}$, as x becomes increasingly negative, because γ is positive,

- For a wide channel, the terms in dP/dh in the formulae for γ are unimportant (and are often not well known), so that $A_0/B_0 \approx h_0$, the channel depth, and for small Froude number this gives

$$\gamma \approx 3\frac{S_0}{h_0}, \tag{8.5}$$

  showing that the rate of exponential decay is small for gently sloping and deep streams and greatest for steep and shallow ones.

- Consider the distance $x_{1/2}$ upstream for the effect of a downstream surface elevation to diminish by a factor of $1/2$. Then $\exp\left(-\gamma x_{1/2}\right) = 1/2$, or

$$x_{1/2} = \frac{\ln 2}{\gamma} \approx \frac{\ln 2}{3}\frac{h_0}{S_0} \approx 0.2\frac{h_0}{S_0}$$

So for a gently-sloping river $S_0 = 10^{-4}$ and $2\ \mathrm{m}$ deep, the effect of any backwater decreases by $1/2$ in a distance of $4\ \mathrm{km}$. To diminish to $1/16$, say, the distance is $16\ \mathrm{km}$. For a steeper river, say $S_0 = 0.0016$ from the example simulated above, where $h_0 \approx 1\ \mathrm{m}$, the "half-length" is about $150\ \mathrm{m}$. This is roughly in agreement with the computed results in Example 7 above.

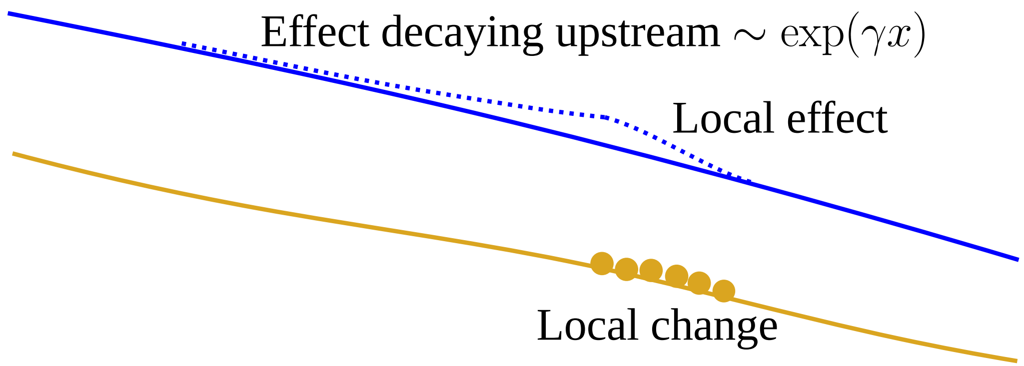
- If the approximate exponential decay solution were shown on that figure, it would not agree closely with the computed results, because the checked-up disturbance is as large as 50% of the depth, when the linear solution is not all that accurate. The beauty of Samuels' result is in its ability to give a quick estimate and an appreciation of the quantities that affect the length of backwater.

General solution for channel

Here we neglect any boundary conditions and consider just the solution due to the forcing function ϕ due to changes in the channel:

$$h = h_0 - \int_x^{\infty} e^{\gamma(x-x')} \phi(x')\, dx' \tag{8.6}$$

This is a simple result: at any point x in subcritical flow, any disturbance is due to the integrated effects of the disturbance function ϕ for all downstream points, from x to ∞, weighted according to the exponential decay function.



Example 8 The effect on a river of a finite length of greater resistance

Consider, as an example, a case where over a finite length L of river, the carrying capacity is reduced by the conveyance K decreasing by a relative amount $K_1/K_0 = -\delta$, such as by local deposition of material, between $x = 0$ and $x = L$, and constant in that interval. Assume F_0^2 negligible and the river wide.

The forcing function form equation (8.4) is:

$$\phi(x) = \begin{cases} 0, & \text{if } x \leqslant 0; \\ -S_0\delta, & \text{if } 0 \leqslant x \leqslant L; \\ 0, & \text{if } x \geqslant L. \end{cases}$$

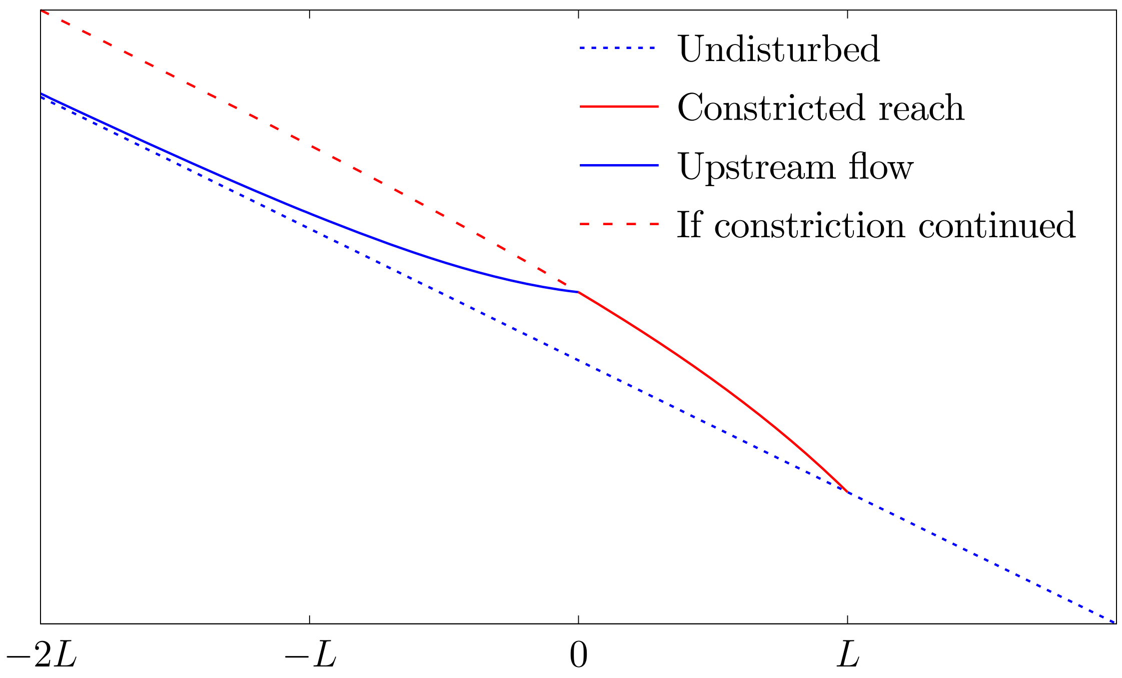
For x downstream, $x \geqslant L$, $\phi(x) = 0$, and $h = h_0$, which is correct in this sub-critical flow, there are no downstream effects.

For x in the section where the changes occur, $0 \leqslant x \leqslant L$, the solution is

$$h = h_0 + S_0\delta \int_x^L e^{\gamma(x-x')}\,dx' = h_0 + \frac{S_0\delta}{\gamma}\left(1 - e^{\gamma(x-L)}\right).$$

For x upstream, $x \leqslant 0$, where there is no extra resistance,

$$h = h_0 + S_0\delta e^{\gamma x} \int_0^L e^{-\gamma x'}\,dx' = h_0 + \frac{S_0\delta}{\gamma} e^{\gamma x}\left(1 - e^{-\gamma L}\right).$$



These solutions are all shown in the figure with an arbitrary vertical scale such that the slope is exaggerated. The calculations were performed for $S_0 = 0.0005$, $h_0 = 1\,\mathrm{m}$, and with a constricted length of $L = 1000\,\mathrm{m}$, with a 10% increase in resistance there, such that $\delta = 0.1$. Using these figures, and with $\gamma = 3S_0/h_0$, the computed backwater at the beginning of the constriction calculated according to the formula was $2.6\,\mathrm{cm}$.

In the reach of increased resistance the surface is raised, as one expects and shows an exponential approach to the changed depth $S_0\delta/\gamma$ if $L \to \infty$.

The abrupt changes of gradient violate our physical assumptions of the long wave equations, but they give us a clear picture of what happens, possibly obvious in retrospect, but hopefully of assistance.

We have obtained an approximate solution to the problem, with little input data necessary.