Appendix A. A general tool for simplifying problems and results – series and Taylor expansions

It is worthwhile here suggesting a method that can often be usefully applied to give simpler results that make problems and results simpler and help our understanding. Sometimes results from hydraulic analyses are complicated, or the importance of results are not clear. If one can assume that a parameter of the problem is small, then one can use power series expansions, often to give a simpler result with more understanding. Usually only one term in the approximation need be taken, when we say that we *linearise*.

Example 1 – Power series

A power series expansion is where we write

$$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2!}\varepsilon^2 + O\left(\varepsilon^3\right), \tag{A-1}$$

where we have introduced the “Big O” notation to show the size of terms we have neglected.

Another common example is that of the logarithmic function

$$\ln(1+\varepsilon) = \log_e(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + O\left(\varepsilon^3\right),$$

another is the exponential function

$$\exp(\varepsilon) = 1 + \varepsilon + \frac{1}{2!}\varepsilon^2 + O\left(\varepsilon^3\right),$$

Example 2 – Taylor series

Here we write the value of a quantity at $x+\delta$ in terms of that at x, and its derivatives. It can be written as the *shift operator* $Ef(x) = f(x+\delta)$, and can be evaluated as the exponential of the derivative operator $D = \mathrm{d}/\mathrm{d}x$ using the power series for exponential

$$f(x+\delta) = e^{\delta D} = \left(1 + \delta\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2!}\delta^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \dots\right) f(x) = f(x) + \delta\frac{\mathrm{d}f}{\mathrm{d}x}(x) + \frac{1}{2!}\delta^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) + \dots. \quad \text{(A-2)}$$

Combining expressions

Often there will be more than one series in a compound expression. Usually these can be arranged so that they appear as a product (multiplication) of two or more series. All such combinations are treated as one might expect, multiplying through and retaining just the terms we need. For example

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1 b_1 + \varepsilon(a_2 b_1 + a_1 b_2) + O\left(\varepsilon^2\right). \quad \text{(A-3)}$$