Hydraulics

Solution Sheet 8 – Dimensional analysis and similitude

1. *Verify that the following expressions are dimensionally homogeneous:*
   a. *the integral energy expression for a length of pipe or channel*
   b. *and the momentum expression for a force of fluid on a control surface.*

   (a) Energy integral equation for a simple inlet-outlet system:

$$-Q_{\text{in}}\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{in}}+Q_{\text{out}}\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{out}}=0.$$

   As the Q factors are common, we can ignore them, and just need to examine the dimensions of

$$\begin{aligned}
&\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\\
&\frac{\text{MLT}^{-2}}{\text{L}^2}\times\frac{\text{L}^3}{\text{M}}\times\frac{\text{T}^2}{\text{L}},\quad \text{L},\quad \frac{\text{T}^2}{\text{L}}\left(\frac{\text{L}^3}{\text{T}}\right)^2\frac{1}{\text{L}^4}\\
&\text{L},\quad \text{L},\quad \text{L}
\end{aligned}$$

   (b) Now the momentum integral equation:

$$\mathbf{P}=\sum_j\left(\rho\beta_j\frac{Q_j^2}{A_j}+\bar{p}_jA_j\right)(-\hat{\mathbf{n}}_j)+\mathbf{F}_{\text{body}}.$$

$$\begin{aligned}
&\rho\beta_j\frac{Q_j^2}{A_j}:\frac{\text{M}}{\text{L}^3}1\left(\frac{\text{L}^3}{\text{T}}\right)^2\frac{1}{\text{L}^2}=\text{MLT}^{-2}\\
&\bar{p}_jA_j:\text{Clearly has units of force MLT}^{-2}\\
&\hat{\mathbf{n}}_j\text{ has no units}\\
&\mathbf{F}_{\text{body}}\text{ has units of force}
\end{aligned}$$

   Hence all terms are consistent.

2. *Assuming that the thrust F of a screw propeller is dependent upon the diameter d, speed of advance V, fluid density ρ, revolutions per second n and coefficient of viscosity μ,*
   a. *Show that the thrust can be expressed by the equation*

$$\frac{F}{\rho V^2d^2}=f(\frac{\mu}{\rho Vd},\frac{dn}{V}),$$

   b. *Interpret each of the terms in the equation.*
   Of relevance are

1	Thrust	F	MLT^{-2}
2	diameter	d	L
3	Speed of advance	V	LT^{-1}
4	Density of the liquid	ρ	ML^{-3}
5	Rotation rate	n	T^{-1}
6	Dynamic viscosity of the liquid	μ	$\text{ML}^{-1}\text{T}^{-1}$

   We have M, L, and T in the variables, so that $m=3$. We have $n=6$ physical quantities so that by the Buckingham Π theorem we will have $n-m=3$ dimensionless variables. We choose V, d, and ρ as our repeating variables, which involve all 3 dimensions.

(1) Thrust F: $\Pi_1 = V^{i_1} d^{i_2} \rho^{i_3} F$, so that the dimensional equation is

$$\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0 = \left(\mathrm{LT}^{-1}\right)^{i_1} \mathrm{L}^{i_2} \left(\mathrm{ML}^{-3}\right)^{i_3} \mathrm{MLT}^{-2},$$

and considering the exponent of each dimension in turn,

$$\begin{aligned} \mathrm{M} &: \quad 0 = i_3 + 1 \\ \mathrm{L} &: \quad 0 = i_1 + i_2 - 3i_3 + 1 \\ \mathrm{T} &: \quad 0 = -i_1 - 2, \end{aligned}$$

From this we have immediately $i_1 = -2$, $i_3 = -1$, and solving the remaining equation, $i_2 = -2$, giving

$$\Pi_1 = \frac{F}{\rho V^2 d^2},$$

which can be described as a dimensionless force.

(2) Rotation rate n: $\Pi_2 = V^{j_1} d^{j_2} \rho^{j_3} n$,

$$\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0 = \left(\mathrm{LT}^{-1}\right)^{j_1} \mathrm{L}^{j_2} \left(\mathrm{ML}^{-3}\right)^{j_3} \mathrm{T}^{--1},$$

$$\begin{aligned} \mathrm{M} &: \quad 0 = j_3 + 0 \\ \mathrm{L} &: \quad 0 = j_1 + j_2 - 3j_3 \\ \mathrm{T} &: \quad 0 = -j_1 - 1, \end{aligned}$$

and $j_3 = 0$, $j_1 = -1$, and $j_2 = 1$, so that

$$\Pi_2 = \frac{nd}{V},$$

which can be thought of as proportional to the ratio of the propeller tip velocity to the boat velocity.

(3) Dynamic viscosity of the liquid μ,

$$\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0 = \left(\mathrm{LT}^{-1}\right)^{k_1} \mathrm{L}^{k_2} \left(\mathrm{ML}^{-3}\right)^{k_3} \mathrm{ML}^{-1}\mathrm{T}^{--1},$$

$$\begin{aligned} \mathrm{M} &: \quad 0 = k_3 + 1 \\ \mathrm{L} &: \quad 0 = k_1 + k_2 - 3k_3 - 1 \\ \mathrm{T} &: \quad 0 = -k_1 - 1, \end{aligned}$$

and $k_3 = -1$, $k_1 = -1$, and $k_2 = -1$, so that

$$\Pi_3 = \frac{\mu}{\rho V d},$$

which is the *inverse* of the Reynolds number of the propeller based on the forward velocity and diameter.

3. *The variables controlling the motion of a ship through water are the drag force F, the speed U, the length l, the fluid density ρ, dynamic viscosity of the water μ, and gravitational acceleration g.*
   a. *Show that the force must be given by an expression of the form*

   $$\frac{F}{\rho U^2 l^2} = f\left(\frac{\mu}{\rho U l}, \frac{gl}{U^2}\right),$$

   where $f()$ is the functional dependence which is to be determined experimentally.
   b. *Interpret each of the terms in the equation.*

c. *A model boat of a scale* $1:50$ *was measured in a towing tank to have a drag force of* $0.27\,\mathrm{N}$ *when towed at a velocity of* $0.91\,\mathrm{m\,s^{-1}}$.

   i. *If it is assumed that most drag is not due to friction but to the waves generated, estimate the drag force on the protoype boat.*

   ii. *Estimate the power required to propel the boat.*

d. *How might the expression in (a) be modified if the ship is in water of finite depth* D*?*

Of relevance are

1	Drag force	F	$\mathrm{MLT^{-2}}$
2	Speed	U	$\mathrm{LT^{-1}}$
3	Length	l	L
4	Density of the liquid	ρ	$\mathrm{ML^{-3}}$
5	Gravitational acceleration	g	$\mathrm{LT^{-1}}$
6	Dynamic viscosity of the liquid	μ	$\mathrm{ML^{-1}T^{-1}}$

We have M, L, and T in the variables, so that $m = 3$. We have $n = 6$ physical quantities so that by the Buckingham Π theorem we will have $n - m = 3$ dimensionless variables. We choose U, l, and ρ as our repeating variables, which involve all 3 dimensions.

This problem is so similar to the last one mathematically (*although the actual physical variables are different*) that we will simply adopt the dimensionless force and viscosity numbers from the previous question

$$\Pi_1 = \frac{F}{\rho V^2 l^2} \quad \text{and} \quad \Pi_3 = \frac{\mu}{\rho V l},$$

which are the dimensionless drag and dimensionless viscosity, the inverse of the ship Reynolds number. This leaves us with

(2) Gravity F: $\Pi_2 = U^{j_1} l^{j_2} \rho^{j_3} g$, so that the dimensional equation is

$$\mathrm{M^0L^0T^0} = \left(\mathrm{LT^{-1}}\right)^{j_1} \mathrm{L}^{j_2} \left(\mathrm{ML^{-3}}\right)^{j_3} \mathrm{LT^{-2}},$$

and considering the exponent of each dimension in turn,

$$\begin{aligned} \mathrm{M} &: \quad 0 = j_3 \\ \mathrm{L} &: \quad 0 = j_1 + j_2 - 3j_3 + 1 \\ \mathrm{T} &: \quad 0 = -j_1 - 2, \end{aligned}$$

From this we have immediately $j_1 = -2$, $j_3 = 0$, and solving the remaining equation, $j_1 = +1$, giving

$$\Pi_2 = \frac{gl}{U^2},$$

which is dimensionless gravity, or the Froude number of the ship raised to the power -2. It is more fundamental in this form. Hence we have

$$\frac{F}{\rho U^2 l^2} = f\left(\frac{\mu}{\rho U l}, \frac{gl}{U^2}\right),$$

(c) If we ignore viscosity, then for dynamical similitude we must have both $F/\rho U^2 l^2$ and gl/U^2 the same in each case. Considering the latter we have

$$\begin{aligned} \frac{gl_m}{U_m^2} &= \frac{gl_p}{U_p^2}, \text{ such that } \frac{U_p}{U_m} = \sqrt{\frac{l_p}{l_m}} = \sqrt{50} = 7.07, \\ U_p &= 7.07 \times U_m = 7.07 \times 0.91 = 6.43 \end{aligned}$$

and for the force,

$$\frac{F_m}{\rho U_m^2 l_m^2} = \frac{F_p}{\rho U_p^2 l_p^2}$$

such that

$$\begin{aligned} \frac{F_p}{F_m} &= \frac{\rho U_p^2 l_p^2}{\rho U_m^2 l_m^2} = \frac{l_p^3}{l_m^3} = 50^3, \\ F_p &= 0.27 \times 50^3 = 33.8\,\mathrm{kN} \end{aligned}$$

Now for the power P, which is equal to force by velocity,

$$\begin{aligned} \frac{P_p}{P_m} &= \frac{F_p}{F_m} \times \frac{U_p}{U_m} = \frac{l_p^{7/2}}{l_m^{7/2}}, \\ P_p &= P_m \times 50^{7/2} = 0.27 \times 0.91 \times 50^{7/2} = 217\,\mathrm{kW} \end{aligned}$$

(d) Now, if the water depth also were added, the only modification would be one more dimensionless number l/D – this is obvious, and there is no need to repeat the analysis:

$$\frac{F}{\rho U^2 l^2} = f\left(\frac{\mu}{\rho U l}, \frac{gl}{U^2}, \frac{l}{D}\right)$$

However, this also means that in the three numbers above, we could replace l by D in any or all of them, and this is where a certain judgement is necessary. For large values of l/D it might be possible that it is the depth D which might be more important in the dimensionless gravity term. The following is a possible re-arrangement:

$$\frac{F}{\rho U^2 l^2} = f\left(\frac{\mu}{\rho U l}, \frac{gD}{U^2}, \frac{l}{D}\right)$$

4. *A V-notch weir is a vertical plate with a notch of angle θ cut into the top of it and placed across an open channel. The liquid in the channel is backed up and forced to flow through the notch. The discharge Q is some function of the elevation H of the upstream liquid surface above the bottom of the notch. The discharge also depends upon gravity g and upon the velocity of approach V_0 to the weir.*

   a. *Use the Buckingham Π theorem to show that there are three governing dimensionless parameters, and that the discharge equation may be expressed in the form:*

   $$\frac{Q}{(gH^5)^{1/2}} = f(V_0/(gH)^{1/2}, \theta),$$

   where $f()$ is the functional dependence which is to be determined experimentally.

   b. *In practice it is found that the approach velocity is not very important. Ignore the effect of that velocity to estimate the ratio of discharge in the full scale weir to that in a geometrically-similar laboratory weir at a scale of $1:10$.*

   Of relevance are

1	Discharge	Q	$\mathrm{L^3T^{-1}}$
2	Angle of notch	θ	1
3	Velocity of approach	V_0	$\mathrm{LT^{-1}}$
4	Elevation above notch	H	L
5	Gravitational acceleration	g	$\mathrm{LT^{-1}}$

   We have only L, and T in the variables, so that $m = 2$. We have $n = 5$ physical quantities so that by the Buckingham Π theorem we will have $n - m = 3$ dimensionless variables. We choose g, H, and θ as our repeating variables. However, it is immediately obvious that θ itself is dimensionless and we will choose this as one of the Π.

(1) Discharge Q: $\Pi_1 = g^{i_1} H^{i_2} Q$, so that the dimensional equation is

$$\mathrm{L}^0\mathrm{T}^0 = \left(\mathrm{LT}^{-2}\right)^{i_1} \mathrm{L}^{i_2} \times \mathrm{L}^3\mathrm{T}^{-1},$$

and considering the exponent of each dimension in turn,

$$\begin{aligned} \mathrm{L} \quad &: \quad 0 = i_1 + i_2 + 3 \\ \mathrm{T} \quad &: \quad 0 = -2i_1 - 1, \end{aligned}$$

From this we have $i_1 = -1/2$, $i_2 = -5/2$, and

$$\Pi_1 = \frac{Q}{\sqrt{gH^5}}.$$

(2) Approach velocity V_0: $\Pi_2 = g^{i_1} H^{i_2} V_0$, so that the dimensional equation is

$$\mathrm{L}^0\mathrm{T}^0 = \left(\mathrm{LT}^{-2}\right)^{i_1} \mathrm{L}^{i_2} \times \mathrm{LT}^{-1},$$

and considering the exponent of each dimension in turn,

$$\begin{aligned} \mathrm{L} \quad &: \quad 0 = i_1 + i_2 + 1 \\ \mathrm{T} \quad &: \quad 0 = -2i_1 - 1, \end{aligned}$$

From this we have $i_1 = -1/2$, $i_2 = -1/2$, and

$$\Pi_2 = \frac{V_0}{\sqrt{gH}},$$

and as we have $\Pi_3 = \theta$, we have

$$\frac{Q}{(gH^5)^{1/2}} = f(V_0/(gH)^{1/2}, \theta).$$

Now, for two geometrically-similar weirs (θ the same) and ignoring the effect of approach velocity, we have

$$\begin{aligned} \frac{Q_m}{(gH_m^5)^{1/2}} &= \frac{Q_p}{(gH_p^5)^{1/2}}, \\ \frac{Q_p}{Q_m} &= \left(\frac{H_p}{H_m}\right)^{5/2} = 10^{5/2} = 316. \end{aligned}$$