Hydraulics

Solution Sheet 7 – Conservation of energy and Bernoulli's theorem

1. *Water flowing in a pipeline* $4\,\mathrm{m}$ *above datum level has a mean velocity of* $12\,\mathrm{m\,s^{-1}}$ *and is at a gauge pressure of* $36\,\mathrm{kPa}$. *If the density of water is* $1000\,\mathrm{kg\,m^{-3}}$, *and the Coriolis coefficient* $\alpha = 1$, *what is*

   a. *the total energy per unit mass of the water, relative to the datum and to atmospheric pressure, and the total head? (Ans:* $147\,\mathrm{J\,kg^{-1}}$, $15\,\mathrm{m}$*)*

   b. *How are these results affected if the Coriolis coefficient* α *is 1.3?*

   The total energy per unit mass is

$$\begin{aligned} gH &= \frac{p}{\rho} + gz + \frac{\alpha}{2}\frac{Q^2}{A^2} \\ &= \frac{36 \times 1000}{1000} + 9.8 \times 4 + \frac{1}{2} \times 12^2 \\ &= 147\,\mathrm{J\,kg^{-1}}(\mathrm{L^2T^{-2}}) \end{aligned}$$

   The total head is $H = 147/9.8 = 15.0\,\mathrm{m}$.
   If $\alpha = 1.3$,

$$\begin{aligned} gH &= \frac{36 \times 1000}{1000} + 9.8 \times 4 + \frac{1.3}{2} \times 12^2 \\ &= 169\,\mathrm{J\,kg^{-1}}(\mathrm{L^2T^{-2}}), \\ H &= 17.2\,\mathrm{m} \end{aligned}$$

2. *A pipe* $300\,\mathrm{m}$ *long tapers from* $1.2\,\mathrm{m}$ *diameter to* $0.6\,\mathrm{m}$ *diameter at its lower end and slopes downwards at a slope of 1 in 100. The pressure at the upper end is* $69\,\mathrm{kPa}$. *Neglecting friction losses and assuming* $\alpha = 1$, *find the pressure at the lower end when the rate of flow is* $5.5\,\mathrm{m^3/\,min}$. *(Ans:* $98.4\,\mathrm{kPa}$*).*

   Energy conservation between points 1 and 2:

$$\begin{aligned} \frac{p_1}{\rho g} + z_1 + \frac{\alpha}{2g}\frac{Q^2}{A_1^2} &= \frac{p_2}{\rho g} + z_2 + \frac{\alpha}{2g}\frac{Q^2}{A_2^2} \\ \frac{69 \times 1000}{1000 \times 9.8} + \underbrace{0}_{\text{Datum}} + \frac{1}{2 \times 9.8}\left(\frac{5.5/60}{\frac{\pi}{4}1.2^2}\right)^2 &= \frac{p_2}{1000 \times 9.8} + \left(0 - \frac{1}{100} \times 300\right) + \frac{1}{2 \times 9.8}\left(\frac{5.5/60}{\frac{\pi}{4}0.6^2}\right)^2 \\ p_2 &= 98350\,\mathrm{Pa} \end{aligned}$$

   Therefore, rounding, the pressure is $98.4\,\mathrm{kPa}$.

3. *A siphon is in the form of a hose, one end of which is immersed in a swimming pool, it passes over a fence* $1.2\,\mathrm{m}$ *higher than the pool surface and its other end (*$2.4\,\mathrm{m}$ *below the pool surface) is allowed to discharge the water down the stormwater drain. What is the velocity of water in the hose, and what is the pressure in the pipe at the point where it passes over the fence? Use* $\alpha = 1.1$. *(Ans:* $6.54\,\mathrm{m\,s^{-1}}$, $-35.3\,\mathrm{kPa}$*)*

   Energy equation between pool surface and hose outlet:

$$\begin{aligned} \frac{p_1}{\rho g} + z_1 + \frac{\alpha}{2g}U_1^2 &= \frac{p_2}{\rho g} + z_2 + \frac{\alpha}{2g}U_2^2 \\ 0 + 0 + 0 &= 0 - 2.4 + \frac{1.1}{2 \times 9.8}U_2^2 \\ U_2 &= 6.54\,\mathrm{m\,s^{-1}} \end{aligned}$$

   Energy equation between pool surface and top of fence, as the mean velocity in the hose is the same everywhere:

$$\begin{aligned} 0 &= \frac{p_2}{1000 \times 9.8} + 1.2 + \frac{1.1}{2 \times 9.8} \times 6.54^2 \\ p_2 &= -35.3\,\mathrm{kPa} \end{aligned}$$

Cavitation is not a danger here for typical water temperatures – only when the height of a pipe approaches $10\,\mathrm{m}$ above the hydraulic grade line – which we will look at later.

4. *The highest residential area in a town is at an elevation of* $250\,\mathrm{m}$. *It is local regulations that the minimum static supply pressure anywhere in town be* $100\,\mathrm{kPa}$. *What must the surface level in the local reservoir be to ensure this without installing a pump? Also, with what velocity would the water issue from a garden hose in that residential area if the water was at that minimum level and* $\alpha = 1.1$*? (Ans:* $260.2\,\mathrm{m}$*,* $13.5\,\mathrm{m\,s^{-1}}$*)*

   The static supply pressure is $100\,\mathrm{kPa}$, the equivalent static head:

$$h = \frac{p}{\rho g} = \frac{100 \times 1000}{1000 \times 9.8} = 10.2\,\mathrm{m},$$

   therefore the elevation of the reservoir is $250 + 10.2 = 260.2\,\mathrm{m}$.

   Applying the integral energy theorem bewtween the top of the reservoir and the outlet from the garden hose:

$$\begin{aligned} \left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{reservoir}} &= \left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{hose}} \\ 0 + 260.2 + 0 &= 0 + 250 + \frac{1.1}{2 \times 9.8}U^2 \\ U &= \sqrt{2 \times 9.8 \times 10.2/1.1} = 13.5\,\mathrm{m\,s^{-1}}. \end{aligned}$$

5. *A jet of water is initially* $12\,\mathrm{cm}$ *in diameter and when directed vertically upwards reaches a maximum height of* $20\,\mathrm{m}$. *Assuming that the jet remains circular in cross-section and* $\alpha = 1$*, determine the volume rate of water flowing and the diameter of the jet at a height of* $10\,\mathrm{m}$. *(Ans:* $0.224\ \mathrm{m^3\,s^{-1}}$*,* $14.3\,\mathrm{cm}$*)*

   Initially determine the discharge:

$$\begin{aligned} \frac{p_1}{\rho g} + z_1 + \frac{\alpha}{2g}\frac{Q^2}{A_1^2} &= \frac{p_2}{\rho g} + z_2 + \frac{\alpha}{2g}\frac{Q^2}{A_2^2} \\ 0 + 0 + \frac{1}{2 \times 9.8}\frac{Q^2}{\left(\frac{\pi}{4} \times 0.12^2\right)^2} &= 0 + 20 + 0 \\ Q &= 0.224\,\mathrm{m^3\,s^{-1}} \end{aligned}$$

   Now we apply the integral energy theorem between nozzle and $10\,\mathrm{m}$ to determine the velocity there, hence the cross-sectional area, and then the diameter:

$$\begin{aligned} \frac{p_1}{\rho g} + z_1 + \frac{\alpha}{2g}\frac{Q^2}{A_1^2} &= \frac{p_3}{\rho g} + z_3 + \frac{\alpha}{2g}\frac{Q^2}{A_3^2} \\ 0 + 0 + \frac{1}{2 \times 9.8}\frac{0.224^2}{\left(\frac{\pi}{4} \times 0.12^2\right)^2} &= 0 + 10 + \frac{1}{2 \times 9.8}\frac{0.224^2}{\left(\frac{\pi}{4} \times D_3^2\right)^2} \\ D_3 &= 0.143\,\mathrm{m}. \end{aligned}$$

6. *A horizontal venturi meter, consisting of a converging portion followed by a throat section of constant diameter and then a diverging portion is used to measure the rate of flow in a pipe. The diameter of the pipe is* $15\,\mathrm{cm}$*, and at the throat section it is* $10\,\mathrm{cm}$. *The pressure difference between the two sections is* $20\,\mathrm{kPa}$*, when oil of density* $900\,\mathrm{kg\,m^{-3}}$ *flows in the pipe. Use the integral energy equation to determine the discharge in the pipe, with* $\alpha = 1.06$. *(Ans:* $56.8\,\mathrm{L\,s^{-1}}$*).*

   Energy between 1 before Venturi and 2 in throat – horizontal $z_1 = z_2$, and it is easier to multiply through by g:

$$\begin{aligned} \frac{p_1}{\rho} + gz_1 + \frac{\alpha}{2}\frac{Q^2}{A_1^2} &= \frac{p_2}{\rho} + gz_2 + \frac{\alpha}{2}\frac{Q^2}{A_2^2} \\ \frac{p_1}{900} + \frac{1.06}{2}\left(\frac{Q}{\frac{\pi}{4}0.15^2}\right)^2 &= \frac{p_1 - 20000}{900} + \frac{1.06}{2}\left(\frac{Q}{\frac{\pi}{4}0.1^2}\right)^2 \\ Q &= 0.0568\,\mathrm{m^3\,s^{-1}} = 56.8\,\mathrm{L\,s^{-1}} \end{aligned}$$

7. *A Pitot tube is used to measure air velocity. If a manometer connected to the instrument indicates a difference in pressure head of* $4\,\mathrm{mm}$ *of water between the entry to the tube and a point on the same streamline upstream, calculate the air velocity. (Density of air is* $1.2\,\mathrm{kg\,m^{-3}}$*) (Ans:* $8.08\,\mathrm{m\,s^{-1}}$*)*

   Consider the *Bernoulli* equation applied between an upstream point on a streamline and the stagnation point on the same streamline at the mouth of the Pitot tube:

$$\begin{aligned} \frac{p_1}{\rho_a} + gz_1 + \frac{1}{2}U_1^2 &= \frac{p_2}{\rho_a} + gz_2 + \frac{1}{2}U_2^2 \\ \frac{p_1}{\rho_a} + 0 + \frac{1}{2}U_1^2 &= \frac{p_2}{\rho_a} + 0 + \frac{1}{2}\times 0^2 \\ U_1 &= \sqrt{2\left(\frac{p_2}{\rho_a} - \frac{p_1}{\rho_a}\right)} \end{aligned}$$

   but we are told that $p_2 - p_1 = \rho_w g \times 0.004$, hence

$$U_1 = \sqrt{2\left(\frac{p_2 - p_1}{\rho_a}\right)} = \sqrt{2\left(\frac{\rho_w g \times 0.004}{\rho_a}\right)} = \sqrt{2\left(\frac{1000 \times 9.8 \times 0.004}{1.2}\right)} = 8.08\,\mathrm{m\,s^{-1}}.$$