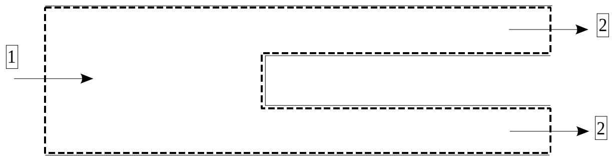
Hydraulics

Solution Sheet 5 – Mass conservation and the continuity equation

1. *Water flows along a pipe of* $50\,\mathrm{mm}$ *diameter with a mean velocity of* $1\,\mathrm{m\,s^{-1}}$ *until it meets a circular rod of* $46\,\mathrm{mm}$ *diameter arranged concentrically in the pipe. Draw a suitable control volume and calculate the mean velocity with which the water flows through the annular gap surrounding the rod. (Ans:* $6.51\,\mathrm{m\,s^{-1}}$*)*



The mass-conservation equation or continuity equation here states

$$
\begin{aligned}
-Q_1 + Q_2 &= 0\\
-U_1 A_1 + U_2 A_2 &= 0\\
-U_1 \frac{\pi}{4} D_1^2 + U_2 \frac{\pi}{4}\left(D_1^2 - D_2^2\right) &= 0\\
U_2 = U_1 \frac{D_1^2}{\left(D_1^2 - D_2^2\right)} &= 1 \times \frac{0.05^2}{0.05^2 - 0.046^2} = 6.51\,\mathrm{m\,s^{-1}}.
\end{aligned}
$$

2. *You are walking and come to a small river that you have to cross, but there is no bridge and no crossing stones. You see that part of it is wide, and part is narrow. That narrow section looks so much easier ... Where should you cross so that you minimise problems?*

   The lesson of the continuity equation for constant discharge is that where cross-sectional area is large, velocities are small, and *vice versa*. We have different possibilities:

   a. As a first approximation we might asssume that the cross-sectional area of the river is roughly constant along it. This means that
      i. where the river is wide, it is also shallow,
      ii. and where it is narrow it is deep,

      so that it is advisable to cross where the river is wide, even though it might look more daunting, so that we do not have to roll up our clothes very far.

   b. If the area does vary, then
      i. where it is narrow,
         I. if it is shallow then the area is smaller, the velocity is larger, and we might even be pulled over by the water,
         II. if it is deep, the area is large, the velocity is small, but we are going to get wet and might have to swim.
      ii. where it is wide, it is probably going to be shallow, and the velocities are still likely to be small.

   Overall, it seems best to cross where it is widest!

3. *Under conditions of turbulent flow in a channel, the mean horizontal velocity* u *at a point is approximately given by*

$$
\frac{u}{u_{\max}} = \left(\frac{z}{d}\right)^{1/7},
$$

   where $u_{\max}$ *is the velocity at the surface,* z *is the distance above the bed, and* d *is the depth of flow. Calculate the volumetric flow rate per unit span and the mean velocity in the channel. (Ans:*

$7/8 \times\ u_{\max} d$, $7/8 \times\ u_{\max}$).

$$
\begin{aligned}
Q &= 1 \times \int_0^d u\, dz = u_{\max} \int_0^d \left(\frac{z}{d}\right)^{1/7} dz = \frac{7}{8} u_{\max} d \\
U &= \frac{Q}{A} = \frac{\frac{7}{8} u_{\max} d}{d} = \frac{7}{8} u_{\max}.
\end{aligned}
$$