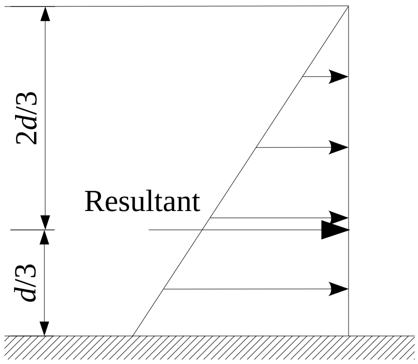
Hydraulics

Solution Sheet 3 – Forces on submerged surfaces

1. *A concrete dam with a vertical upstream face impounds water of density ρ to a total depth of d, with gravitational acceleration g.*

   a. *Sketch a graph showing the variation of pressure with elevation.*



   b. *Calculate the force on the dam per unit length.* We have the formula for the force on any plane

$$F_x = \rho g A_x \bar{h}_x.$$

   In this case, the area of a rectangle, $A_x = d \times 1$, as we are calculating per unit width, and $\bar{h} = d/2$, as the centroid is at the centre of the rectangle, hence

$$F_x = \tfrac{1}{2}\rho g d^2.$$

   This is a well-known and commonly-used result.

   c. *Calculate the depth of the centre of pressure, and add it to your sketch, showing the resultant of the force.*

   We can use the formula that we obtained in the additional sheet $M_x = h_{\text{CP}}\, F_x = \rho g I$, where I is the second moment of area about an axis in the surface. As the axis is about one side of a rectangle of width 1 and depth d, we use the formula from Table 1 on the Revision Sheet of Moments

$$h_{\text{CP}} = \frac{I}{A_x \bar{h}_x} = \frac{\text{2nd Moment of area about axis through surface}}{\text{1st Moment of area about axis at the surface}} = \frac{\frac{1}{3}\times 1 \times d^3}{\frac{1}{2}\times 1 \times d^3} = \frac{2}{3}d.$$

   Hence the centre of pressure lies $2d/3$ below the surface, which is also a well-known and useful result.

2. *Find the resultant force and the centre of pressure on*

   a. *A vertical square plate of* $1.8\,\mathrm{m}$ *sides, the centre of the plate* $1.2\,\mathrm{m}$ *below the surface.* (Ans.:) $38.1\,\mathrm{kN}$, $1.43\,\mathrm{m}$.

$$F = \rho g A \bar{h} = 1000 \times 9.8 \times 1.8^2 \times 1.2 = 38.1\,\mathrm{kN}.$$

$$h_{CP} - \bar{h} = \frac{I_G}{A\bar{h}} = \frac{1.8^4/12}{1.8^2 \times 1.2} = 0.225\,\mathrm{m},$$

$$h_{CP} = 1.2 + 0.225 \approx 1.43\,\mathrm{m}.$$

   b. *A vertical circular plate of* $1.8\,\mathrm{m}$ *diameter, the centre of the circle* $1.2\,\mathrm{m}$ *below the surface.*

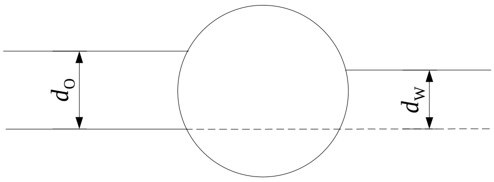
(Ans.: $29.9\,\mathrm{kN}$, $1.37\,\mathrm{m}$.)

$$F = \rho g A\bar{h} = 1000 \times 9.8 \times \pi/4 \times 1.8^2 \times 1.2 = 29.9\,\mathrm{kN}.$$

$$h_{CP} - \bar{h} = \frac{I_G}{A\bar{h}} = \frac{\pi \times 1.8^4/64}{\pi 1.8^2/4 \times 1.2} = 0.169\,\mathrm{m},$$

$$h_{CP} = 1.2 + 0.169 \approx 1.37\,\mathrm{m}.$$

3. *The cylindrical boom shown is required to retain a spill of oil of depth d_{o} and relative density σ_{o} by floating in seawater of relative density σ_{w}.*



a. *Show that the seawater on the other side of the boom rises to a depth of $d_{\mathrm{w}} = d_{\mathrm{o}}\sigma_{\mathrm{o}}/\sigma_{\mathrm{w}}$ above the oil-water interface.*

The pressure on the underside of the oil (pressure equals density of fluid $\times g \times$depth) is $\rho\sigma_{\mathrm{o}} g d_{\mathrm{o}}$. Applying the hydrostatic pressure equation between a point in the seawater just under the oil and a point on the other side of the boom at the same elevation,

$$\begin{aligned} \rho\sigma_{\mathrm{o}} g d_{\mathrm{o}} &= \text{pressure due to seawater} = \rho\sigma_{\mathrm{w}} g d_{\mathrm{w}}, \\ \therefore\ & d_{\mathrm{w}} = d_{\mathrm{o}}\sigma_{\mathrm{o}}/\sigma_{\mathrm{w}} \end{aligned}$$

b. *Show that the horizontal force on the boom per unit length is*

$$\tfrac{1}{2}\rho\sigma_{\mathrm{o}} g d_{\mathrm{o}}^2 \left(1 - \sigma_{\mathrm{o}}/\sigma_{\mathrm{w}}\right),$$

where ρ is the density of fresh water.

To calculate the force per unit length we exploit our knowledge (actually about to be obtained in the next question) that the force per unit length on a depthvertical rectangular shape, here the projection of the boom, is $1/2\times$density$\times g \times$depth^2:

$$\begin{aligned} \text{Force to right due to oil above interface} &= \tfrac{1}{2}\rho\sigma_{\mathrm{o}} g\, d_{\mathrm{o}}^2 \\ \text{Force to left due to seawater above level of interface} &= \tfrac{1}{2}\rho\sigma_{\mathrm{w}} g\, d_{\mathrm{w}}^2. \end{aligned}$$

Below the level of the interface the problem is symmetric, hence, the net force is

$$F = \tfrac{1}{2}\rho g \left(\sigma_{\mathrm{o}}\, d_{\mathrm{o}}^2 - \sigma_{\mathrm{w}}\, d_{\mathrm{w}}^2\right),$$

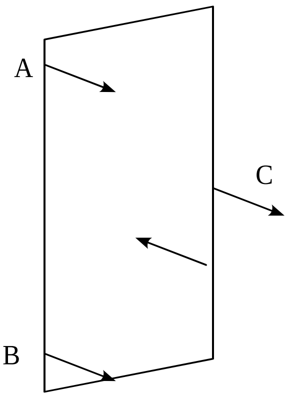
however using the result for d_{w} from part (a),

$$\begin{aligned} F &= \tfrac{1}{2}\rho g \left(\sigma_{\mathrm{o}}\, d_{\mathrm{o}}^2 - \sigma_{\mathrm{w}}\, d_{\mathrm{o}}^2\sigma_{\mathrm{o}}^2/\sigma_{\mathrm{w}}^2\right) \\ &= \tfrac{1}{2}\rho g\sigma_{\mathrm{o}}\, d_{\mathrm{o}}^2 \left(1 - \sigma_{\mathrm{o}}/\sigma_{\mathrm{w}}\right). \end{aligned}$$

c. *In which direction is the force? What implication does this have for the stability of such flexible booms in retaining oil spills? What plan shape do you think a flexible boom would take up?*

The force is to the right, as $\sigma_{\mathrm{o}}/\sigma_{\mathrm{w}}$ is less than 1. This means that the boom will will be pushed outwards by the oil, so that the configuration is generally stable, and we expect it to tend to a circle in plan.

4.



A vertical bulkhead in a ship has a door which has to be designed such that it will not be forced open if part of the ship fills with seawater of density $1025\,\mathrm{kg\,m^{-3}}$. *The door is* $2\,\mathrm{m}$ *high and* $1\,\mathrm{m}$ *wide, and the sea surface is assumed to be* $1\,\mathrm{m}$ *above the top of the door.*

a. *Calculate the force on the door and its position.*

The total force on the door is

$$F = \rho g A \bar{h} = 1025 \times 9.8 \times 2 \times 1 \times (1 + 2/2) = 40.2\,\mathrm{kN}$$

The position of the centre of pressure is

$$h_{CP} - \bar{h} = \frac{I_G}{A\bar{h}} = \frac{1 \times 2^3/12}{2 \times 1 \times (1 + 2/2)} = 0.167\,\mathrm{m},$$

hence it is $1.167\,\mathrm{m}$ below the top of the door. By symmetry, it is in the left-right centre of the door.

b. *The door is fastened by two hinges A and B on one vertical edge,* $15\,\mathrm{cm}$ *from top and bottom, and by a latch C in the centre of the other vertical edge. Calculate the forces on each hinge and on the latch when one face of the bulkhead is subject to water pressure (Ans.:* $20.1\,\mathrm{kN}$, $6.1\,\mathrm{kN}$, $14.0\,\mathrm{kN}$*).*

Now take moments about the axis AB - the moment arm of the force at C is twice that of the water force, hence $F_\mathrm{C} = 40.2/2 = 20.1\,\mathrm{kN}$.

Now take moments about a horizontal axis through B. The moment arm of F_A is $2 - 2 \times 0.15 = 1.7\,\mathrm{m}$. The moment arm of F_C is $1 - 0.15 = 0.85\,\mathrm{m}$. The water force is $0.167\,\mathrm{m}$ below the centroid, hence it has a moment arm of $0.85 - 0.167 = 0.683\ \mathrm{m}$.Hence

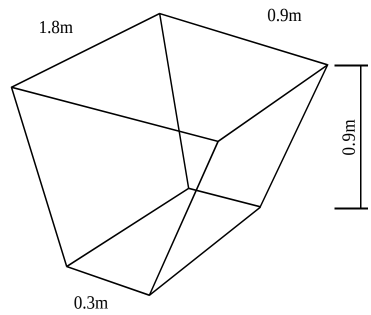
$$\begin{aligned} F_\mathrm{A} \times 1.7 + F_\mathrm{C} \times 0.85 - 40.2 \times 0.683 &= 0 \\ F_\mathrm{A} \times 1.7 &= -20.1 \times 0.85 + 40.2 \times 0.683 \\ F_\mathrm{A} &= 6.1\,\mathrm{kN} \end{aligned}$$

To evaluate the remaining force we just consider horizontal equilibrium and obtain

$$F_\mathrm{B} = 40.2 - 20.1 - 6.1 = 14.0\,\mathrm{kN}$$

5. *A mixing tank with trapezoidal ends is* $1.8\,\mathrm{m}$ *long,* $0.9\,\mathrm{m}$ *deep and* $0.9\,\mathrm{m}$ *wide at the top, tapering to* $0.3\,\mathrm{m}$ *wide at the base. If the tank is completely filled with water, calculate*

   a. *The total weight force of water in the tank.*



The area of one end is

$$1/2 \times 0.9 \times (0.3 + 0.9) = 0.54\,\mathrm{m}^2$$

Hence the volume is $0.54 \times 1.8 = 0.972\ \mathrm{m}^3$, and the total weight is $\rho g V = 1000 \times 9.8 \times 0.972 \approx 9.5\,\mathrm{kN}$.

b. *The total force exerted by water on the base.*

$$\begin{aligned} F &= \text{Weight of water which would occupy the volume between base and surface} \\ &= \rho g V = 1000 \times 9.8 \times 0.9 \times 0.3 \times 1.8 \approx 4.8\,\mathrm{kN} \end{aligned}$$

c. *The total force exerted on one end and where it acts. To do this it will be necessary to calculate the first and second moments of area of the trapezium which forms the face about an axis along the top of the trapezium, the water level.*

i. *The first moment of area $M_\mathrm{O} = A\bar{h}$ can be calculated by using known properties of rectangles and triangles. It is suggested that you calculate it that way ($M_\mathrm{O} = 0.2025\,\mathrm{m}^3$).*

ii. *Then, to check – and to prepare for the next part – set up and evaluate an integral for that first moment of area in terms of the width b of the trapezium as a function of depth h.*

iii. *Then, set up and evaluate a similar integral for the second moment of area I_O, so that you can use the expression $h_\mathrm{CP} = I_\mathrm{O}/M_\mathrm{O}$.*

Solution:

i. To do this we first have to calculate $A\bar{h}$, which is the first moment of area of the trapezium about the surface. We can use the fact that the centroid of a triangle is $1/3$ of its height from its base.

$$\begin{aligned} A\bar{h} &= A\bar{h}\big|_{\text{square}} + 2 \times A\bar{h}\big|_{\text{triangle}} \\ &= 0.9 \times 0.3 \times \tfrac{1}{2} \times 0.9 + 2 \times \tfrac{1}{2} \times 0.3 \times 0.9 \times \tfrac{1}{3} \times 0.9 \\ &= 0.2025\,\mathrm{m}^3 \end{aligned}$$

ii. The width of the trapezium at h below the surface is $b = 0.9 - 0.6h/0.9 = 0.9 - 2h/3$.

$$\begin{aligned} M_\mathrm{O} &= \int_0^{0.9} bh\,dh \\ &= \int_0^{0.9} (0.9 - 2h/3)\,h\,dh \\ &= 0.2025\,\mathrm{m}^3 \end{aligned}$$

Hence $F = \rho g A\bar{h} = 1000 \times 9.8 \times 0.2025 = 1.98 \approx 2.0\,\mathrm{kN}$.

iii. Now to calculate its position we use

$$s_{CP} = \frac{\rho g \sin\theta\, I_\mathrm{O}}{\rho g \sin\theta\, A\bar{s}} = \frac{I_\mathrm{O}}{A\bar{s}},$$

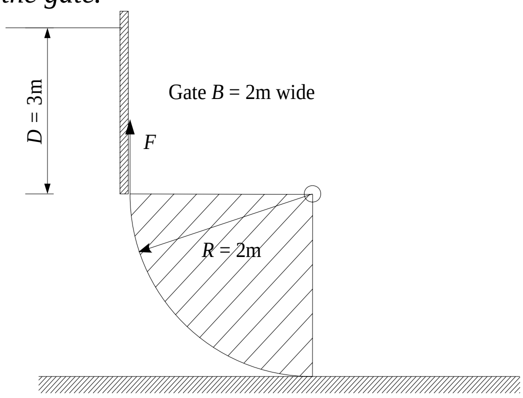
so we must also calculate $I_{\rm O}$ of the trapezoid.

$$\begin{aligned} I_{\rm O} &= \int_0^{0.9} bh^2\,dh \\ &= \int_0^{0.9} (0.9 - 2h/3)\,h^2\,dh \\ &= 0.1094\,{\rm m}^4, \end{aligned}$$

and so

$$h_{CP} = \frac{0.1094}{0.2025} = 0.54\,{\rm m} \text{ from the surface}$$

6. *A sluice gate consists of a radial gate of radius* $2\,{\rm m}$ *pivoted at its centre O, as shown in the figure. Calculate the magnitude and direction of the resultant force on the gate due to the water, and the net moment required to open the gate.*



The horizontal component on the gate is equal to the horizontal force on the projection onto a vertical plane of height R and width B with its top edge D below the surface. The force is

$$F_x = \rho g A_x \bar{h}_x = \rho g B R \left(D + \frac{R}{2} \right).$$

The position of the centre of pressure is that of the centre of pressure of the projection onto a vertical plane, which is a rectangle of dimensions $B \times R$, however the top edge is still D below the surface, so we *cannot* just use the formula $\frac{1}{3}bd^3$. We could use the parallel axes theorem, however it is more intuitive here to calculate the net second moment of a large rectangle of total depth $D + R$ with edge at the surface minus that of a rectangle of depth D but also with edge at the surface:

$$I = \tfrac{1}{3}B\,(D+R)^3 - \tfrac{1}{3}BD^3 = BR\left(\tfrac{1}{3}R^2 + RD + D^2\right).$$

In case anybody is disappointed that we didn't use the Parallel Axes Theorem, here it is, using the second moment of area of a rectangle about an axis through the centroid $\frac{1}{12}BR^3$ *plus the area times the square of the distance between surface axis and centroid* $BR \times (D + R/2)^2$*:*

$$I = \tfrac{1}{12}BR^3 + BR \times \left(D + \frac{R}{2} \right)^2 = BR\left(\tfrac{1}{3}R^2 + RD + D^2\right)$$

showing that the intuitive method worked. Now,

$$h_{\rm CP} = \frac{I}{A_x \bar{h}_x} = \frac{\frac{1}{3}R^2 + RD + D^2}{D + \frac{R}{2}}.$$

Now the vertical force is equal to the weight force on the fluid which would occupy the region between the level of the free surface and the surface on which we are to calculate the force. In this case this is the volume occupied by the quadrant of the gate *plus the rectangular region above that*.

$$F_z = \rho g V = \rho g B \left(\frac{\pi}{4} R^2 + DR \right) = \rho g B R \left(\frac{\pi}{4} R + D \right)$$

This force acts through the centre of gravity of that volume – *and is upwards*. To find this we take moments of volume about an axis through O (to do this we need to know that the distance of the centroid of a semi-circle or quadrant is $4/(3\pi)$ times the radius from the centre of the circle):

$$\begin{aligned}\text{Total moment of vertical force about O} &= \rho g B\left(\tfrac{1}{4}\pi R^2 \times \frac{4R}{3\pi} + DR \times \tfrac{1}{2}R\right) \\ &= \rho g B R^2\left(\frac{R}{3} + \frac{D}{2}\right)\end{aligned}$$

Thus, the vertical force acts a horizontal distance of $\bar{x}$ from O:

$$\bar{x} = \frac{\text{Moment of vertical force}}{F_z} = \frac{\rho g B R^2\left(\frac{R}{3} + \frac{D}{2}\right)}{\rho g B R\left(\frac{\pi}{4}R + D\right)} = \frac{R\left(\frac{R}{3} + \frac{D}{2}\right)}{\frac{\pi}{4}R + D}$$

Now we calculate the net moment of the horizontal and vertical forces about O:

$$\begin{aligned}\text{Net moment} &= \rho g B R^2\left(\frac{R}{3} + \frac{D}{2}\right) - F_x\left(h_{\text{CP}} - D\right) \\ &= \rho g B R^2\left(\frac{R}{3} + \frac{D}{2}\right) - \rho g B R\left(D + \frac{R}{2}\right)\left(\frac{\frac{1}{3}R^2 + RD + D^2}{D + \frac{R}{2}} - D\right) \\ &= 0\text{! As it should be in this radial gate case.}\end{aligned}$$

Now we put numbers in: $B = 2\,\mathrm{m}$, $D = 3\,\mathrm{m}$, $R = 2\,\mathrm{m}$, and $\rho = 1000\,\mathrm{kg\,m^{-3}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$:

$$\begin{aligned}F_x &= \rho g B R\left(D + \frac{R}{2}\right) = 1000 \times 9.8 \times 2 \times 2 \times (3 + 2/2) = 157\,\mathrm{kN} \\ h_{\text{CP}} &= \frac{\frac{1}{3}R^2 + RD + D^2}{D + \frac{R}{2}} = 4.083\,\mathrm{m} \\ F_z &= \rho g B R\left(\frac{\pi}{4}R + D\right) = 179\,\mathrm{kN} \\ \bar{x} &= \frac{R\left(\frac{R}{3} + \frac{D}{2}\right)}{\frac{\pi}{4}R + D} = 0.948\,\mathrm{m}\end{aligned}$$

Thus, a horizontal force of $157\,\mathrm{kN}$ acts to the right, a distance of $4.083 - 3 = 1.083\,\mathrm{m}$ below the pivot, and the vertical force of $179\,\mathrm{kN}$ acts vertically upwards (the gate is above the water) a horizontal distance of $0.948\,\mathrm{m}$ from the pivot.

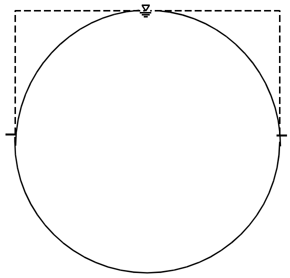
The resultant force is $\sqrt{157^2 + 179^2} = 238\,\mathrm{kN}$, at an angle to the horizontal of $\arctan(179/157) = 0.851\text{rad} \approx 49°$.

Net moment of pressure forces about pivot

$$\text{Net moment of pressure forces about pivot} = 157 \times 1.083 - 179 \times 0.948 \approx 0$$

to our order of accuracy. This result is expected, as every element of the pressure force is perpendicular to the arc, so that the line of action of the resultant is through the pivot, and there is no moment, the reason for using radial gates.

7. *A spherical container is made up of two hemispheres, the joint between the two halves being horizontal. The sphere is completely filled with water through a small hole in the top. It is found that 50kg of water are required for this purpose. If the two halves of the container are not secured together, what must be the mass of the upper hemisphere if it just fails to lift off the lower hemisphere? (Ans.:* $12.5\,\mathrm{kg}$*).*



The force in the direction of gravity is equal to the weight force on the fluid which would occupy a volume between the surface on which the force is to be calculated and the plane of the free surface. That is, it is the region between the upper hemisphere, and the level of the water surface, which forms a cylinder of radius r and height r. The direction of the force is upwards.

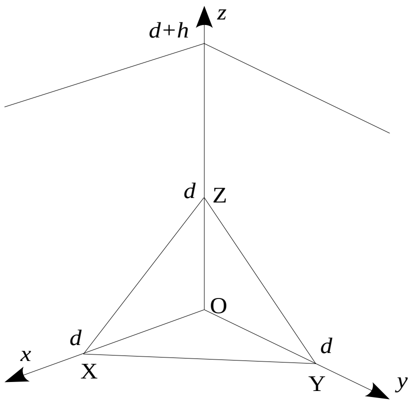
$$\begin{aligned} \text{The volume of this region} &= \underbrace{\pi r^2 \times r}_{\text{Vol. of cylinder}} - \underbrace{\tfrac{1}{2} \times \tfrac{4}{3}\pi r^3}_{\text{Vol. of hemisphere}} \\ &= \tfrac{1}{3}\pi r^3, \end{aligned}$$

hence the mass of water in that region is $m = \rho\frac{1}{3}\pi r^3$. However the mass of water which fills the sphere is $50\,\mathrm{kg}$, hence

$$\rho\tfrac{4}{3}\pi r^3 = 50,$$

and $m = 50/4 = 12.5\,\mathrm{kg}$.

8. *The corner of a tank is bevelled by equal dimensions d as shown in the figure. It is filled to h above the top of the bevel. What is the force on the triangular corner?*



$$\begin{aligned} \text{Force in } x \text{ direction} &= \text{Force on projection OYZ} \\ &= \rho g A_{OYZ}\,\bar{h}_{OYZ} = \rho g \times \tfrac{1}{2}d^2 \times \left(h + \tfrac{2}{3}d\right) = \frac{\rho g d^2}{6}\left(3h + 2d\right) \\ &= \text{Force in } y \text{ direction by symmetry} \end{aligned}$$

$$\begin{aligned} \text{Force in } z \text{ direction} &= \text{Weight force on prism above XYZ} \\ &= \rho g \times \text{ Volume} = \rho g \times \frac{d^2}{2} \times \tfrac{1}{3}(h + d + h + d + h) = \frac{\rho g d^2}{6}\left(3h + 2d\right). \end{aligned}$$

Thus, each of the force components is the same. This is what we would expect for such a surface whose direction cosines are the same for all 3 directions.