Gradually-varied flow equation

The differential equation governing the water profile in a channel is

$$\frac{dh}{dx} = \frac{\tilde{S} - Q^2/K^2}{1 - \beta F^2}, \qquad \text{(Gradually-varied flow equation)}$$

where h is the water depth, $\tilde{S}$ is the bottom slope, Q is the discharge, K is the conveyance, and βF^2 (F the Froude number) expresses the fluid inertia effects, which can be ignored for a channel on a sufficiently mild slope. For a wide channel of constant mild slope, using the Gauckler-Manning-Strickler equation, the differential equation can then be written

$$\frac{dh}{dx} = S_0 \left(1 - \left(\frac{h_0}{h}\right)^{10/3}\right), \qquad \text{(Approximation to GVFE)}$$

where h_0 is the *normal depth* for the channel, that depth when the water surface is parallel to the bed, which satisfies the equation

$$q = \frac{1}{n} h_0^{5/3} \sqrt{S_0},$$

where q is the discharge per unit width $(\mathrm{m}^2\,\mathrm{s}^{-1})$ and n is Manning's resistance coefficient.

A wide canal has a bed slope of 1×10^{-4}, a Manning $n = 0.02$, and flows normally with a depth of $2\,\mathrm{m}$. It is proposed to install a barrage on it which will raise the water level at the barrage to $h = 3\,\mathrm{m}$. You are to investigate how far upstream the enhanced water levels will affect the channel and enable irrigation of the surrounding countryside.

1. Check that the term $\beta F^2 = \beta q^2/gh^3$ will be sufficiently small that the Approximation to GVFE is accurate enough, ($\beta \approx 1.05$).
2. Solve the Approximation to GVFE numerically using Euler's method, starting with $h(0) = 3\,\mathrm{m}$ and stepping upstream backwards in x (*i.e.* Δx is negative) with steps of $1000\,\mathrm{m}$ until the water level is within $1\,\mathrm{cm}$ of the normal depth.
3. How far upstream is it? Are you surprised?
4. Plot your results on a carefully-labelled graph, using Excel or any other plotting software. For clarity, it will be best to plot just h, which will be decaying to a constant value upstream.

   While that is good for present purposes of accuracy comparison, it is also good to generate a plot of the actual surface elevation to give a more physical feel. Plot the surface elevation also, which is h plus the local bed elevation.

   Plot all answers to the following questions on the same figures.
5. Repeat your calculations with steps of $500\,\mathrm{m}$. How do your two answers compare – at $x = -20\,\mathrm{km}$ for example?
6. Refine your solution at every $1000\,\mathrm{m}$, using Richardson extrapolation, which in this case can be written for a point X:

   $$h_{\text{Improved}}(X) = 2h_{\Delta/2}(X) - h_{\Delta}(X).$$
7. You could repeat the Euler solution with smaller values of the step to check this.
8. Explore the nature of convergence by considering several values of the space step and comparing results at say, $-20\,\mathrm{km}$. Plot a graph and add the Richardson result, nominally at step length of 0.

9. Use the analytical solution given in lectures.

$$h = h_0 + (h_1 - h_0)\, e^{\mu_0 x}, \tag{1}$$

where h_1 is the depth at $x = 0$ (I know, it is not such a good notation, but I used 0 for the undisturbed solution), and μ_0 is a constant decay rate given by

$$\mu_0 \approx \frac{10}{3}\frac{S_0}{h_0}. \tag{2}$$

It does not work so well near the gate, as the checked-up depth of 50% greater than the normal depth is probably too much for the linearisation to be accurate. Further away it is more accurate and gives a good indication of the solution.

10. Experiment with different slope, say a steep slope of $S_0 = 1 \times 10^{-3}$, and see how the length of the backwater changes. You will have to use smaller steps!

11. In the lectures we have been emphasising the value of a small step, so the huge step magnitude recommended might come as a surprise ... it is actually possible because in scaled terms the step is small. We could re-write the equation as

$$\frac{dh}{d\left(S_0 x\right)} = \ldots$$

and the magnitudes of the variable $S_0 x$ are more reasonable. In fact, $S_0 x$ is the bed elevation relative to the point $x = 0$, so we could think of the re-written differential equation as

$$\frac{dh}{dz_{\text{Bed}}} = \ldots,$$

and solve it in those terms.

Corrections

Dear Students,

I have made a mess of both the Question Sheet and my Worked Answer. The second equation on the Question Sheet was wrong – the exponent should have been $10/3$ rather than $5/3$. If we had used Chézy-Weisbach then the exponent would have been 3, which was what I had in my original worked solution – but I was using Manning resistance, so it was confused ...

I have corrected the two previous pages above. Now I suggest you just use my Worked Solution at:

URL: *http://johndfenton.com/Lectures/Computations+Hydraulics/GVFE.xls*.

We may as well use the full formulation of the GVFE, the first equation on the Question Sheet, which uses the square of the Froude number (we can test the importance of that term by setting $\beta = 0$ in our calculations). We have from the G-M-S equation for uniform flow

$$q = \frac{1}{n} h_0^{5/3} \sqrt{S_0}$$

and the GVFE for a wide channel becomes

$$\frac{dh}{dx} = \frac{S_0 \left(1 - \left(\frac{h_0}{h}\right)^{10/3}\right)}{1 - \frac{\beta q^2}{g h^3}}, \qquad \text{(GVFE for Wide Channel)}$$

which is very little more complicated than the simplification I proposed.

- You will see that the analytical approximation is not very accurate for this case with the large depth imposed at $x = 0$. I suggest you experiment with different values – try changing cell C5 to say, 2.5, and you will see that it is rather more accurate.
- Also try $\beta = 0$ in C7: you will see that we could have used the original simplification, as the term is small.
- Change the slope to the rather steeper $S_0 = 0.001$, and you will find that the backwater is much shorter, as suggested by equation (2). The results will go wrong - you will have to change the step size for each method to $1/10$ of the original value.