June 9, 2010

Computational Hydraulics

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Abstract

The aim is to give an understanding of the numerical approximation and solution of physical systems, especially in open channel hydraulics, but with more general application in hydraulics and fluid mechanics generally.

The marriage of hydraulics and computations has not always been happy, and often unnecessarily complicated methods have been developed and used, when a little more computational sophistication would have shown the possibility of using rather simpler methods.

This course considers a number of problems in hydraulics, and it is shown how general methods, based on computational theory and practice, can be used to give simple methods that can be used in practice.

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1. Introduction – numerical methods

In the first part of the course we considered methods for common problems, including solutions of nonlinear equations; systems of equations; interpolation of data including piecewise polynomial interpolation; approximation of data; differentiation and integration; numerical solution of ordinary differential equations. The lecture notes are at **URL:** *http://johndfenton.com/Lectures/Numerical-Methods/Numerical-Methods.pdf*.

For the remainder of the course we will be considering problems in hydraulics, largely open-channel hydraulics, and will be developing methods to solve those problems computationally. Initially the problems considered are those where there is a single independent variable, and we solve ordinary differential equations, notably for calculating the passage of floods through reservoirs and for steady flow in open channels. Subsequently we consider unsteady river problems such that variation with space and time are involved, and we solve partial differential equations. We consider model equations that describe important phenomena in hydraulics and fluid mechanics, and develop methods for them. Then these methods are carried over to the full equations of open channel hydraulics.

It will be found that the methods developed here are based on general methods of computational mechanics, and are rather simpler than many methods used in computational hydraulics by large software companies and organisations. It is hoped that these lectures will empower people and give them confidence that they can solve problems for themselves, incorporating a spirit of modelling, and not be dependent on large software packages whose properties are often spurious.

2. Reservoir routing

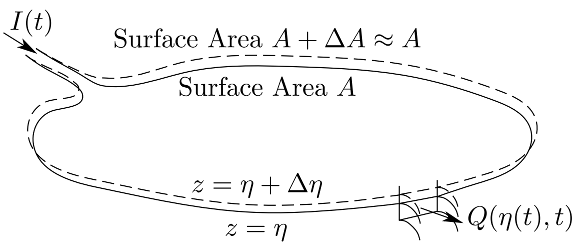

Figure 2-1. Reservoir or tank, showing surface level varying with inflow, determining the rate of outflow

Consider the problem shown in figure 2-1, where a generally unsteady inflow rate I enters a reservoir or a storage tank, and we have to calculate what the outflow rate Q is, as a function of time t. The action of the reservoir is usually to store water, and to release it more slowly, so that the outflow is delayed and the maximum value is less than the maximum inflow. Some reservoirs, notably in urban areas, are installed just for this purpose, and are called *detention* reservoirs or storages.

Also consider the volume conservation equation, stating that the rate of change of volume in a reservoir is equal to the difference between inflow and outflow rates:

$$\frac{dS}{dt} = I(t) - Q(\eta(t), t), \tag{2.1}$$

in which S is the volume of water stored in the reservoir, t is time, $I(t)$ is the volume rate of inflow, which is a known function of time or known at points in time, and $Q(\eta(t), t)$ is the volume rate of outflow, which is usually a known function of the surface elevation $\eta(t)$, itself a function of time as shown, and the extra dependence on time is if the outflow device such as a weir or a gate is moved. The procedure of solving this differential equation is called *Level-pool Routing*.

The process can be visualised as in figure 2-2. Initially it is assumed that the flow is steady, when outflow equals the inflow. Then, a flood comes down the river and the inflow increases substantially as shown. It is assumed that the spillway cannot cope with this increase and so the water level rises in the reservoir until at the point O when the outflow over the spillway now balances the inflow. At this point, where $I = Q$, equation (2.1) gives $dS/dt = 0$ so that the surface elevation in the reservoir also has a maximum, and so does the outflow over the spillway, so that the outflow is a maximum when the inflow equals the outflow. After this, the inflow might reduce quickly, but it

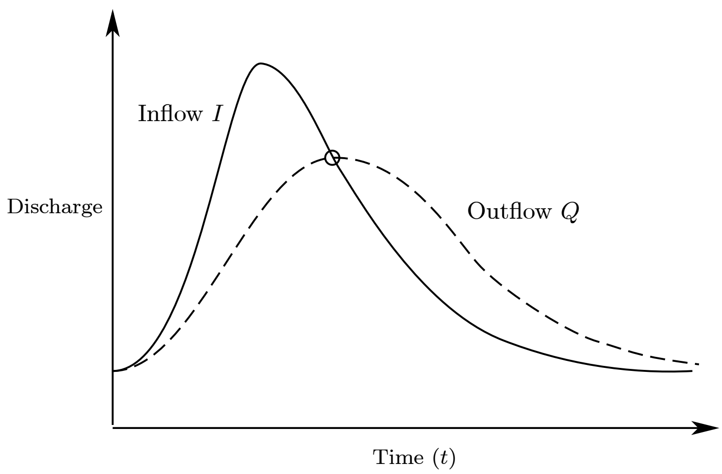

Figure 2-2. Typical inflow and outflow hydrographs for a reservoir

still takes some time for the extra volume of water to leave the reservoir.

2.1 The traditional "Modified Puls" method

The traditional method of solving the problem is to relate the storage volume S to the surface elevation η, which can be done from knowledge of the variation of the plan area A of the reservoir surface and evaluating

$$S(\eta) = \int^{\eta} A(z)\,dz, \tag{2.2}$$

usually by a low-order numerical approximation for various surface elevations. Equation (2.1) can be then be written

$$\frac{dS}{dt} = I(t) - Q(S(\eta(t)), t), \tag{2.3}$$

so that it is an ordinary differential equation for S as a function of time t that could be solved numerically.

It can be solved numerically by any one of a number of methods, of which the most elementary are very simple, such as Euler's method. Strangely, the fact that the problem is merely one of solving a differential equation seems not to have been recognised. The traditional method of solving equation (2.3), described in almost all books on hydrology, is unnecessarily complicated. The differential equation is approximated by a forward difference approximation of the derivative and then a trapezoidal approximation for the right side, so that it is written

$$\frac{S(t+\Delta) - S(t)}{\Delta} = \tfrac{1}{2}\left(I(t) + I(t+\Delta)\right) - \tfrac{1}{2}\left(Q(S(t+\Delta), t+\Delta) + Q(S(t), t)\right),$$

where Δ is a finite step in time. The equation can be re-arranged to give

$$\frac{2S(t+\Delta)}{\Delta} + Q(S(t+\Delta), t+\Delta) = I(t) + I(t+\Delta) + \frac{2S(t)}{\Delta} - Q(S(t), t) \tag{2.4}$$

At a particular time level t all the quantities on the right side can be evaluated. The equation is then a nonlinear equation for the single unknown quantity $S(t+\Delta)$, the storage volume at the next time step, which appears transcendentally on the left side. There are several methods for solving such nonlinear equations and the solution is in principle not particularly difficult. However textbooks at an introductory level are forced to present procedures for solving such equations (by graphical methods or by inverse interpolation) which tend to obscure with mathematical and numerical detail the underlying simplicity of reservoir routing. At an advanced level a number of practical difficulties may arise, such that in the solution of the nonlinear equation considerable attention may have to be given to pathological cases. As the methods are iterative, several function evaluations of the right side of equation (2.4) are necessary at each time step.

2.2 An alternative form of the governing equation

Another form of the differential equation can be simply obtained. Figure 2-1 shows the reservoir or tank surface, showing the surface level initially at $z = \eta$ and the level some time later at $z = \eta + \Delta\eta$. Of course in the limit $\Delta\eta \to 0$ the change in storage dS is given by

$$\frac{dS}{d\eta} = A(\eta), \tag{2.5}$$

in terms of the plan area A of the water surface at elevation η. Substituting into equation (2.1) an equivalent form of the storage equation is obtained:

$$\frac{d\eta}{dt} = \frac{I(t) - Q(\eta, t)}{A(\eta)} = f\left(t, \eta\right), \tag{2.6}$$

which is a differential equation for the surface elevation itself, and we have introduced the symbol $f\left(.\right)$ for the right side of the equation. This equation has been presented by Chow, Maidment & Mays (1988, Section 8.3), and by Roberson, Cassidy & Chaudhry (1988, Section 10.7), but as a supplementary form to equation (2.3). In fact it has advantages over that form, and this formulation is be preferred. It makes no use of the storage volume S, which then does not have to be calculated. Also, the dependence of outflow Q on surface elevation is usually a simple expression from a weir-flow formula or the like. Usually where outflow is *via* outlet pipes and spillways, it can be expressed as a simple mathematical function of η, usually involving terms like $(\eta - z_{\text{outlet}})^{1/2}$ and/or $(\eta - z_{\text{crest}})^{3/2}$, where z_{outlet} is the elevation of the pipe or tailrace outlet to atmosphere and z_{crest} is the elevation of the spillway crest. The dependence on t can be obtained by specifying the vertical gate opening or valve characteristic as a function of time, usually as a coefficient multiplying these powers of η. In general the η formulation requires a table for A and η, obtained from planimetric information from contour maps, to give $A(\eta)$ by interpolation.

2.3 Solution as a differential equation

The lecturer (Fenton 1992) adopted the differential equation (2.6) and emphasised that it was just a differential equation that could be solved by any method for differential equations, most much simpler than the modified Puls method. When he presented it at a conference in Christchurch in New Zealand, a friend of his said at question time "But this is trivial! I always solve it like that. Doesn't everybody?". The answer, then as now, was "strangely and regrettably, no".

2.3.1 Euler's method

This is the simplest but least-accurate of all methods, being of first-order accuracy only. It is

$$\eta_{i+1} = \eta_i + \Delta f(t_i, \eta_i) + O\left(\Delta^2\right) = \eta_i + \Delta \frac{I(t_i) - Q(\eta_i, t_i)}{A(\eta_i)} + O\left(\Delta^2\right), \tag{2.7}$$

where we use the notation $\eta_i = \eta\left(i\Delta\right)$ for the solution at time step i, and $f\left(.\right)$ for the right side of the differential equation as shown. This makes the presentation of the next higher approximation simpler.

2.3.2 Heun's method

The scheme is evaluated in two steps and can be written:

$$\eta^*_{i+1} = \eta_i + \Delta\, f\left(t_i, \eta_i\right), \tag{2.8a}$$

$$\eta_{i+1} \approx \eta_i + \frac{\Delta}{2}\left(f\left(t_i, \eta_i\right) + f\left(t_{i+1}, \eta^*_{i+1}\right)\right) + O\left(\Delta^2\right). \tag{2.8b}$$

2.3.3 Richardson extrapolation

For simple Euler time-stepping solutions of ordinary differential equations, if we perform two simulations, one with a time step Δ and then one with $\Delta/2$, we have that at any step a more accurate solution, denoted by η^+ is

$$\eta^+(t) = 2\,\eta(t, \Delta/2) - \eta(t, \Delta) + O\left(\Delta^3\right), \tag{2.9}$$

where the numerical solution at time t has been shown as a function of the step. This is very simply implemented.

2.3.4 Higher-order methods

Any method or software for solving ordinary differential equations can be used. Fenton (1992) considered several, including higher-order Runge-Kutta methods, but for most purposes those mentioned here are adequate.

2.4 An example

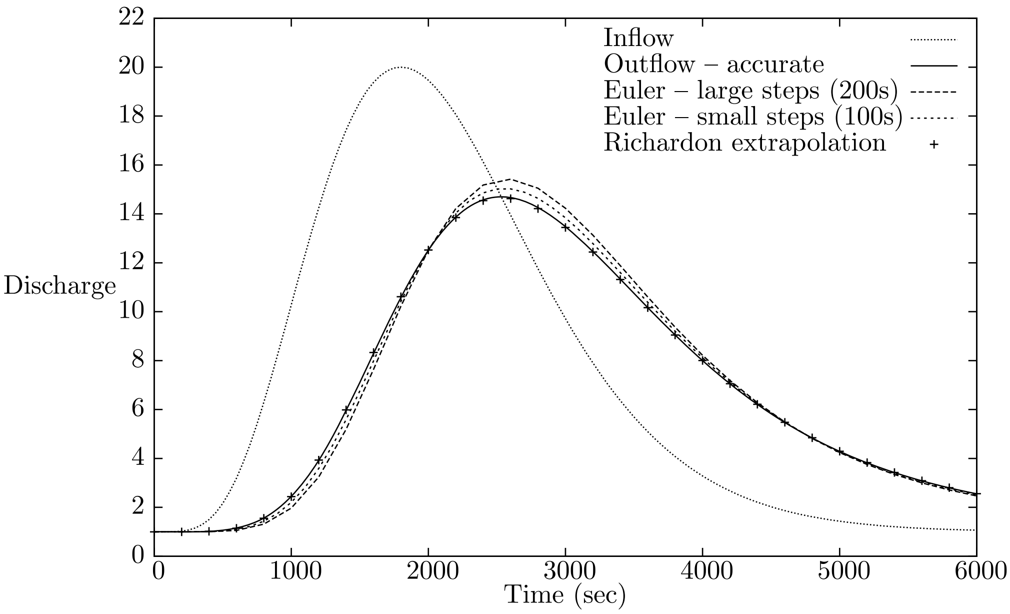

Figure 2-3. Computational results for the routing of a sudden storm through a small detention reservoir

Consider a small detention reservoir, square in plan, with dimensions 100m by 100m, with water level at the crest of a sharp-crested weir of length of $b = 4\,\mathrm{m}$, where the outflow over the sharp-crested weir can be taken to be

$$Q(\eta) = 0.6\sqrt{g}b\eta^{3/2}, \tag{2.10}$$

where $g = 9.8\,\mathrm{ms}^{-2}$.The surrounding land has a slope (V:H) of about 1:2, so that the length of a reservoir side is $100 + 2 \times 2 \times \eta$, where η is the surface elevation relative to the weir crest, and

$$A(\eta) = (100 + 4\eta)^2 .$$

The inflow hydrograph is:

$$I(t) = Q_{\mathrm{min}} + (Q_{\mathrm{max}} - Q_{\mathrm{min}}) \left(\frac{t}{T_{\mathrm{max}}} e^{1-t/T_{\mathrm{max}}} \right)^5 , \tag{2.11}$$

where the event starts at $t = 0$ with Q_{min} and has a maximum Q_{max} at $t = T_{\mathrm{max}}$. This general form will be useful throughout this course, as it mimics a typical storm, with a sudden rise, and slower fall. In this example we consider a typical sudden local storm event, with $Q_{\mathrm{min}} = 1\,\mathrm{m}^3\mathrm{s}^{-1}$, and $Q_{\mathrm{max}} = 20\,\mathrm{m}^3\mathrm{s}^{-1}$ at $T_{\mathrm{max}} = 1800\,\mathrm{s}$.

The problem was solved with an accurate 4th-order Runge-Kutta scheme, and the results are shown as a solid line on figure 2-3, to provide a basis for comparison. Next, Euler's method (equation 2.7) was used with 30 steps of $200\,\mathrm{s}$, with results that are barely acceptable. Halving the time step to $100\,\mathrm{s}$ and taking 60 steps gave the better results shown. It seems, as expected from knowledge of the behaviour of the global error of the Euler method, that it has been halved at each point. Next, applying Richardson extrapolation, equation (2.9), gave the results shown by the crosses. They almost coincide with the accurate solution, and cross the inflow hydrograph with an apparent horizontal gradient, as required, whereas the less-accurate results do not. Overall, it seems that the simplest Euler method can be used, but is better together with Richardson extrapolation. In fact, there was nothing in this example that required large time steps – a simpler approach might have been just to take rather smaller steps.

The role of the detention reservoir in reducing the maximum flow from $20\,\mathrm{m}^3\mathrm{s}^{-1}$ to $14.7\,\mathrm{m}^3\mathrm{s}^{-1}$ is clear. If one wanted a larger reduction, it would require a longer spillway. It is possible in practice that this problem might have been solved in an inverse sense, to determine the spillway length for a given maximum outflow.

3. The equations of open channel hydraulics

In a course preliminary to this one[1], we went to a lot of trouble to obtain the long-wave equations for rivers and canals, making the general assumption that flow was essentially one-dimensional, observing that such channels are much longer than they are wide or deep, and that variation along the stream is gradual.

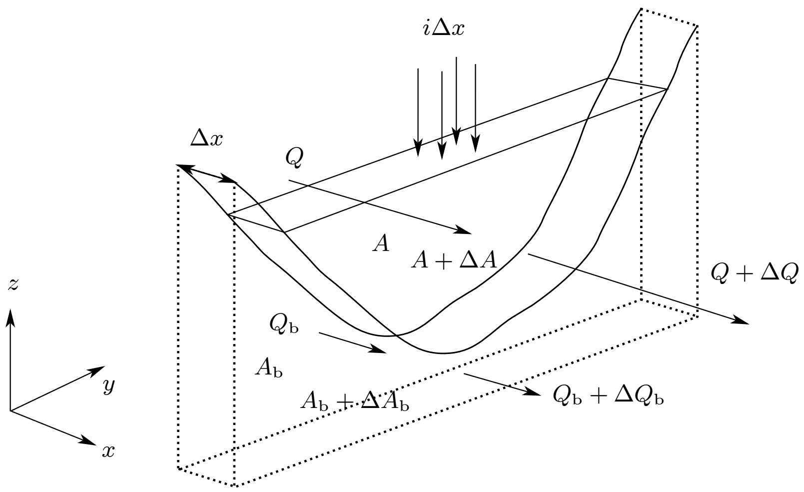

Figure 3-1. Elemental length of channel showing control volumes

Consider Figure 3-1, showing an elemental slice of channel of length Δx with two stationary vertical faces across the flow. It includes two different control volumes. The surface shown by solid lines includes the channel cross section, but not the moveable bed, and is used for mass and momentum conservation of the channel flow. The surface shown by dotted lines contains the soil moving as bed load. Each is modelled separately, subject to a mass conservation equation, and each to a relationship that determines the flux. In this course we will not consider the movement of soil.

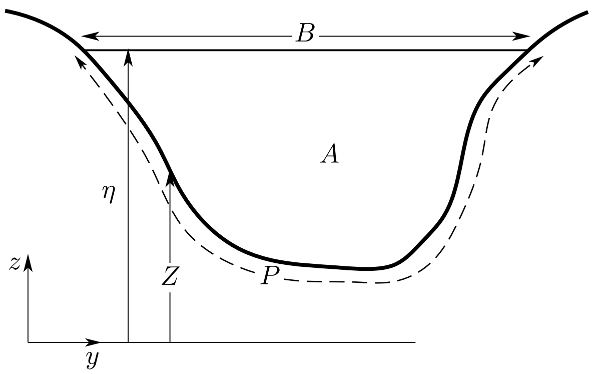

Figure 3-2. Cross-section of channel showing important dimensions

The most important dependent quantities that we need to calculate are Q, the *discharge* or *volume flux*, as shown in figure 3-1, and η, the elevation of the free surface, as shown in the cross-section in figure 3-2. The equations that will be considered are in terms of other geometric quantities shown on that figure:

- A – cross-sectional area
- B – top width of the surface
- P – the wetted perimeter around which the resistance to the flow acts.
- Z – the elevation of the bed at any point, which we will see appears only in determining the mean slope.

In fact, it is surprising that so few geometric quantities are involved!

[1] The lecture notes are available here: **URL:** *http://johndfenton.com/Lectures/RiverEngineering/River-Engineering.pdf*

3.1 Mass conservation equation

If rainfall, seepage, or tributaries contribute an inflow volume rate i per unit length of stream, the mass conservation equation can be obtained

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=i. \tag{3.1}$$

Remarkably for hydraulics, this is almost exact. The only approximation has been that the channel is straight. This say that if the discharge Q is varying along the stream, and/or if there is inflow i, then the cross-sectional area A will change with time, to accommodate the volume required.

We usually work in terms of water surface elevation stage η, which is easily measurable and which is practically more important. We make a significant assumption here, but one that is usually accurate: the water surface is horizontal across the stream. Now, if the surface changes by an amount $\delta\eta$ in an increment of time δt, then the area changes by an amount $\delta A = B\,\delta\eta$, where B is the width of the stream surface. Taking the usual limit of small variations in calculus, we obtain $\partial A/\partial t = B\,\partial\eta/\partial t$, and the mass conservation equation can be written

$$B\frac{\partial \eta}{\partial t}+\frac{\partial Q}{\partial x}=i. \tag{3.2}$$

This is a partial differential equation in terms of distance along the channel x and time t, for the surface elevation $\eta(x,t)$, which we have assumed is horizontal across the channel, and the discharge $Q(x,t)$.

3.2 Momentum conservation equation

If we consider the fluid momentum in the x direction in the elemental slice of the above figure, we obtain another partial differential equation in x and t, which is surprisingly simple in view of the complexity of the problem:

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\beta\frac{Q^2}{A}\right)+gA\frac{\partial \eta}{\partial x}=-\Lambda P\frac{Q\,|Q|}{A^2}. \tag{3.3}$$

The terms are:

- $\partial Q/\partial t$ – comes from the rate of change of momentum in the control volume with time
- $\partial\left(\beta Q^2/A\right)/\partial x$ – comes from the variation of momentum flux due to fluid velocity along the channel. The quantity β is an empirical coefficient, modelling the velocity distribution across the channel. It is about 1.05 and in many places taken to be 1.0.
- $gA\partial\eta/\partial x$ – comes from the pressure forces in the water. The pressure distribution has been assumed to be hydrostatic, such that pressure is proportional to the depth of water above any point. If the surface slopes, such that $\partial\eta/\partial x$ is finite (and almost always negative), then in the water, along any horizontal line, there is a net downstream pressure gradient, which when integrated over the channel section, gives this term.
- $\Lambda PQ\,|Q|\,/A^2$ – we consider the resistance to motion in the channel as being caused by a shear force τ on the boundary, with Weisbach's empirical formula:

  $$\tau=\frac{\lambda}{8}\rho\left(\frac{Q}{A}\right)^2, \tag{3.4}$$

  where λ is the dimensionless Weisbach resistance factor, ρ is the density, and Q/A is the mean velocity along the pipe or channel. It is convenient for us to consider channels which are not very steep, to introduce the parameter $\Lambda=\lambda/8$, and to integrate this stress around the wetted perimeter P, giving the result shown, where we have written $Q\,|Q|$ rather than Q^2 to allow for possible negative values of Q in tidal estuaries.

In fact, equation (3.3) is not ready for use, as expanding the second term would give a contribution $\partial A/\partial x$, and we need to express this in terms of the free surface gradient. It can be shown that

$$\frac{\partial A}{\partial x}=B\left(\frac{\partial \eta}{\partial x}+\tilde{S}\right), \tag{3.5}$$

where the symbol $\tilde{S}$ for the mean downstream bed slope across the section has been introduced such that

$$\tilde{S} = -\frac{1}{B}\int_B \frac{\partial Z}{\partial x}\,\mathrm{d}y, \tag{3.6}$$

where the negative sign has been used such that in the usual case when Z decreases with x, this mean downstream bed slope at a section is positive. In the usual case where bed topography is poorly known, a reasonable local approximation or assumption is made to give a value of $\tilde{S}$.

3.3 The long wave equations

The momentum equation (3.3) can then be expressed in terms of $\partial\eta/\partial x$. Writing it and the mass-conservation equation again, we have the pair of partial differential equations

$$\frac{\partial \eta}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = \frac{i}{B}, \tag{3.7}$$

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2 B}{A^2}\right)\frac{\partial \eta}{\partial x} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Lambda P\frac{Q\,|Q|}{A^2}. \tag{3.8}$$

These are the *long wave* equations, sometimes called the *Saint-Venant* equations. They are the basis for most open channel hydraulics.

3.3.1 Other resistance formulations – Chézy and Gauckler-Manning-Strickler

The simplest model of a river is that the channel is prismatic, the flow is *steady* ($\partial/\partial t = 0$) and it is *uniform*, with a constant bed and surface slope S_0 such that $\partial\eta/\partial x = -\tilde{S} = -S_0$. The momentum equation (3.8) gives

$$\Lambda P\frac{Q^2}{A^2} = gAS_0,$$

giving the Weisbach equation for steady uniform flow

$$\frac{Q}{A} = \sqrt{\frac{g}{\Lambda}\frac{A}{P}S_0}. \tag{3.9}$$

Other (and more traditional) formulations of the resistance term include those of Chézy and Gauckler-Manning-Strickler. For them to agree for steady uniform flow, Λ can be expressed in terms of the Chézy coefficient C, the Manning coefficient n, and the Strickler coefficient k_{St} respectively, the latter two being in *SI* units:

$$\Lambda = \frac{g}{C^2} = \frac{gn^2P^{1/3}}{A^{1/3}} = \frac{g}{k_{\mathrm{St}}^2}\frac{P^{1/3}}{A^{1/3}}, \tag{3.10}$$

giving the familiar results for steady uniform flow (note that Chézy is the same form as Weisbach):

$$\text{Chézy} \quad : \quad \frac{Q}{A} = C\sqrt{\frac{A}{P}S_0}$$

$$\text{Gauckler-Manning-Strickler} \quad : \quad \frac{Q}{A} = \frac{1}{n}\left(\frac{A}{P}\right)^{2/3}\sqrt{S_0} = k_{\mathrm{St}}\left(\frac{A}{P}\right)^{2/3}\sqrt{S_0}$$

3.3.2 Conveyance and Friction Slope

It is convenient to introduce the *conveyance* K, so that the resistance term in the momentum equations appears as

$$-\Lambda P\frac{Q\,|Q|}{A^2} = -gA\frac{Q\,|Q|}{K^2}. \tag{3.11}$$

From equation (3.10), the various definitions of K become

$$K = \sqrt{\frac{g}{\Lambda}\frac{A^3}{P}} = C\sqrt{\frac{A^3}{P}} = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}} = k_{\mathrm{St}}\frac{A^{5/3}}{P^{2/3}}, \tag{3.12}$$

showing that is a convenient shorthand that includes the resistance coefficient of whatever law is being used, plus cross-sectional geometry terms. It has units of discharge, $\mathrm{L}^3\mathrm{T}^{-1}$.

A simple and important result for steady uniform flow is that

$$Q = K_0 \sqrt{S_0}, \tag{3.13}$$

where the subscript 0 denotes the conveyance corresponding to the normal depth h_0 of the uniform flow.

Many textbook presentations write the friction term in terms of a dimensionless quantity $S_{\mathrm{f}} = Q\,|Q|\,/K^2$, called the "friction slope", possibly better known as "resistance slope", so that the resistance term in the momentum equations appears as $-gAS_{\mathrm{f}}$. It is also known as the "slope of the energy grade line", or the "head gradient", which gives an uninformative and misleading picture, for in our momentum-based approach it is neither of those things. In textbook derivations of the steady equations S_{f} is actually calculated from the slope of the energy grade line, which it should not be.

3.3.3 The nature of the long wave equations

They are a pair of equations that can be written as a vector evolution equation

$$\frac{\partial \mathbf{u}}{\partial t} + C\frac{\partial \mathbf{u}}{\partial x} = \mathbf{r}\,(\mathbf{u})\,,$$

where $\mathbf{u}$ is the vector of unknowns, for example, $[\eta, Q]$, C is a 2×2 matrix with algebraic coefficients, and $\mathbf{r}$ is the vector of right side terms, due to inflow, slope, and resistance.

It can be shown that the system is hyperbolic, although this mathematical terminology seems not very useful for us. The implication of that is that solutions are of a wave-like nature. We will see that the behaviour of disturbances is more complicated than we might expect or is often stated. This arises because the right sides are functions of the dependent variables, that we have written here as $\mathbf{r}\,(\mathbf{u})$. In particular we will see that a common interpretation of the system in terms of *characteristics*, with the solution that of travelling waves with simple properties, is incorrect. The solution is actually more complicated: disturbances travel at speeds which depend on their length, and show diffusion as well.

4. Steady uniform flow in prismatic channels

Steady flow does not change with time; uniform flow is where the depth does not change along the waterway. For this to occur the channel properties also must not change along the stream, such that the channel is prismatic, and this occurs only in constructed canals. However in rivers if we need to calculate a flow or depth, it is common to use a cross-section which is representative of the reach being considered, and to assume it constant for the approximate application of theory. This is the simplest problem we consider!

The Weisbach and Chézy equations and the Gauckler-Manning-Strickler forms give formulae for the discharge Q in terms of resistance coefficient, slope S_0, area A, and wetted perimeter P:

$$\text{Weisbach-Chézy} \quad : \quad Q = \sqrt{\frac{8g}{\lambda}}\frac{A^{3/2}}{P^{1/2}}\sqrt{S_0} = \sqrt{\frac{g}{\Lambda}}\frac{A^{3/2}}{P^{1/2}}\sqrt{S_0} = C\frac{A^{3/2}}{P^{1/2}}\sqrt{S_0} \tag{4.1}$$

$$\text{G-M-S} \quad : \quad Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_0} = k_{\mathrm{St}}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_0} \tag{4.2}$$

in which both A and P are functions of the flow depth. Each equation show how flow increases with cross-sectional area and slope and decreases with wetted perimeter. The maximum depth is the *normal depth*, and determining it is a common problem.

Trapezoidal sections: Most canals are excavated to a trapezoidal section, and this is often used as a convenient approximation to river cross-sections too. In many of the problems in this course we will consider the case of trapezoidal sections. We will introduce the terms defined in Figure 4-1: the bottom width is W, the depth is h, the top width is B, and the *batter slope*, defined to be the ratio of H:V dimensions is γ. From these the following

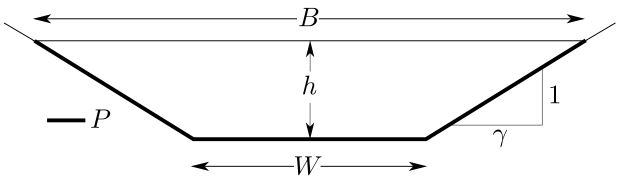

Figure 4-1. Trapezoidal section showing important dimensions

important section properties are easily obtained:

$$\begin{aligned}\text{Top width} &: \quad B = W + 2\gamma h \\ \text{Area} &: \quad A = h\,(W + \gamma h) \\ \text{Wetted perimeter} &: \quad P = W + 2\sqrt{1+\gamma^2}h.\end{aligned}$$

(*Ex. Obtain these relations*).

4.1 Computation of normal depth

If the discharge, slope, and the appropriate roughness coefficient are known, any of equations (4.1)-(4.2) is a transcendental equation for the normal depth h, which can be solved by the methods for solving transcendental equations described earlier.

In the case of wide channels, (*i.e.* channels rather wider than they are deep, $h \ll B$, which is a common case) neither the wetted perimeter P nor the breadth B vary much with depth h. Hence the quantity $A(h)/h$ also does not vary strongly with h. Hence we can rewrite the G-M-S expression:

$$Q = \frac{1}{n}\frac{A^{5/3}(h)}{P^{2/3}(h)}\sqrt{S_0} = \frac{\sqrt{S_0}}{n}\frac{(A(h)/h)^{5/3}}{P^{2/3}(h)} \times h^{5/3},$$

where most of the variation with h is contained in the last term $h^{5/3}$, and by solving for that term we can re-write the equation in a form suitable for direct iteration

$$h = \left(\frac{Qn}{\sqrt{S_0}}\right)^{3/5} \times \frac{P^{2/5}(h)}{A(h)/h},$$

where the first term on the right is a constant for any particular problem, and the second term is expected to be a relatively slowly-varying function of depth, so that the whole right side varies slowly with depth – a primary requirement that the direct iteration scheme be convergent and indeed be quickly convergent.

Experience with typical trapezoidal sections shows that this works well and is quickly convergent. However, it also works well for flow in circular sections such as sewers, where over a wide range of depths the mean width does not vary much with depth either. For small flows and depths in sewers this is not so, and a more complicated method such as the secant method might have to be used.

Example 4.1 Calculate the normal depth in a trapezoidal channel of slope 0.001, Manning's coefficient $n = 0.04$, bottom width $10\,\mathrm{m}$, with batter slopes $2:1$, carrying a flow of $20\,\mathrm{m}^3\,\mathrm{s}^{-1}$. We have $A = h\,(10 + 2\,h)$, $P = 10 + 4.472\,h$, giving the scheme

$$\begin{aligned} h &= \left(\frac{Qn}{\sqrt{S_0}}\right)^{3/5} \times \frac{(10 + 4.472\,h)^{2/5}}{10 + 2\,h} \\ &= 6.948 \times \frac{(10 + 4.472\,h)^{2/5}}{10 + 2\,h}\end{aligned}$$

and starting with $h = 2$ we have the sequence of approximations: 2.000, 1.609, 1.639, 1.637 – quite satisfactory in its simplicity and speed.

5. Steady gradually-varied flow – backwater computations

A common problem in river engineering is, for example, how far upstream water levels might be changed, and hence flooding possibly enhanced, due to downstream works such as the installation of a bridge or other obstacles. Far away from the obstacle or control, the flow may be uniform, but generally it is variable. The transition between conditions at the control or point of known level, and where there is uniform flow is described by the *Gradually-Varied Flow Equation*, which is an ordinary differential equation for the water surface height. The solution will approach uniform flow if the channel is prismatic, but in general we can treat non-prismatic waterways also. The steady flow approximation is often used as a first approximation, even when the flow is unsteady, such as in floods.

5.1 The differential equation

Consider the mass conservation equation for steady flow, when $\partial/\partial t \equiv 0$, and equation (3.7) becomes

$$\frac{dQ}{dx} = i,$$

with the solution obtained by integration

$$Q(x) = Q_0 + \int_{x_0}^{x} i \, \mathrm{d}x,$$

where at an upstream station x_0 the discharge is Q_0, the extra discharge just being given by the integral of the inflow i.

The momentum equation (3.8) for $\partial Q/\partial t = 0$ and $\partial Q/\partial x = i$, and assuming Q positive, becomes

$$\left(gA - \beta \frac{Q^2 B}{A^2}\right) \frac{d\eta}{dx} = \beta \frac{Q^2 B}{A^2} \tilde{S} - \Lambda P \frac{Q^2}{A^2}, \quad (5.1)$$

which is a first-order ordinary differential equation for $\eta(x)$, provided we have evaluated $Q(x)$, and that we know how the geometric quantities A, B and P depend on surface elevation at each point. This is the *Gradually-Varied Flow Equation* (GVFE). The equation may be solved numerically using any of a number of methods available for solving ordinary differential equations.

It is surprising that books on open channels do not recognise that the problem of numerical solution of the gradually-varied flow equation is actually a standard numerical problem, although practical details may make it more complicated. Instead, such texts use methods such as the ”Direct step method” and the ”Standard step method”, which can become complicated. There are several software packages such as HEC-RAS which use such methods, but solution of the gradually-varied flow equation is not a difficult problem to solve for specific problems in practice if one recognises that it is merely the solution of a differential equation.

In sub-critical (relatively slow) flow, the effects of any control can propagate back up the channel, and so it is that the numerical solution of the gradually-varied flow equation also proceeds in that direction. On the other hand, in super-critical flow, all disturbances are swept downstream, so that the effects of a control cannot be felt upstream, and numerical solution also proceeds downstream from the control.

No inflow: If there is no inflow, $i = 0$ and $Q = Q_0$, a constant, throughout. Dividing both sides of equation (5.1) by gA gives

$$\frac{d\eta}{dx} = F^2 \frac{\beta \tilde{S} - \Lambda P/B}{1 - \beta F^2} = \frac{\beta \tilde{S} - \Lambda P/B}{1/F^2 - \beta}, \quad (5.2)$$

where, unusually for lectures on flow with a free surface, it has taken us until now to define the *Froude number*

$$F^2 = \frac{Q^2 B}{gA^3} = \frac{(Q/A)^2}{g\,(A/B)},$$

the ratio of the *mean* velocity Q/A squared to g times the *mean* depth A/B. In this course we call F^2 the Froude number, and not F, as the latter quantity occurs quite rarely, and F^2 expresses the real relative importance of inertia terms.

The GVFE in the form of equation (5.2) seems simple – deceptively simple. For example, β can be taken as a constant; $\tilde{S}$ might be a function of x, but we probably do not have enough information to express it as a function of η; many open channels are much wider than their depth, and so $P \approx B$ and $P/B \approx 1$. This leaves most of the functional variation with η on the right side in the term $1/F^2 = gA^3/Q^2B$ in which, for practical river problems the dependence of A and B on the local elevation η is actually quite complicated.

Constant slope: As a special case, consider a channel with a bed of constant slope $\tilde{S} = S_0$. It is simpler to use as a variable the depth of flow h, where $h = \eta - Z$, where Z is the elevation of the bed at a section, so that $dZ/dx = -S_0$. Equation (5.2) becomes

$$\frac{d\eta}{dx} = \frac{dh}{dx} + \frac{dZ}{dx} = \frac{dh}{dx} - S_0 = \frac{\beta S_0 - \Lambda P/B}{1/F^2 - \beta}.$$

Solving for dh/dx and introducing the conveyance gives the GVFE for a prismatic canal of constant slope:

$$\frac{dh}{dx} = \frac{S_0 - Q^2/K^2}{1 - \beta F^2}. \tag{5.3}$$

5.2 Properties of gradually-varied flow and the governing equations

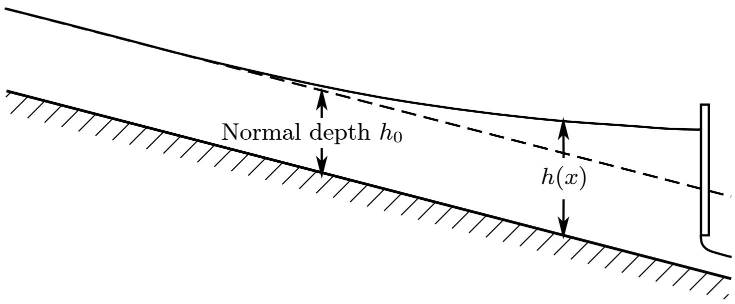

Figure 5-1. Subcritical flow retarded by a gate, showing typical behaviour of the free surface and, if the channel is prismatic, decaying upstream to normal depth

- The equation and its solutions are important, in that they tell us how far the effects of a structure or works in or on a stream extend upstream or downstream.
- It is an ordinary differential equation of first order, hence one boundary condition must be supplied to obtain the solution. In sub-critical flow, this is the depth at a downstream control; in super-critical flow it is the depth at an upstream control.
- If the channel is prismatic, far from the control, the flow is uniform, and the depth is said to be *normal*.
- In general the boundary depth is not equal to the normal depth, and the differential equation describes the transition from the one to the other. The solutions look like exponential decay curves, and below we will show that they are, to a first approximation. Figure 5-1 shows a typical sub-critical flow in a prismatic channel, where the depth at a control is greater than the normal depth.
- The differential equation is nonlinear, and the dependence on h is complicated, such that analytical solution is not possible, and we will usually use numerical methods.
- However, a small-disturbance approximation can be made, the resulting analytical solution is useful in providing us with some insight into the quantities which govern the extent of the upstream or downstream influence.
- If the flow approaches critical flow, when $\beta F^2 \to 1$, then $dh/dx \to \infty$, and the surface becomes vertical. This violates the assumption we made that the flow is gradually varied and the pressure distribution is hydrostatic. This is the one great failure of our open channel hydraulics at this level, that it cannot describe the transition between sub- and super-critical flow.

5.3 When we do not compute – approximate analytical solution

Whereas the numerical solutions give us numbers to analyse, sometimes very few actual numbers are required, such as merely estimating how far upstream water levels are raised to a certain level, the effect of downstream works on flooding, for example. Here we introduce a different way of looking at a physical problem in hydraulics, where we obtain an approximate mathematical solution so that we can provide equations which reveal to us more of the nature of the problem than do numbers. This work was originally done by Samuels (1989).

This is carried out by ”linearising” about the uniform flow in a prismatic channel, *i.e.* by considering small disturbances to that flow. Consider the water depth to be written

$$h(x) = h_0 + h_1(x), \tag{5.4}$$

where we use the symbol h_0 for the constant normal depth, and $h_1(x)$ is a relatively small departure of the surface from that. We use the governing differential equation (5.3) (although our notation has obscured the fact that F and K are functions of h). Substitute equation (5.4) into the equation and writing numerator and denominator as Taylor series in h_1:

$$\frac{dh_0}{dx} + \frac{dh_1}{dx} = \frac{\left(S_0 - Q^2/K^2\right)_0 + h_1\left(\frac{d}{dh}\left(S_0 - Q^2/K^2\right)\right)_0 + \text{terms in } h_1^2}{\left(1-\beta F^2\right)_0 + h_1\left(\frac{d}{dh}\left(1-\beta F^2\right)\right)_0 + \text{terms in } h_1^2}.$$

Now, as h_0 is constant, $dh_0/dx = 0$. Also, from equation (3.13), $\left(S_0 - Q^2/K^2\right)_0 = 0$ and the first term in the numerator is zero. Now evaluating $d\left(S_0 - Q^2/K^2\right)/dh = 2Q^2/K^3\, dK/dh$, and considering just the first term top and bottom, neglecting all higher order powers of h_1 as it is small, we find

$$\frac{dh_1}{dx} \approx h_1 \frac{2Q^2/K_0^3\ (dK/dh)_0}{1-\beta F_0^2} = \mu_0\, h_1, \tag{5.5}$$

where

$$\mu_0 = \frac{2S_0}{1-\beta F_0^2}\ \frac{1}{K}\frac{dK}{dh}\bigg|_0, \tag{5.6}$$

and where we have used $Q^2/K_0^2 = S_0$.

Equation (5.5) is a linear differential equation which we can solve analytically by separation of variables, giving

$$h_1 = Ce^{\mu_0 x}, \quad \text{and} \quad h = h_0 + Ce^{\mu_0 x}, \tag{5.7}$$

where C is a constant which would be evaluated by satisfying the boundary condition at the control, and where μ_0 is a constant decay rate given by equation (5.6).

This shows that the water surface is actually approximated by an exponential curve passing from the value of depth at the control to normal depth. As dK/dh is positive, and for subcritical flow $1-\beta F_0^2$ is also positive, equation (5.6) shows that μ_0 is positive, and far upstream as $x \to -\infty$, the water surface decays to normal depth. For supercritical flow, $1-\beta F_0^2 < 0$, μ_0 is negative, and the water surface approaches normal depth downstream.

Now we obtain an approximate expression for the rate of decay μ_0. From the Gauckler-Manning-Strickler formula for a wide channel, a common approximation, we can show that $K \sim h^{5/3}$, $dK/dh \sim 5/3 \times h^{2/3}$, and for slow flow $\beta F_0^2 \ll 1$, we find

$$\mu_0 \approx \frac{10}{3}\frac{S_0}{h_0}. \tag{5.8}$$

The larger this number, the more rapid is the decay with x. The formula shows that more rapid decay occurs with steeper slopes (large S_0), and smaller depths (h_0). Hence, generally the water surface approaches normal depth more quickly for steeper and shallower streams, and the effects of a disturbance can extend a long way upstream for mild slopes and deeper water.

Let us use equation (5.8) to calculate the distance upstream that the disturbance decays by $1/2$, that is, $\exp(\mu_0 x) = 0.5$. We find

$$\frac{10}{3}\frac{S_0 x}{h_0} = \ln 0.5 \quad \text{giving} \quad \frac{x}{h_0} = \frac{3\ln 0.5}{10}\frac{1}{S_0}.$$

For $S_0 = 10^{-4}$ and a stream $2\,\mathrm{m}$ deep, the distance is $4\,\mathrm{km}$. For the stream disturbance to decay to $1/16 = (1/2)^4$ of the original, this distance is $4 \times 4\,\mathrm{km} = 16\,\mathrm{km}$. These are possibly surprising results, showing how far the backwater effect extends.

5.4 Numerical solution of the gradually-varied flow equation

Consider the gradually-varied flow equation (5.3)

$$\frac{dh}{dx} = \frac{S_0 - Q^2/K^2}{1 - \beta F^2}$$

where $F^2(h) = Q^2 B(h)/gA^3(h)$. The equation is a differential equation of first order, and to obtain solutions it is necessary to have a boundary condition $h = h_0$ at a certain $x = x_0$, which will be provided by a control. The problem may be solved using any of a number of methods available for solving ordinary differential equations which are described in books on numerical methods.

The direction of solution is very important. For mild slope (sub-critical flow) cases the surface decays somewhat exponentially to normal depth upstream from a downstream control, whereas for steep slope (super-critical flow) cases the surface decays exponentially to normal depth downstream from an upstream control. This means that to obtain numerical solutions we will always solve (a) for sub-critical flow: from the control *up*stream, and (b) for super-critical flow: from the control *down*stream.

5.4.1 Euler's method

The simplest (Euler) scheme to advance the solution from (x_i, h_i) to $(x_i + \Delta x_i, h_{i+1})$ is

$$x_{i+1} \approx x_i + \Delta x_i, \quad \text{where } \Delta x_i \text{ is negative for subcritical flow,} \tag{5.9a}$$

$$h_{i+1} \approx h_i + \Delta x_i \left.\frac{dh}{dx}\right|_i = h_i + \Delta x_i \frac{S_0 - Q^2/K^2(h_i)}{1 - \beta F^2(h_i)} + O\left((\Delta x_i)^2\right). \tag{5.9b}$$

This is the simplest but least accurate of all methods – yet it might be appropriate for open channel problems where quantities may only be known approximately. One can use simple modifications such as Heun's method to gain better accuracy, or use Richardson extrapolation – or even more simply, just take smaller steps Δx_i.

5.4.2 Richardson extrapolation

There is an interesting method for obtaining more accurate solutions from computational schemes for almost any physical problem. Applied to the Euler scheme in the present context, as the local truncation error is $O\left((\Delta x_i)^2\right)$, so that after a number of steps proportional to $1/\Delta x_i$, the actual solution has an error $O\left((\Delta x_i)^1\right)$ so that $n = 1$, a first-order scheme. In the present problem, if we use a constant space step Δ to obtain the first solution 1, then another constant space step ***half*** that $\Delta/2$, requiring twice the number of steps, then $r = 1/2$, and equation (2.9) gives for a better estimate of the solution

$$h_i \approx 2h_{2i}(\Delta/2) - h_i(\Delta), \tag{5.10}$$

which is trivially applied to each or any of the steps. The notation $h_{2i}(\Delta/2)$ is intended to show that the same point in physical space is used; with half the step size it will now take twice the number of steps to reach that point.

5.4.3 Heun's method

In this case the value of h_{i+1} calculated from Euler's method, equation (5.9b), is used as a first estimate of the depth at the next point, written h^*_{i+1}, then the value of the derivative at that point $\left(x_{i+1}, h^*_{i+1}\right)$ is calculated. Heun's method is then to use the mean slope over the step, the mean of the initial value and that at the other end of the interval calculated by the Euler step. Then, the change over the step is calculated, multiplying that mean slope by the step length. That is,

$$\begin{aligned} h_{i+1} &\approx h_i + \frac{\Delta x_i}{2}\left(\left.\frac{dh}{dx}\right|_{(x_i,h_i)} + \left.\frac{dh}{dx}\right|_{\left(x_{i+1},h^*_{i+1}\right)} \right) \\ &= h_i + \frac{\Delta x_i}{2}\left(\frac{S_0 - Q^2/K^2(h_i)}{1-\beta F^2(h_i)} + \frac{S_0 - Q^2/K^2(h^*_{i+1})}{1-\beta F^2(h^*_{i+1})} \right) + O\left((\Delta x_i)^3\right). \end{aligned} \tag{5.11}$$

Now, the error of a single step is proportional to the third power of the step length and the error at any point will be proportional to the second power.

Neither of these two methods are presented in hydraulics textbooks as alternatives, yet they are simple and flexible, and reveal the nature of what we are doing. The step Δx_i can be varied at will, to suit possible irregularly spaced cross-sectional data. In many situations, where $F^2 \ll 1$, we can ignore the βF^2 term in the denominators, giving a notationally simpler scheme.

5.4.4 Predictor-corrector method – Trapezoidal method

This is simply an iteration of the last method, whereby the step in equation (5.11) is repeated several times, at each stage setting h^*_{i+1} equal to the updated value of h_{i+1}. This gives an accurate and convenient method, and it is surprising that it has not been used.

5.4.5 Higher-order methods

One of the aims here has been to emphasise that all that is being done is to solve numerically a differential equation, and any method can be used, for which reference can be made to any book on numerical solution of ordinary differential equations. There are sophisticated methods such as high-order Runge-Kutta methods and predictor-corrector methods. However, in the case of open channel hydraulics there will usually be some variation of parameters along the channel that such sophistication is unnecessary.

5.5 Traditional methods

Here we present methods for comparison as they are given in textbooks.

5.5.1 Derivation of the gradually-varied flow equation using energy

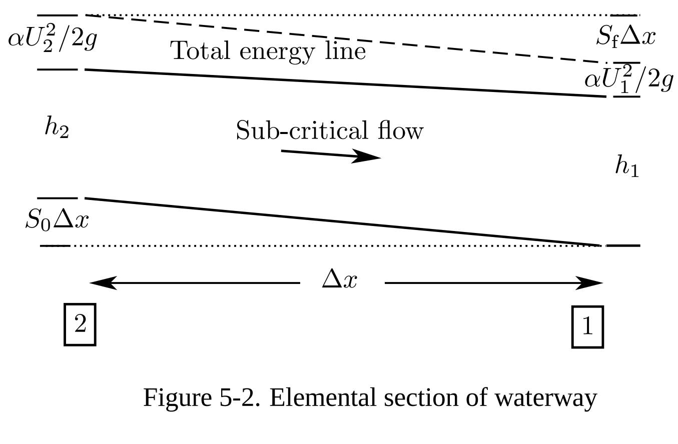

Figure 5-2. Elemental section of waterway

Consider the elemental section of waterway of length Δx shown in Figure 5-2. We have shown stations 1 and 2 in what might be considered the reverse order, but for the more common sub-critical flow, numerical solution of the governing equation will proceed back up the stream. Considering stations 1 and 2:

$$\begin{aligned} \text{Total head at 2} &= H_2 \\ \text{Total head at 1} &= H_1 = H_2 - H_L, \end{aligned}$$

and we introduce the concept of the *friction slope* S_f which is the gradient of the total energy line such that

$H_L = S_{\mathrm{f}} \times \Delta x$. This gives

$$H_1 = H_2 - S_{\mathrm{f}}\,\Delta x,$$

and if we introduce the Taylor series expansion for H_1:

$$H_1 = H_2 + \Delta x\,\frac{dH}{dx} + \ldots,$$

substituting and taking the limit $\Delta x \rightarrow 0$ gives

$$\frac{dH}{dx} = -S_{\mathrm{f}}, \tag{5.12}$$

an ordinary differential equation for the head as a function of x, *and* we use the approximation that the friction slope is given by Q^2/K^2.

5.5.2 Direct step method

Textbooks do present the Direct Step method, which is applied by taking steps in the height and calculating the corresponding step in x. It has practical disadvantages, such that it is applicable only to prismatic sections, results are not obtained at specified points in x, and as uniform flow is approached the Δx become infinitely large. However it is a surprisingly accurate method.

The reciprocal of equation (5.12) is

$$\frac{dx}{dH} = -\frac{1}{S_{\mathrm{f}}}.$$

The numerical method as set out in textbooks is to approximate the differential equation (5.12) by the finite difference expression

$$\begin{aligned}
\Delta x_i &= \frac{\Delta H_{b_i}}{S_0 - \overline{Q^2/K^2\left(H_b\right)}} && \text{(5.13a)}\\
&= \frac{\Delta H_{b_i}}{S_0 - \frac{1}{2}Q^2\left(K_i^{-2} + K_{i+1}^{-2}\right)} && \text{(5.13b)}
\end{aligned}$$

where the overbar in equation (5.13a) indicates the mean of the friction slope at beginning and end of the computational interval, which finds its mathematical expression in equation (5.13b), where the shorthand K_i has been used for $K\left(H_{bi}\right)$.

While this is a plausible approximation, it is not mathematically consistent. It is an apparent attempt to develop a Trapezoidal method. Applying Heun's method as formally presented in equation (5.11) automatically leads to the Trapezoidal scheme which in this case gives

$$x_{i+1} = x_i + \frac{\Delta H_{b,i}}{2}\left(\frac{1}{S_0 - Q^2/K_i^2} + \frac{1}{S_0 - Q^2/K_{i+1}^2}\right) + O\left(\left(\Delta H_{b,i}\right)^3\right), \tag{5.14}$$

The term $O\left(\ldots\right)$ is a Landau order symbol, showing in this case that the local truncation error is proportional to the third power of the step, which is a strong result and explains the accuracy of the method. Since the use of a step size of $\Delta H_{b,i}$ over the whole computational domain requires a *number* of steps proportional to $1/\Delta H_{b,i}$, the global error in this case will be of order $\left(\Delta H_{b,i}\right)^2$, thus the global error, or accumulated error at the end of that integration interval will be of this order, so that halving the step should improve the global accuracy by about a factor of 4.

In view of the method presented here, the method is no longer applicable only to prismatic sections, but the practical disadvantages remain that results are not obtained at specified points in x, and as uniform flow is approached the Δx become infinitely large.

5.5.3 Standard step method

The nomenclature "standard" is not very descriptive. Presumably it refers to finding the solution for η at specified values of x, rather than the other way round, for which the term "direct", as above, is even worse. This is an *implicit* method, requiring numerical solution of a transcendental equation at each step. It can be used for irregular channels, and is rather more general. In this case, the distance interval Δx is specified and the corresponding depth change calculated. In the Standard step method the procedure is to write

$$\Delta H = -S_{\mathrm{f}}\,\Delta x,$$

and then write it as

$$H_2(h_2) - H_1(h_1) = -\frac{\Delta x}{2}\left(S_{\mathrm{f}1} + S_{\mathrm{f}2}\right),$$

for sections 1 and 2, where the mean value of the friction slope is used. This gives

$$\alpha\frac{Q^2}{2gA_2^2} + Z_2 + h_2 = \alpha\frac{Q^2}{2gA_1^2} + Z_1 + h_1 - \frac{\Delta x}{2}\left(S_{\mathrm{f}1} + S_{\mathrm{f}2}\right),$$

where Z_1 and Z_2 are the elevations of the bed. This is a transcendental equation for h_2, as this determines A_2, P_2, and $S_{\mathrm{f}2}$. Solution could be by any of the methods we have had for solving transcendental equations, such as direct iteration, bisection, or Newton's method.

Although the Standard step method is an accurate and stable approximation, the lecturer considers it unnecessarily complicated, as it requires solution of a transcendental equation at each step. It would be much simpler to use a simple explicit Euler or Heun's method as described above.

5.6 Comparison of schemes

To compare the performance of the various numerical schemes, Example 10-1 of Chow (1959, p250) was solved using each. All quantities specified by Chow were converted to SI units and rounded to the numbers shown here: a flow of $11.33\,\mathrm{m^3\,s^{-1}}$ passes down a trapezoidal channel of gradient $S_0 = 0.0016$, bed width $6.10\,\mathrm{m}$ and channel side slopes 0.5, $g = 9.8\,\mathrm{m\,s^{-2}}$, the quantity α or $\beta = 1.10$, and Manning's $n = 0.025$. At $x = 0$ the flow is backed up to a depth of $1.524\,\mathrm{m}$, and the backwater curve was computed for $1000\,\mathrm{m}$. Results for the water surface profile are shown in Figure 5-3, while Figure 5-4 shows the errors. Some 10 computational steps were used for each scheme.

The basis of accuracy is shown by the solid line, from a highly-accurate Runge-Kutta 4th order method (see, for example, p372 of Yakowitz & Szidarovszky, 1989, or almost any other book on numerical methods). It is not recommended here as a method, however, as it makes use of information from three intermediate points at each step, information which in non-prismatic channels is usually not available. The rest of the results are shown in reverse order of accuracy. The dotted line is that with the same numerical method, but where the roughness n was changed by -5%, to give an idea of the effect of uncertainty in knowledge of that quantity. The maximum error, of about $-3\,\mathrm{cm}$, in the normal depth, is greater than any of the other methods, so that a preliminary conclusion is that if the roughness is not known to within 5%, almost certainly the case in practice, then any of the methods can be used.

It can be seen that Euler's method, eqn (5.9), was the least accurate, as expected. As it is a first-order scheme, halving the step size would halve the error. Actually doing just that and then applying Richardson extrapolation, equation (5.10), gave the second most accurate of all the methods, with an error of about $1\,\mathrm{mm}$. The most accurate of all was the Trapezoidal method, namely using Heun's method, equation (5.11) and iterating the final step. All the other methods, Heun's, and the two inverse formulations, equations (5.13) and (5.14), gave errors intermediate between the two extremes. It is interesting that the two traditional methods were accurate, notably the traditional inverse formulation over the modified version presented in this work; and the Trapezoidal method, the basis for the so-called "standard" method.

The results show the disadvantages of the inverse formulation (Direct Step), that the distance between computational points becomes large as uniform flow is approached, and the points are at awkward distances. In this example relatively few steps were chosen (roughly 10) so that the numerical accuracies of the methods could be distinguished visually. The computational effort was very small indeed.

In this problem the analytical solution (5.7) gave poor results. This was because the depth at the control was rather larger (50%) than the normal depth, and the linearisation adopted, for small departures from normal depth, was not accurate. In general, however, it does give a simple approximate result for the rate of decay and how far upstream the effects of the control extend. For many practical problems, this accuracy and simplicity may be enough.

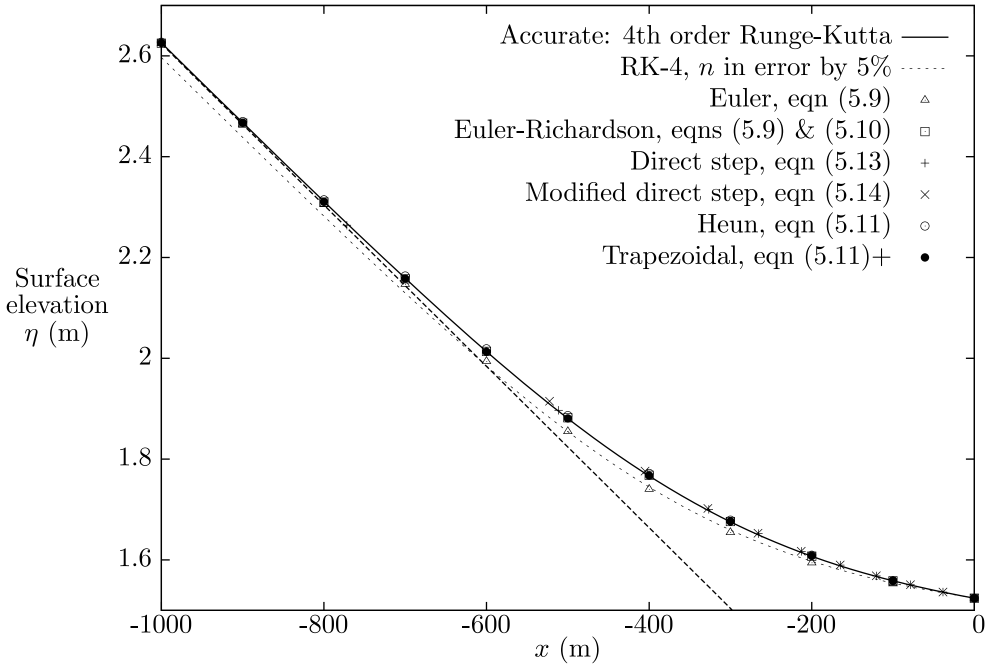

Figure 5-3. Backwater curve computed with various schemes; the dashed line is the surface for uniform flow.

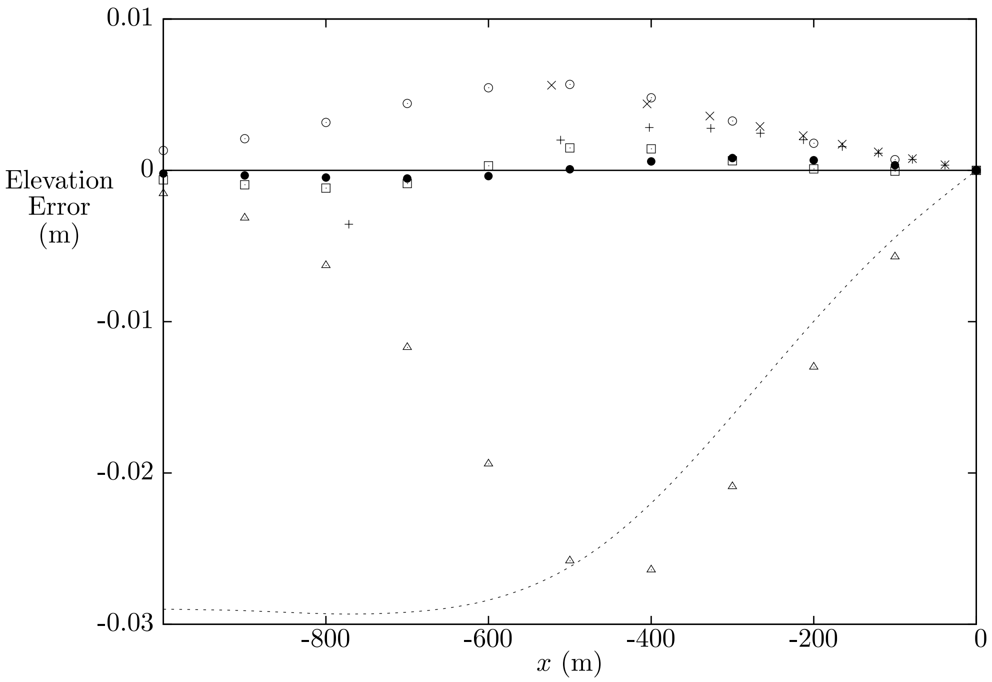

Figure 5-4. Errors for the different schemes; symbols as for Figure 5-3.

6. Model equations and theory for computational hydraulics and fluid mechanics

In §3.3.3 we saw that the long wave equations could be written as a vector evolution equation

$$\frac{\partial \mathbf{u}}{\partial t} + C\frac{\partial \mathbf{u}}{\partial x} = \mathbf{r}\left(\mathbf{u}\right),$$

where $\mathbf{u}$ is the vector of unknowns, for example, $[\eta, Q]$, C is a 2×2 matrix with algebraic coefficients, and $\mathbf{r}$ is the vector of right side terms, due to inflow, slope, and resistance. In this case the matrix C is a generalised velocity, and it is possible to obtain the eigenvalues of the matrix to obtain propagation velocities.

The combination of a time derivative plus a velocity times a space derivative, called an *advective derivative*, occurs throughout fluid mechanics and hydraulics – the Navier-Stokes equations: all fluid motion, in fact, including the equations of meteorology, oceanography, and in our case, the long wave equations. Possibly more computational power around the world is used in the numerical solution of such equations than in any other, especially in the large scale numerical solution of the equations of the atmosphere.

The advective derivative describes the time rate of change of some quantity (such as heat or momentum) by following it, while moving with a velocity field. Numerical solution of it is surprisingly non-trivial, as we are about to see.

6.1 The advection equation

To introduce the subject and demonstrate the numerical difficulties that can occur, firstly we consider the one-dimensional advection equation:

$$\frac{\partial \phi}{\partial t} + u\left(x, t, \phi\right)\frac{\partial \phi}{\partial x} = 0, \tag{6.1}$$

where $\phi\left(x, t\right)$ is some passive scalar, and $u\left(x, t, \phi\right)$ is a velocity, possibly a wave speed, and possibly even dependent on the dependent variable ϕ.

A typical problem is to solve the advection equation when we know $\phi\left(x, 0\right)$, that is, the distribution of ϕ with x at some initial time $t = 0$, and we also know what $\phi\left(0, t\right)$ is, namely how it is varying at the upstream boundary. We want to obtain the solution for all x and t.

6.1.1 Exact solution for constant velocity

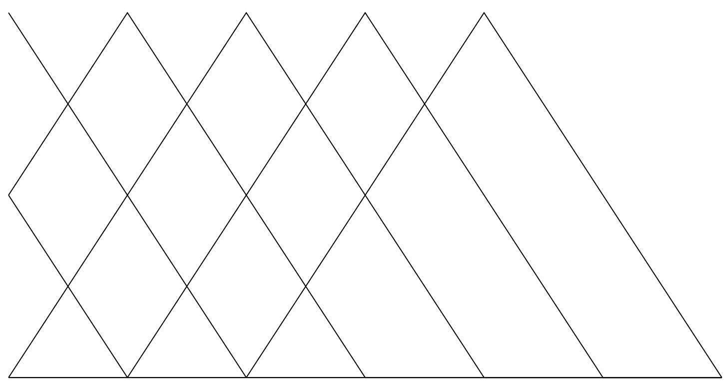

Figure 6-1. Exact solution of advection equation for triangular wave

In the case of a constant velocity $u(x, t) = U$, the equation has a simple analytical solution $\phi(x, t) = f(x - Ut)$, where the function $f(t)$ is given by the history of ϕ at the upstream boundary, $f(t) = \phi(0, t)$, and to obtain the value at any general place and time (x, t) we just substitute $f(x - Ut)$. The solution corresponds to a simple "wave" travelling at a speed of U. We can easily verify that this is the solution, by using the Chain Rule for partial differentiation:

$$\frac{\partial \phi}{\partial t} = \frac{df}{d(x-Ut)} \times \frac{\partial (x-Ut)}{\partial t} = -U \times f'(x-Ut)$$
$$\frac{\partial \phi}{\partial x} = \frac{df}{d(x-Ut)} \times \frac{\partial (x-Ut)}{\partial x} = f'(x-Ut),$$

where $f'(x-Ut) = df(x-Ut)/d(x-Ut)$. Substituting these values into equation (6.1) shows that it is satisfied exactly. Figure 6-1 shows the exact solution of a triangular wave being advected with no diffusion.

6.1.2 An advective numerical scheme

In situations where the velocity is not constant, then numerical solutions have to be made. It is rare that such a simple equation has to be solved numerically, but here we include numerical schemes as models for rather more complicated problems. The previous exact solution scheme suggests the following scheme:

$$\phi(x, t+\Delta) \approx \phi(x - u(x,t,\phi)\Delta, t), \tag{6.2}$$

where the errors can be shown by a *consistency analysis*, introduced below, to be $O(\Delta^2)$, which means that neglected terms are of the order of Δ^2. This is an advective scheme, which attempts to build in the nature of the solution. It can be interpreted as

> *To obtain the solution at some point x at a later time $t+\Delta$, take the known value of the velocity at (x,t), namely $u(x,t)$, and at a distance upstream given by this velocity times the time step, interpolate the value.*

In the case of a constant velocity $u(x,t) = U$ this would be exact, for the value at $(x, t+\Delta)$ is precisely that which was upstream at $(x - U\Delta, t)$. However, if the velocity is variable, it is not exact, and errors are proportional to the square of the time step.

Such advective schemes are to much to be preferred in fluid mechanics, hydraulics, and hydrology, as they mimic the behaviour of solutions as well as the equation, rather than mimicking just the behaviour of the equation. Schemes which do not incorporate the advective nature of the solution can have some unpleasant characteristics, as we now demonstrate.

The interpolation can be done by any scheme – a simple one is to fit a quadratic to three computational points. The lecturer prefers using cubic splines, which are a very powerful way of using a series of cubics.

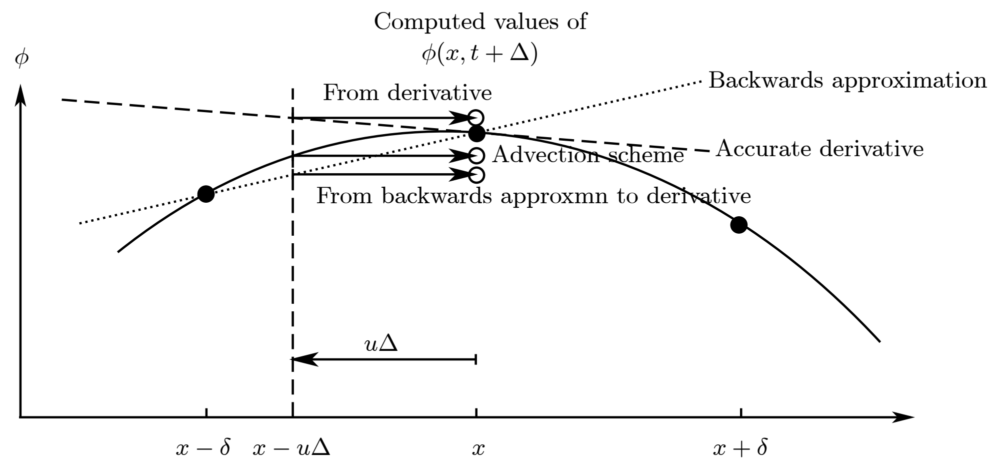

Figure 6-2. Physical nature of three computational schemes for solving the advection equation

Figure 6-2 shows this scheme and two other finite-difference-based schemes. We will use it to demonstrate the inferior properties of the other schemes.

6.1.3 Forwards-Time-Accurate-Space schemes

Here we consider a family of schemes which approximate the derivative in x accurately, The time-stepping scheme is only first-order, although the spatial approximation might be accurate. To evaluate the derivative accurately a high-order scheme using splines or Fourier series or a centred space scheme could be used. For our purposes here it doesn't matter which scheme is used. These schemes do not exploit the travelling-wave nature of the solutions, but rather just approximate all the derivatives of the partial differential equation:

$$\frac{\phi(x, t+\Delta) - \phi(x, t)}{\Delta} + u\Delta \frac{\partial \phi}{\partial x}(x, t) \approx 0.$$

This can be re-written as the scheme

$$\phi(x, t+\Delta) \approx \phi(x, t) - u\Delta \frac{\partial \phi}{\partial x}(x, t). \tag{6.3}$$

This scheme can be interpreted as "the change in ϕ is equal to $-u\Delta$ times the approximation to the derivative", or, "travel along the line with gradient that of the approximation back a distance $u\Delta$, and that is the updated value". This can be seen on figure 6-2. We have deliberately drawn this near a maximum in x, such that the tangent is always above the interpolating function. This shows that when the solution is updated, the value at $t + \Delta$ will be *greater* than the previous maximum, and shows that the scheme will be *unstable*, as maxima will grow. All such schemes are unuseable for any value of $u\Delta$. This phenomenon is well-known in numerical methods for solving partial differential equations. It is paradoxical, that a good approximation to the derivative gave us bad results. We will see later that the converse also holds – a bad approximation gives us a useable scheme!

It is interesting that this is the first-order Taylor expansion in x of the potentially-exact scheme, equation (6.2). These are traditionally much more common throughout computational hydraulics than advection schemes. One wonders why.

6.1.4 The most obvious finite difference scheme: Forward-Time-Centre-Space (FTCS)

A special case of the "accurate space" scheme is the "Forwards-Time-Centre-Space" scheme. Finite difference approximations to derivatives are used throughout engineering to provide numerical solutions of partial differential equations. Here, we adopt the typical types of approximations to the derivatives used in finite difference approximations. Using the accurate centre space approximation

$$\frac{\partial \phi}{\partial x} = \frac{\phi(x+\delta, t) - \phi(x-\delta, t)}{2\delta},$$

we substitute into the Forwards-Time-Accurate-Space scheme (6.3), and rearranging gives the FTCS scheme for computing the updated value at $(x, t+\Delta)$:

$$\phi(x, t+\Delta) = \phi(x, t) - \frac{u(x, t, \phi)\,\Delta}{2\delta}\left(\phi(x+\delta, t) - \phi(x-\delta, t)\right), \tag{6.4}$$

so that the scheme can be represented as "calculate the centre difference approximation $\partial\phi/\partial x \approx (\phi(x+\delta, t) - \phi(x-\delta, t))/2\delta$, and then calculate the change in value at x by calculating the distance $u\Delta$ and the change $u\Delta \times \partial\phi/\partial x$". The "stencil", showing the points that are involved in this calculation, is shown in figure 6-3.

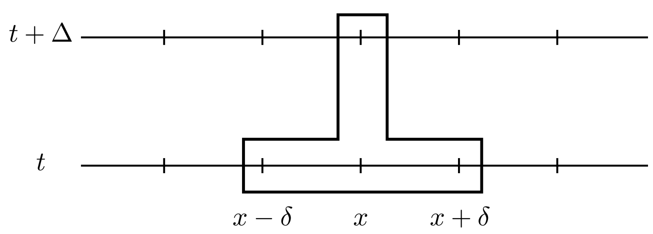

Figure 6-3. Computational stencil for FTCS scheme

The behaviour is the same as in the previous section, namely, it is unconditionally *un*stable. Figure 6-4 shows such a numerical solution for an initially triangular distribution for $C = u\Delta/\delta = 0.75$, the same problem as in Figure 6-1, but here solved numerically. The parameter C is an important one in computational hydraulics, the *Courant Number*, which expresses how far the solution should be advected in a single time step relative to the space step. In this case, the solution should be carried 3/4 of a space step in a time step. We have found that this simple and obvious scheme is unstable, and is unable to be used at all, as was suggested by Figure 6-2.

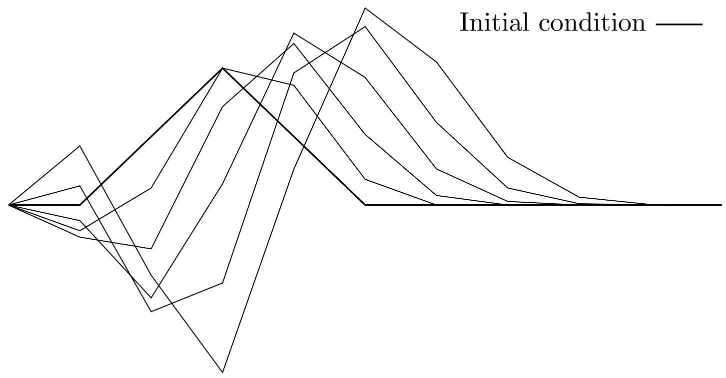

Figure 6-4. Unstable numerical solution of advection equation with FTCS scheme

6.1.5 Forwards-Time-Backwards-Space scheme

Finally we consider a simple Forwards Time Backwards Space scheme, where the derivative is approximated by a backwards difference approximation

$$\phi\left(x,t+\Delta\right)=\phi\left(x,t\right)-\frac{u\Delta}{\delta}\left(\phi\left(x,t\right)-\phi\left(x-\delta,t\right)\right), \tag{6.5}$$

as shown in Figure 6-2. The dotted line shows the backwards difference approximation to the gradient, giving the corresponding updated point as an open circle. It can be seen that the solution is now lower than the accurate advection solution. This shows the phenomenon of *numerical diffusion*, due to such a poor level of approximation, which occurs in many computational schemes.

If we were free to choose $u\Delta=\delta$ the solution would be exact, as the point we would update from is the exact solution at $x-\delta$. However, u is usually a function of time and space and this cannot be satisfied at all points. If we were to take $u\Delta>\delta$, then as can be seen on Figure 6-2 the gradient line is now *above* the exact solution, and the scheme would be unstable. It is common for this limitation to occur in computational schemes that are conditionally stable. It is called the Courant-Friedrichs-Levy criterion, which can be written in terms of the Courant number C:

$$C=\frac{u\Delta}{\delta}\leqslant 1\quad\text{for stability,}$$

whose essential meaning is "for stability of this scheme, the computational wave in a single time step should not travel more than a single space step".

6.2 Convergence, stability, and consistency

In view of some of the insight we have obtained, we now consider some theory for the nature of the numerical solution of finite difference approximations. We are concerned with the conditions that must be satisfied if the solution of the finite difference equations is to be a reasonably accurate approximation to the solution of the corresponding partial differential equation.

6.2.1 Convergence

A finite difference approximation to a differential equation is said to be convergent if its solution converges to the solution of the differential equation in the limit as $\Delta\to 0$ and $\delta\to 0$.

> ***Lax' Equivalence Theorem:*** *if a linear difference equation is* ***consistent*** *with a properly-posed linear initial-value problem, then* ***stability*** *is the necessary and sufficient condition for* ***convergence****.*

6.2.2 Consistency

If the local truncation error at a mesh point goes to zero as the mesh lengths tend to zero, the difference equation is said to be **consistent** with the partial differential equation. We examine consistency using Taylor expansions of the difference equation.

Example 6.1 Consistency of the Forwards-time-Backwards-Space scheme for the advection equation

Consider the scheme (6.5):

$$\phi(x,t+\Delta)=\phi(x,t)-\frac{u\Delta}{\delta}\left(\phi(x,t)-\phi(x-\delta,t)\right).$$

Expanding both sides as Taylor series:

$$\phi+\Delta\phi_t+\frac{1}{2}\Delta^2\phi_{tt}+O\left(\Delta^3\right)=\left(1-\frac{u\Delta}{\delta}\right)\phi+\frac{u\Delta}{\delta}\left(\phi-\delta\phi_x+\frac{1}{2}\delta^2\phi_{xx}+O\left(\delta^3\right)\right),\qquad(6.6)$$

where subscripts denote partial differentiation, giving

$$\phi_t+u\phi_x+\frac{1}{2}\Delta\phi_{tt}+O\left(\Delta^2\right)=\tfrac{1}{2}u\delta\phi_{xx}+O\left(\delta^2\right).\qquad(6.7)$$

Clearly in the limit $\Delta,\delta\to 0$ this has the limiting result

$$\phi_t+u\phi_x=0,$$

which is the differential equation we are solving.

However, by considering higher-order terms some additional insight is obtained. If we write equation (6.7) as

$$\phi_t+u\phi_x=O\left(\Delta,\delta\right),$$

then differentiating with respect to *x* and then with respect to *t* gives

$$\phi_{tx}+u\phi_{xx}=O\left(\Delta,\delta\right)\quad\text{and}\quad\phi_{tt}+u\phi_{xt}=O\left(\Delta,\delta\right),$$

and eliminating the cross-derivatives gives

$$\phi_{tt}=u^2\phi_{xx}+O\left(\Delta,\delta\right),$$

and substituting into equation (6.7) gives

$$\begin{aligned}\phi_t+u\phi_x&=\tfrac{1}{2}u\delta\phi_{xx}-\frac{1}{2}\Delta u^2\phi_{xx}+O\left(\Delta^2,\delta^2\right)\\&=\tfrac{1}{2}u\delta\left(1-\frac{u\Delta}{\delta}\right)\phi_{xx}+O\left(\Delta^2,\delta^2\right).\end{aligned}$$

The second derivative term on the right means that this is actually a diffusion equation, and we have the interesting result that our FTBS is actually not solving the advection equation but an equation with a diffusion term, showing that our scheme exhibits numerical diffusion. The diffusion coefficient is $u\Delta\left(1-C\right)/2$, which disappears in the limit as the time step goes to zero. What is more interesting, however, is that if $u\Delta>\delta$, such that the Courant number is greater than one, $C>1$, the scheme has a negative coefficient of diffusion, and is unstable, as we have seen!

6.2.3 Stability using Fourier series – von Neumann's method

Now we examine the effect that the time stepping has on the nature of our solution relative to the analytical solution. We suppose that the solution to the difference equation can be written

$$\phi(x,t)=A(t)\;e^{ikx},$$

such that the variation in x is a sine wave with wavelength $L=2\pi/k$. This is not as arbitrary as it appears at first, as we can in theory represent any (periodic) variation in *x* as a Fourier series, and as we are considering linear equations only, we can just restrict ourselves to a single term in the Fourier series such as this one. Substituting into our FTBS computational scheme

$$\phi(x,t+\Delta)=\phi(x,t)-\frac{u\Delta}{\delta}\left(\phi(x,t)-\phi(x-\delta,t)\right)$$

gives

$$A(t+\Delta)\ e^{ikx} = A(t)\ e^{ikx} - \frac{u\Delta}{\delta}\left(A(t)\ e^{ikx} - A(t)\ e^{ik(x-\delta)}\right),$$

and dividing through by $A(t)\ e^{ikx}$gives

$$r = \frac{A(t+\Delta)}{A(t)} = 1 - C + C\,e^{-ik\delta},$$

where r is the factor by which the solution changes at each time step.We consider the magnitude of the *amplification factor* $|A(t+\Delta)/A(t)|$ by multiplying by the complex conjugate:

$$\begin{aligned} rr_* &= \left|\frac{A(t+\Delta)}{A(t)}\right|^2 = \left(1 - C + C\,e^{-ik\delta}\right)\left(1 - C + C\,e^{+ik\delta}\right) \\ &= (1-C)^2 + C(1-C)\left(e^{-ik\delta} + e^{+ik\delta}\right) + C^2 \\ &= 1 - 2C + 2C^2 + 2C(1-C)\sin k\delta. \end{aligned}$$

The criterion for stability is that the amplitude ratio should be less than or equal to one, that is,

$$rr_* = \left|\frac{A(t+\Delta)}{A(t)}\right|^2 \leq 1,$$

which gives

$$1 - 2C + 2C^2 + 2C(1-C)\sin k\delta \leq 1,$$

or,

$$2C(1-C)(\sin k\delta - 1) \leq 0,$$

such that the term on the left must be negative. The first factor $2C$ is always positive, and the last factor $\sin k\delta - 1$ is always negative or zero, so that the only way that the whole left term can be negative or zero is that the remaining term $1 - C$ must be positive or zero, giving the stability criterion for the FTBS scheme:

$$C \leq 1, \quad \text{or} \quad u\Delta \leq \delta.$$

This criterion is the *Courant-Friedrichs-Levy stability criterion*, which occurs in many computational schemes. The physical interpretation of it is that the time step should be such that in one such step the computational solution should not be advected a distance greater than the space step.

6.3 The diffusion equation

Thus far we have ignored the important physical phenomenon of diffusion. The process of diffusion occurs because of a continuous process of random particle movements, and leads to viscosity, amongst other effects. The diffusion equation, obtained when the advective velocity of the medium is zero, is

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}, \qquad \text{(Diffusion Equation)}$$

and is well-known to describe many physical quantities in nature, including the flow of heat and electrical charge. Diffusion has the characteristic of smoothing out all variation. A typical analytical solution that shows the essential behaviour, is the Gaussian function, describing the diffusion of an initial single spike of concentration/heat/pollution:

$$\phi = \frac{1}{\sqrt{4\pi\nu t}} \exp\left(-\frac{x^2}{4\nu t}\right),$$

with the behaviour shown in figure 6-5. Note the doubling of the time at each stage – the later behaviour is relatively slow.

Forwards-Time-Centre-Space scheme: The best-known numerical scheme is where the time derivative in the diffusion equation is approximated by a forward difference, and the diffusive term by a centre-difference

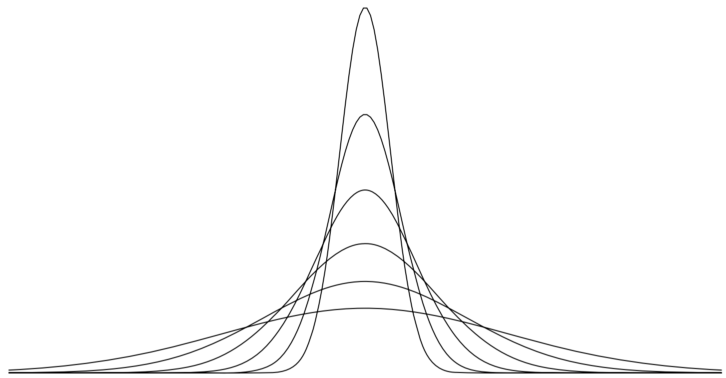

Figure 6-5. Diffusion of a concentration spike at times $t = 1/4$, $1/2$, 1, 2, 4, and 8,

expression. We obtain

$$\frac{\phi\left(x, t+\Delta\right)-\phi\left(x, t\right)}{\Delta}=\nu\frac{\phi\left(x+\delta, t\right)-2\phi\left(x, t\right)+\phi\left(x-\delta, t\right)}{\delta^{2}},$$

which gives the scheme

$$\phi\left(x, t+\Delta\right)=D\,\phi\left(x-\delta, t\right)+\left(1-2D\right)\,\phi\left(x, t\right)+D\,\phi\left(x+\delta, t\right), \tag{6.8}$$

in which D is the computational diffusion number $D = \nu\Delta/\delta^2$. This is widely used, notably in civil engineering, to solve the consolidation equation in geomechanics, which is simply the diffusion equation.

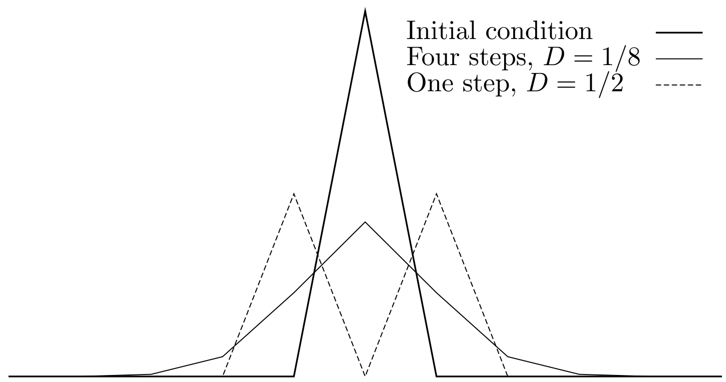

Figure 6-6. FTCS solutions for a single spike of concentration, taking four steps of $D = 1/8$ and one with $D = 1/2$

We consider the finite difference expression (6.8) applied to the case of a single pulse of concentration, with the analytical solution as shown in figure 6-5. Numerical results are shown in figure 6-6. Four steps of $D = 1/8$ give the physically reasonable solution shown. However, in the limiting case for stability $D = 1/2$ the solution has "snapped through" far too much and gives a physically nonsensical result. Clearly, accuracy rather than stability determines the desirable step size here. Of course, there are many other schemes which could be tried. There are *many* papers in the technical literature on this problem.

Now we perform a stability analysis. Let $\phi\left(x, t\right) = A\left(t\right)\ e^{ikx}$, then equation (6.8) gives

$$\begin{aligned} r &= \frac{A(t+\Delta)}{A(t)} = De^{-ik\delta}+\left(1-2D\right)+De^{+ik\delta} \\ &= 1-2D\left(1-\cos k\delta\right) \end{aligned}$$

Squaring and imposing the limit for stability $rr^* \leq 1$ gives the condition

$$D\left(1-\cos k\delta\right)\left(D\left(1-\cos k\delta\right)-1\right)\leq 0.$$

The first factor is positive, the second factor is positive or zero for all $k\delta$, so that for stability the last factor must be negative. That is,

$$D\,(1-\cos k\delta)-1\le 0,$$

giving

$$D\le\frac{1}{1-\cos k\delta},$$

and the minimum value of the right side is 1/2, giving the criterion

$$D=\frac{\nu\Delta}{\delta^2}\le\frac{1}{2}$$

for stability, which is a well-known result.

6.4 Advection-diffusion combined

Consider the advection-diffusion equation containing both advection and diffusion terms:

$$\frac{\partial\phi}{\partial t}+u(x,t,\phi)\frac{\partial\phi}{\partial x}=\nu\frac{\partial^2\phi}{\partial x^2},$$

where in most physical problems the diffusivity parameter ν (viscosity in fluid mechanics) is a constant.

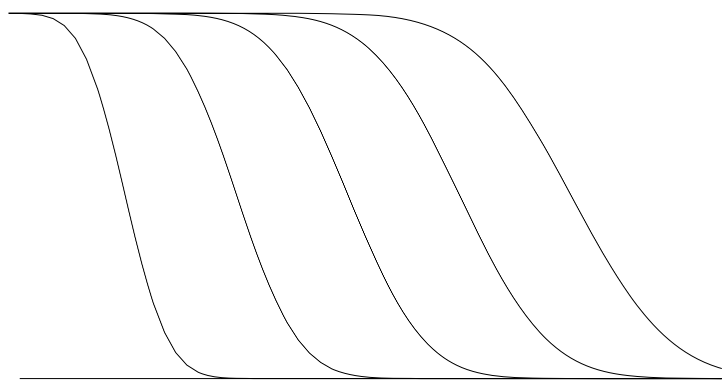

Figure 6-7. Solution of advection-diffusion equation

The combined effects of advection and diffusion can be seen in figure 6-7, where initially concentration is everywhere zero. At the upstream end a constant non-zero concentration is suddenly introduced. The advection transports the solution; the diffusion has the effect of smoothing out the behaviour.

6.4.1 Forward Time Centred Space scheme

Performing a Von Neumann stability analysis, the result is obtained for the amplification factor

$$r=\frac{A(t+\Delta)}{A(t)}=1-2D\,(1-\cos k\delta)-iC\sin k\delta, \tag{6.9}$$

from which, after considerable difficulty, it can be shown that for stability, two criteria are obtained. The first is a limitation on the computational number D:

$$\frac{\nu\Delta}{\delta^2}\le\frac{1}{2},$$

which is independent of the flow velocity, and is valid for pure diffusion as well. The second becomes

$$\frac{u^2\Delta}{\nu}\le 2,$$

and it can be seen how difficult and strange the behaviour of diffusion can make numerical schemes. To satisfy the first criterion, the time step allowed is inversely proportional to diffusion, the more diffusion, the smaller the time step, which feels reasonable. However, to satisfy the second criterion, the allowable time step is proportional to the amount of diffusion, thus, strangely, the less diffusion there is, the smaller is the time step allowed for stability, and in the limit of vanishing diffusion, the scheme is unconditionally unstable, as has been already discovered!

6.4.2 A simple advection-oriented scheme

The previous results suggested that the combination of advection and diffusion can be difficult to compute. Many of these difficulties are overcome if the advective nature of solutions are incorporated, as we saw for the advection equation. A simple scheme which the lecturer advocates is simply using the advective nature, writing the scheme

$$\begin{aligned}\phi(x, t+\Delta) &= \left(1+\nu\Delta\,\frac{\partial^2}{\partial x^2}\right)\phi(x-u\Delta, t) \\ &= \phi(x-u\Delta, t)+\nu\Delta\,\frac{\partial^2\phi}{\partial x^2}(x-u\Delta, t)\,.\end{aligned}$$

This can be interpreted as ”interpolate to find the value of ϕ upstream a distance $u\Delta$ as well as its second derivative there, and combine them as shown to give the updated value allowing for the diffusion of the advected solution”.

Using a von Neumann stability analysis, we obtain a stability criterion which is similar to that for the pure diffusion equation. By incorporating advection ”exactly” we have overcome any difficulties with the combination of advection and diffusion as we found above. If we used the FTCS scheme for the diffusion part we would obtain the same criterion as for the pure diffusion case.

7. Wave and flood propagation in rivers and canals

7.1 Governing equations

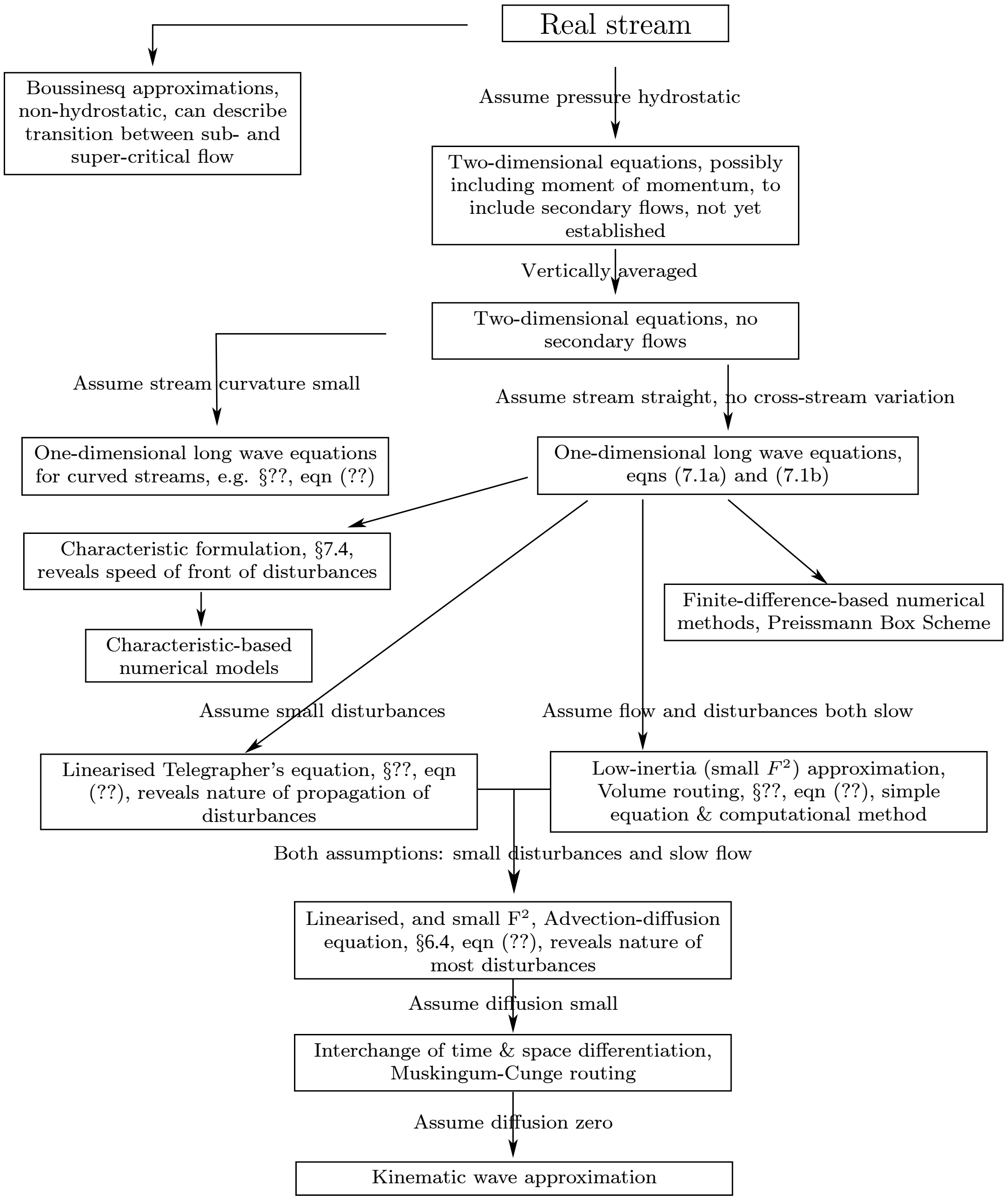

Figure 7-1. Hierarchy of one-dimensional open channel theories and approximations

Figure 7-1 shows how a number of theories relate to each other. The arrows generally show the direction of increasing approximation. The assumptions made in each case are shown as text without a box. There have been a variety of approximations used, because the core theory, the one-dimensional long wave equations, have been believed difficult and problematical to solve. Here we will present a method that overcomes a number of perceived problems, such that it is not necessary to go to the approximations that have long been considered an important part of computational hydraulics.

The problem is shown in Figure 7-2, showing the x axis corresponding to distance along the river, the t or time axis and computational points. To commence numerical solution it is necessary to know the initial conditions – what the values of Q and η are along the x axis, or at least at the computational points. These initial conditions

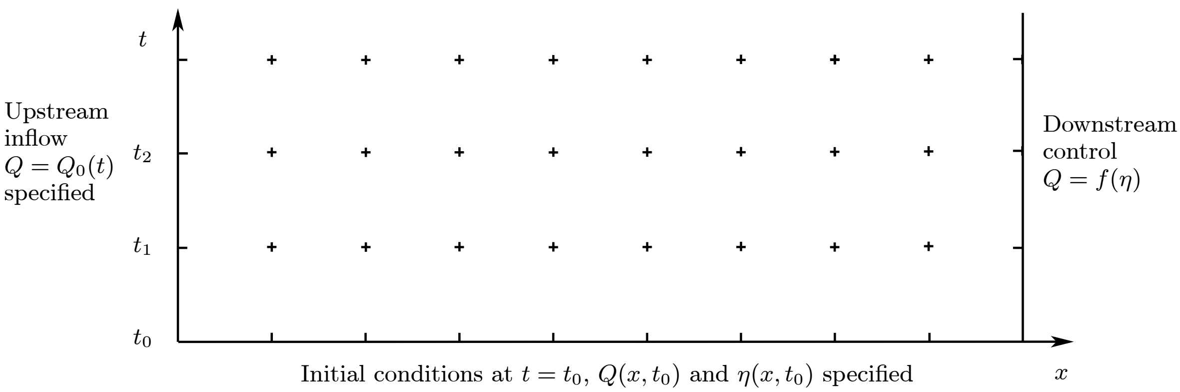

Figure 7-2. (x, t) axes showing initial and boundary conditions and a typical computational grid

are complemented by the upstream boundary condition, usually given in the form of a discharge hydrograph, $Q = Q_0(t)$, and the downstream boundary condition, which is usually that at some control such as a weir, where $Q = f(\eta)$ might be specified.

Consider the long wave equations, (3.7) and (3.8):

$$\frac{\partial \eta}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} = \frac{i}{B}, \tag{7.1a}$$

$$\frac{\partial Q}{\partial t} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} + \left(gA - \beta\frac{Q^2 B}{A^2}\right)\frac{\partial \eta}{\partial x} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Lambda P\frac{Q\,|Q|}{A^2}. \tag{7.1b}$$

We will consider numerical methods to solve them, and a couple of approximations to them that have been used sufficiently often that we should mention them, and an approximation that is quite useful in practice.

The one-dimensional long wave equations require relatively little detailed information, given the complexity of the problem. The process of integrating across the cross-section, by which the equations reflect conservation of mass and momentum, has been able to be performed in relatively simple terms.

7.1.1 Dependent variables

They are most conveniently written in terms of the two dependent variables:

1. Discharge Q – most hydraulic and hydrologic investigations specify the rate of volume transport of water input into a stream, often based on simplified and approximate analyses, but it is the volume transport that is more important than mean velocity, which could also have been used. The rate of outflow from the system is also important, whether being used in further models downstream or in the design of hydraulic works.

   Here we can also mention the lateral inflow volume rate i per unit length of stream, which may have to be specified, although it can often be ignored.

2. Surface elevation η – usually this is very important, for it is easily measured, and it is important in determining whether or nor a stream will overtop its banks and flood, or what areas of land can be irrigated.

For structures on the stream, there is often a definite and well-known relationship between discharge and surface elevation, such as in the formulae for the discharge of a weir.

7.1.2 Geometrical information

- B – surface width: ideally one should now the cross-sectional geometry such that this can be calculated as a function of surface elevation
- A – cross-sectional area: also should be known as a function of η
- P – wetted perimeter of the section, also a function of η, but this requires even more detailed knowledge of the underwater topography. For wide channels, it might simply be assumed that $P \approx B$.
- $\tilde{S}$ – bed slope, the mean across the section: usually this is not well known, and an approximate value is

estimated.

A common approximation is to assume that the stream, even if a natural one, has a cross-section that is approximated by a trapezoidal section, giving

$$\begin{aligned} \text{Top width} &: \quad B = W + 2\gamma h \\ \text{Area} &: \quad A = h\,(W + \gamma h) \\ \text{Wetted perimeter} &: \quad P = W + 2\sqrt{1+\gamma^2}h, \end{aligned}$$

in which $h = \eta - Z$, where Z is the local elevation of the bottom of the channel, which in turn might be able to be approximated by a formula such as $Z(x) = Z(x_0) - xS_0$.

7.1.3 Fluid flow parameters

It is surprising that so few parameters enter. The problem which is being modelled is that of a turbulent shear flow and the resistance that the boundary exerts on the flow.

- β – momentum correction factor, which is a relatively trivial parameter. It enters because the square of the velocity has been integrated across the section to calculate momentum transport by the fluid. A typical value is 1.05, which could easily be approximated by 1. The terms which contain β can be shown to be of a relative magnitude of the square of the Froude number F^2, often not important.
- Λ – the resistance parameter. In the momentum equation (7.1b) it appeared as $-\Lambda PQ\,|Q|\,/A^2$, which is one of the dominant terms. Like the slope $\tilde{S}$, the resistance Λ is often not well-known at all. Yen has given a convenient formula for the Weisbach coefficient λ, which Fenton (2010) re-wrote, and as $\Lambda = \lambda/8$ the formula becomes:

$$\Lambda = \frac{0.166}{\ln^2\left(\dfrac{\varepsilon}{12.} + \left(\dfrac{2.}{R}\right)^{0.9}\right)}, \tag{7.2}$$

where $\varepsilon = k_s/\,(A/P)$ is the relative roughness of the boundary, which is the equivalent sand-grain diameter k_s divided by the hydraulic radius A/P; and where $R = Q/P\nu$ is the channel Reynolds number. In many situations the Reynolds number dependence can be neglected. Results are as shown in figure 7-3.

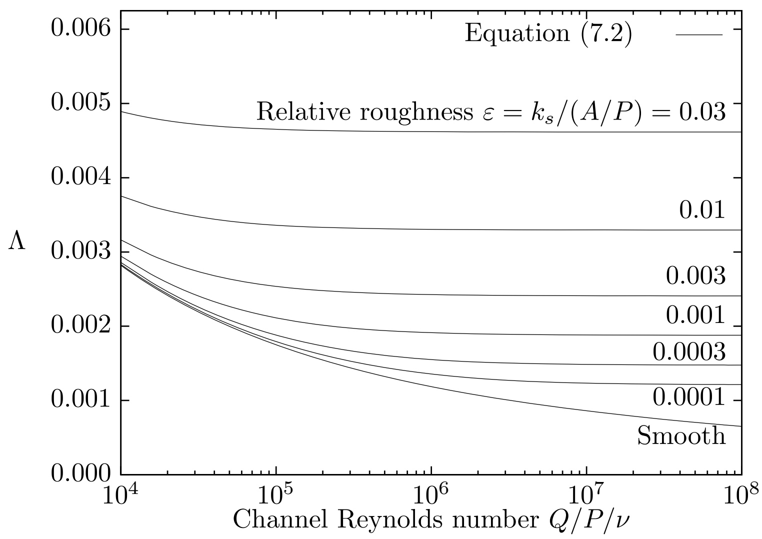

Figure 7-3. Dependence of λ on relative roughness and channel Reynolds number

Other formulations of the resistance term include those of Chézy and Gauckler-Manning-Strickler. For them to agree for steady uniform flow, Λ can be expressed in terms of the Chézy coefficient C, the Manning coefficient n, and the Strickler coefficient k_{St} respectively, the latter two being in *SI* units:

$$\Lambda = \frac{g}{C^2} = \frac{gn^2}{(A/P)^{1/3}} = \frac{g}{k_{\mathrm{St}}^2\,(A/P)^{1/3}}. \tag{7.3}$$

There is little theory or sophisticated experimental results for these parameters. There are two books that provide a catalogue of stream types and resistance parameters n and C for a number of rivers from New Zealand and the USA.

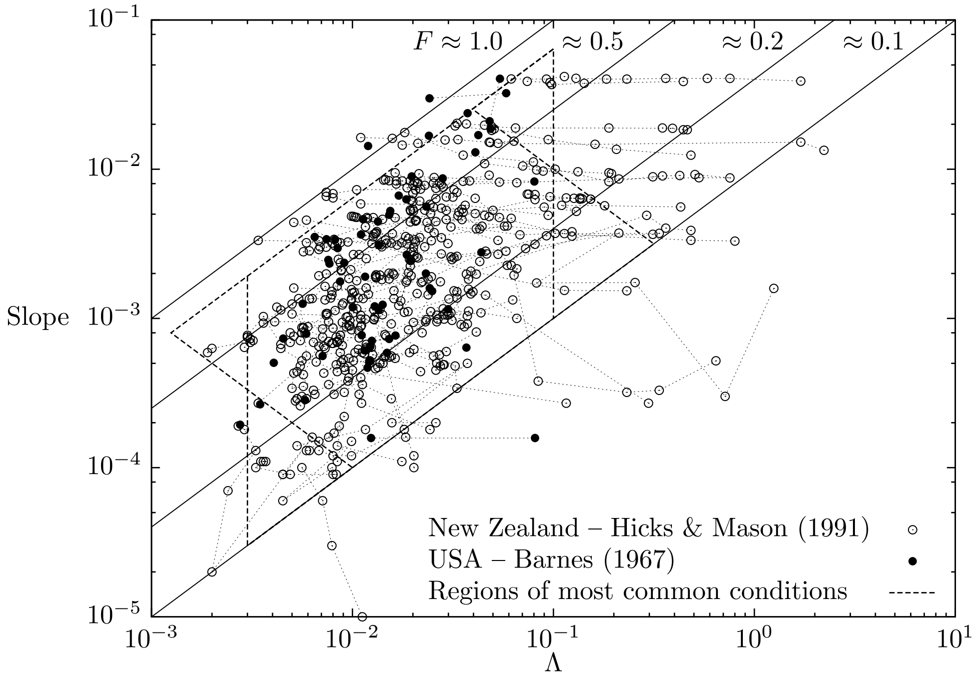

Figure 7-4. Resistance-Slope plot of rivers from New Zealand and the USA

Figure 7-4 shows the slopes and the resistance coefficient Λ for a number of rivers from New Zealand and the USA. The results for New Zealand are taken from 78 river and canal reaches (Hicks & Mason 1991). The data for the USA were taken from 50 natural streams (Barnes 1967). From all that data, most values lie in the range $0.003 \lesssim \Lambda \lesssim 0.1$. It can be seen that for a number of the rivers, the roughnesses are rather greater than included in figure 7-3. The figure shows that the values are arranged between limits according to Froude number, for, from equation (3.9), the Weisbach equation for steady uniform flow

$$\frac{Q}{A} = \sqrt{\frac{g}{\Lambda}\frac{A}{P}S},$$

it can be shown that

$$F = \frac{Q/A}{\sqrt{gA/B}} = \sqrt{\frac{S}{\Lambda}\frac{B}{P}} \approx \sqrt{\frac{S}{\Lambda}},$$

for a wide channel such that $P \approx B$. It can be seen that the data mostly fall between the lines corresponding to slope to resistance coefficient ratios of $S/\Lambda \approx F^2$ for F between 0.1 and 0.8. For lateral boundaries there are two possibilities, shown by the two dashed parallelograms. One has vertical lines, suggesting that most data points are bounded by $0.003 \lesssim \Lambda \lesssim 0.1$. However, another possibility is that the lines with a negative slope form the boundaries. Those lines are such that $S = 10^{-6}/\Lambda$ and $S = 10^{-3}/\Lambda$. These are mostly for natural streams. In canal problems smaller slopes might be encountered.

7.2 Initial conditions

Usually there is some initial flow in the channel that is assumed $Q(x, t_0)$ which is constant if there is no inflow. The next step is to determine the initial distribution of surface elevation η. The conventional method is to solve the Gradually-varied flow equation, using the equations and methods described in §5, as well as the downstream boundary condition, which is about to be described. A simpler method is to use the unsteady equations and computation scheme that will be used later anyway – simply start with an approximate solution for $\eta(x, t_0)$ and let the unsteady dynamics take over, allowing disturbances to propagate downstream and out of the computational domain until the solution is steady. Then, for example, the main computation could be started, such as a flood inflow hydrograph.

7.3 Boundary conditions

7.3.1 Upstream

It is usually the upstream boundary condition that drives the whole model, where a glood or wave enters, via the specification of the time variation of $Q = Q(x_0, t)$ at the boundary. The surface elevation there is obtained as part of the computations.

A model inflow hydrograph is that given previously as equation (2.11):

$$Q(x_0, t) = Q_{\text{min}} + (Q_{\text{max}} - Q_{\text{min}}) \left(\frac{t}{T_{\text{max}}} e^{1-t/T_{\text{max}}} \right)^5,$$

where the event starts at $t = 0$ with Q_{min} and has a maximum Q_{max} at $t = T_{\text{max}}$.

Usually the location of the upstream boundary condition is well-defined, such as just below an upstream structure or maybe the entry of a major tributary.

7.3.2 Downstream

If a control exists: If there is a structure downstream that restricts or controls the flow, the choice of location and the nature of the control are both made easy. For example, a weir might have a flow formula such as

$$Q = 0.6\sqrt{g}b\ (\eta - z_c)^{3/2}$$

where b is the crest length and z_c is the elevation of the crest. Through formulae such as these, we then have the general expression $Q = f(\eta)$, and the way that this is implemented is that η is updated from the mass conservation equation, and the formula used to give the value of Q.

Open boundary condition: often the computational domain might be truncated without the presence of a control, such as just below a town, for which the danger of flooding is to be investigated, but where there is no control structure on the stream. In this case the problem is altogether less well solved.

One solution is to use a uniform flow boundary condition there, and so, from the Chézy-Weisbach formula, for example, for steady uniform flow

$$Q = A\sqrt{\frac{g}{\Lambda}\frac{A}{P}\tilde{S}},$$

and to assume that this holds even if the flow is unsteady and non-uniform, as it generally is as disturbance pass out from the computational domain. As A and P are known functions of surface elevation there, this becomes a formula $Q = f(\eta)$, just as if it were a control.

The lecturer prefers a different approach, and this is simply to treat the boundary as if it were just any other part of the river (which it is!) and to use both long wave equations to update both η and Q there, calculating the necessary derivatives $\partial\eta/\partial x$ and $\partial Q/\partial x$ from upstream finite difference formulae. He has found it works very well in practice, but a senior hydraulic engineer was horrified when he heard this, believing it to violate the physics of the problem. To the lecturer it is a sensible step. One still has to truncate somewhere. If one has truncated the computational domain, one has already abandoned any idea of information coming from downstream anyway. So, if all information is coming from upstream, we just use that and compute η and Q from the equations, using approximations to derivatives from the conditions immediately upstream. To me, it is more sensible than applying the wrong boundary condition such as a uniform flow boundary condition at the downstream end.

7.4 The method of characteristics

This method is described in many books, and for completeness is included here. The lecturer believes that it is something of an accident of history, and that the deductions that emerge from it are misleading and have caused several misunderstandings about the nature of wave propagation in open channels.

The two long wave equations (7.1a) and (7.1b), which are partial differential equations, can be expressed as four ordinary differential equations. Two of the differential equations are for paths for $x(t)$, a path known as a *charac-*

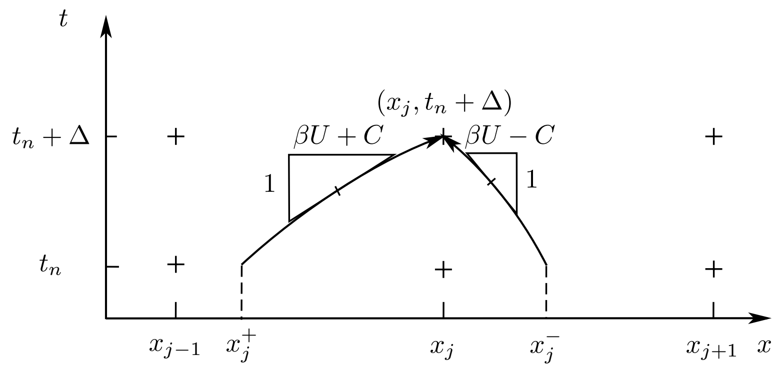

Figure 7-5. Part of (x, t) plane, showing information arriving from upstream and downstream at velocities $\beta U \pm C$

teristic:

$$\frac{dx}{dt} = \beta U \pm C, \tag{7.4}$$

where $U = Q/A$ is the mean fluid velocity in the waterway at that section and the velocity C is

$$C = \sqrt{\frac{gA}{B} + U^2 \left(\beta^2 - \beta\right)},$$

often described as the "long wave speed". It is, as equation (7.4) shows, actually the speed of the characteristics relative to the flowing water. There are two contributions of $\pm C$, corresponding to both upstream and downstream propagation of information. Two characteristics that meet at a point are shown on Figure 7-5. The "downstream" or "+" characteristic has a velocity at any point of $\beta U + C$. In the usual case where U is positive, both parts are positive and the term is large. As shown on the diagram, the "upstream" or "-" characteristic has a velocity $\beta U - C$, which is usually negative and smaller in magnitude than the other. Not surprisingly, upstream-propagating disturbances travel more slowly. The characteristics are curved, as all quantities determining them are not constant, but functions of the variable A, B, and Q.

The other two differential equations for η and Q can be established from the long wave equations:

$$B\left(-\beta\frac{Q}{A} \pm C\right)\frac{d\eta}{dt} + \frac{dQ}{dt} = \beta\frac{Q^2 B}{A^2}\tilde{S} - \Lambda P\frac{Q\,|Q|}{A^2} + i\left(-\beta\frac{Q}{A} \pm C\right), \tag{7.5}$$

On each of the two characteristics given by the two alternatives of equation (7.4), each of these two equations holds, taking the corresponding plus or minus signs in each case. To advance the solution numerically means that the four differential equations (7.4) and (7.5) have to be solved over time, usually using a finite time step Δ. Figure 7-5 shows the nature of the process on a plot of x against t.

The usual computational problem is, for a time $t_{n+1} = t_n + \Delta$, and for each of the discrete points x_j, to determine the values of x_j^+ and x_j^- at which the characteristics cross the previous time level t_n. From the information about η and Q at each of the computational points at that previous time level, the corresponding values of η_j^+, η_j^-, Q_j^+, and Q_j^- are calculated and then used as initial values in the two differential equations (7.5) which are then solved numerically to give the updated values $\eta\left(x_j, t_{n+1}\right)$ and $Q\left(x_j, t_{n+1}\right)$, and so on for all the points at t_{n+1}.

The advantage of characteristics has been believed to be that numerical schemes are relatively stable. The lecturer is unconvinced that they are any more stable then simple finite difference approximations to the original partial differential equations, but this remains to be proved conclusively.

The use of characteristics has led to a widespread misconception in hydraulics where C is understood to be the speed of propagation of waves. It is not – it is the speed of *characteristics*. If surface elevation were constant on a characteristic there would be some justification in using the term ”wave speed” for the quantity C, as disturbances travelling at that speed could be observed. However as equation (7.5) holds, in general neither η (surface elevation – the quantity that we see), nor Q, is constant on the characteristics and one does not have observable disturbances or discharge fluctuations travelling at C relative to the water. While C may be the speed of propagation of information in the waterway relative to the water, it cannot properly be termed the wave speed as it would usually be understood.

7.5 The Preissmann Box scheme

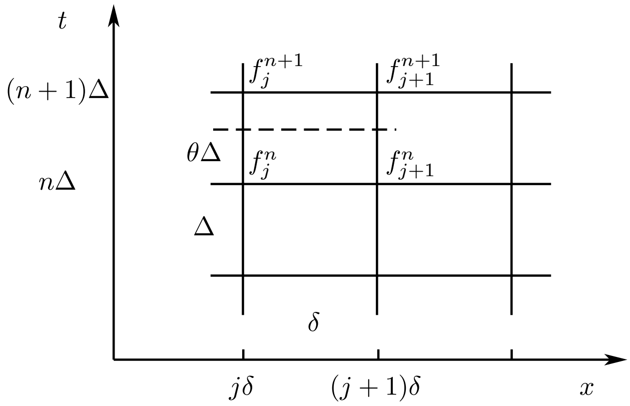

Figure 7-6. Implicit Preissmann Box scheme

The most popular numerical method for solving the equations in time are Implicit Box (Preissmann) models, where the derivatives are replaced by finite-difference equivalents, giving algebraic equations for point values of elevation and discharge in space and time. At time t_n a pair of equations are written for each computational module, giving two equations for two unknowns at time t_{n+1}. Such equations are written all along the x axis (*i.e.* for every computational point), giving a system of $2N$ nonlinear equations in $2N$ unknowns. The method is complicated, and several well-known commercial programs are available.

Consider the finite difference approximations, where δ is a spatial step length, Δ a time step, θ is a weighting parameter, and the point f_j^n is for a spatial point number j and time level n:

$$\begin{aligned}
\frac{\partial f}{\partial x}(j,n) &\approx \frac{1}{\delta}\left[\theta\left(f_{j+1}^{n+1}-f_j^{n+1}\right)+(1-\theta)\left(f_{j+1}^n-f_j^n\right)\right], \\
\frac{\partial f}{\partial t}(j,n) &\approx \frac{1}{2\Delta}\left[\left(f_{j+1}^{n+1}-f_{j+1}^n\right)+\left(f_j^{n+1}-f_j^n\right)\right], \\
\bar{f}(j,n) &\approx \frac{1}{2}\left[\theta\left(f_{j+1}^{n+1}+f_j^{n+1}\right)+(1-\theta)\left(f_{j+1}^n+f_j^n\right)\right],
\end{aligned}$$

in which θ is a weighting coefficient, which determines how much weight is attached to values at time $n+1$ and how much to those at n. Now, in the equations (7.1a) and (7.1b) we use these expressions for all derivatives and also the averaged quantity for quantities that occur algebraically. We obtain a set of complicated algebraic equations in the values of Q and η at the corners of a rectangle in space-time, which can be used for numerical solutions of the full nonlinear equations. Note that values at two points to be determined have entered the equations: f_j^{n+1} and f_{j+1}^{n+1}. This means that it is not possible to solve the equations explicitly, and one obtains at each time step a system of $2N$ simultaneous nonlinear equations that have to be solved. The lecturer uses a multi-dimensional secant method, which requires successive use of solutions of a system of nonlinear equations.

The lecturer thinks this is a terrible scheme. It is very complicated. It is, however, very robust and stable, and large time steps can be taken. The scheme can be shown to be neutrally stable if $\theta = \frac{1}{2}$ is taken, but it is only marginally stable. In practice, one uses a larger value, such as $\theta = 0.6$, and the scheme is stable because it is computationally-diffusive.

7.6 Explicit Forward-Time-Centre-Space scheme

7.6.1 The scheme

Initially, consider the Forward-Time approximation to equations (7.1), which is generic, in that we have not specified a means of computing the spatial derivatives:

$$\eta(x,t+\Delta)=\eta(x,t)+\Delta\left(\frac{i}{B}-\frac{1}{B}\frac{\partial Q}{\partial x}\right)_{(x,t)}, \tag{7.6}$$

$$Q(x,t+\Delta)=Q(x,t)+\Delta\left(\beta\frac{Q^2B}{A^2}\tilde{S}-\Lambda P\frac{Q\left|Q\right|}{A^2}-2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}-\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial\eta}{\partial x}\right)_{(x,t)}. \tag{7.7}$$

We could use, say, cubic splines to approximate the derivatives, and the lecturer has found that the scheme works quite well. In the interest of simplicity, however, it is better to use the Centre-Space expressions for the derivatives:

$$\left(\frac{\partial\eta}{\partial x}\right)_{(x,t)} = \frac{\eta(x+\delta,t)-\eta(x-\delta,t)}{2\delta}$$

$$\left(\frac{\partial Q}{\partial x}\right)_{(x,t)} = \frac{Q(x+\delta,t)-Q(x-\delta,t)}{2\delta}.$$

The computational stencil for the scheme is shown in figure 7-7. It is quite simple to apply

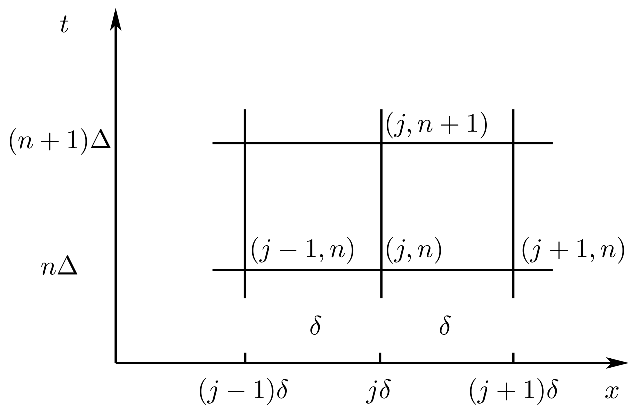

Figure 7-7. Computational stencil for explicit Forward-Time-Centre-Space scheme

Liggett & Cunge (1975) showed that the scheme was unconditionally unstable. This had some important implications, for it meant that the world was forced into using complicated schemes such as the Preissmann Box scheme. The lecturer has recently discovered that their analysis is wrong, and that the scheme has a quite acceptable stability limitation, and it opens up the possibility for simpler computations of floods and flows in open channels. The Preissmann Box Scheme allows much larger time steps, but it is very complicated to apply.

7.6.2 Stability limits

The theory is non-trivial: for present purposes it is probably best just to try and see if the computations are stable. As an example, we consider a $100\,\mathrm{m}$ wide channel, with $L=100\,\mathrm{km}$, $N=100$ computational points, and a flow that rises from $100\,\mathrm{m}^3\mathrm{s}^{-1}$ to a peak of $1000\,\mathrm{m}^3\mathrm{s}^{-1}$ after 24 hours and then diminishes again. We computed for $96\,\mathrm{h}$. A number of different cases of resistance Λ and slope were considered. In each case the approximate limit to stability was found by trial and error. Results are shown in figure 7-8. It can be seen that over the range of most natural rivers and canals, shown by the dashed lines corresponding to the region of most streams on figure 7-4 the allowable time steps might be $100\,\mathrm{s}$. For streams on very small slopes, the steps are quite small, being something like $5\,\mathrm{s}$. This does not seem to be a problem on modern computers, for computational times for each simulation varied between less than a second to several seconds.

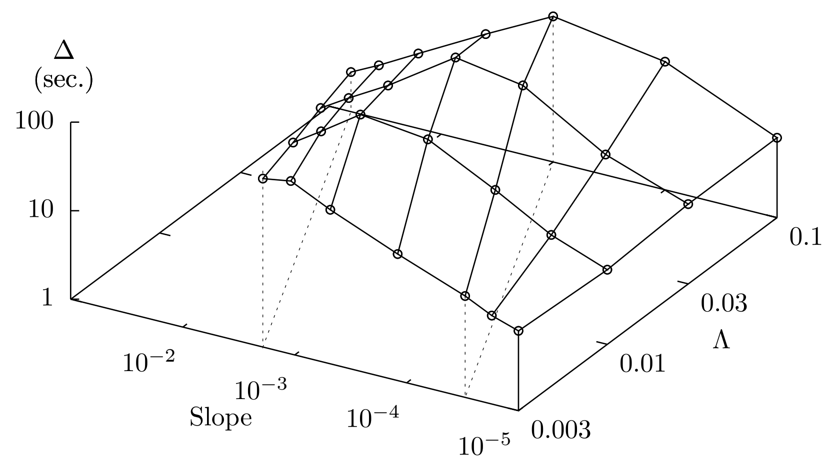

Figure 7-8. Stability limits determined by trial and error for a model river

7.7 The slow-change approximation – advection-diffusion and kinematic theories

There is a family of approximations that can be studied and used, to give rather simpler equations, and certainly simpler numerical methods, than the Preissmann Box Scheme. However the FTCS scheme of the previous section is so general and simple that it might be considered the method of choice. Nevertheless the approximations are included here for completeness, as they traditionally form parts of computational hydraulics courses.

It has been widely believed that the essential approximation that is made is that the square of the Froude number, F^2, is small, so that they apply to slow-moving flows. However that belief has come from non-dimensionalising arguments that postulated that the time scale of variation was dictated by the velocity of the stream. In fact, the time scale of variation is dictated by the time scale of variation of input to the system – how fast floods rise or how fast control gates open.

To examine the implications of that statement we will consider the long wave equations written in terms of cross-sectional area A rather than surface elevation η. By substituting the relations

$$\frac{\partial A}{\partial t} = B\frac{\partial \eta}{\partial t} \quad \text{and} \quad \frac{\partial A}{\partial x} = B\left(\frac{\partial \eta}{\partial x} + \tilde{S}\right), \tag{7.8}$$

into equations (7.1), we obtain

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = i, \tag{7.9a}$$

$$\frac{\partial Q}{\partial t} + \left(\frac{gA}{B} - \beta\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x} + 2\beta\frac{Q}{A}\frac{\partial Q}{\partial x} = gA\tilde{S} - \Lambda P\frac{Q\,|Q|}{A^2}. \tag{7.9b}$$

For slowly-varying input to the system the time derivative $\partial Q/\partial t$ in equation (7.9b) can be shown to be much smaller than the other terms, and not surprisingly, the spatial derivative $\partial Q/\partial x$ is also smaller, as motion is varying slowly in both time and space. This argument cannot be used to simplify equation (7.9a) for both derivatives in general are of the same magnitude. However, in the momentum equation (7.9b), if we neglect both derivatives of Q, the equation is approximated by

$$\left(\frac{gA}{B} - \beta\frac{Q^2}{A^2}\right)\frac{\partial A}{\partial x} = gA\tilde{S} - \Lambda P\frac{Q^2}{A^2}, \tag{7.10}$$

which contains Q only algebraically, and the equation can be solved for Q:

$$Q=\sqrt{\frac{gA^3}{\Lambda P}}\sqrt{\frac{\tilde{S}-\frac{1}{B}\dfrac{\partial A}{\partial x}}{1-\frac{\beta}{\Lambda P}\dfrac{\partial A}{\partial x}}}=K\sqrt{\frac{\tilde{S}-\frac{1}{B}\dfrac{\partial A}{\partial x}}{1-\frac{\beta}{\Lambda P}\dfrac{\partial A}{\partial x}}}, \tag{7.11}$$

where the conveyance K has been introduced for convenience, from equation (3.12).

Now to use the value of Q from equation (7.11) in a mass-conservation equation to give a single equation in a single unknown, there are two ways that we can proceed.

1. The first is to substitute Q into the remaining mass conservation equation (7.9a) to give the equation

$$\frac{\partial A}{\partial t}+\frac{\partial}{\partial x}\left(K\sqrt{\frac{\tilde{S}-\dfrac{1}{B}\dfrac{\partial A}{\partial x}}{1-\dfrac{\beta}{\Lambda P}\dfrac{\partial A}{\partial x}}}\right)=i. \tag{7.12}$$

   Routing problems could be solved by solving for A down the channel and if necessary using equation (7.11) at any stage to calculate the discharge Q. However the free surface elevation η has more direct significance. Equations (7.8) can be used, and equation (7.12) becomes the equation in terms of surface elevation η:

$$\frac{\partial \eta}{\partial t}+\frac{1}{B}\frac{\partial}{\partial x}\left(K\sqrt{\frac{-\partial\eta/\partial x}{1-\dfrac{\beta B}{\Lambda P}\left(\dfrac{\partial \eta}{\partial x}+\tilde{S}\right)}}\right)=\frac{i}{B}. \tag{7.13}$$

2. In many problems, however, the discharge is specified as a boundary condition, which does not naturally lead to a boundary condition on η. It is helpful to introduce the concept of *upstream volume* V, or the volume of water upstream of a point (x,t) and which will pass that point, that is related to A and Q by:

$$\frac{\partial V}{\partial x}=A \quad \text{and} \quad \frac{\partial V}{\partial t}=\int^x i(x')\,\mathrm{d}x'-Q. \tag{7.14}$$

   The first of those relations is obvious, the x-derivative of the volume is the cross-sectional area. The second states that the rate of change of volume is equal to the rate at which it is flowing into the system given by the integral, less the flow which is passing the point, thereby no longer being upstream. Substituting A and Q from equation (7.14) into the mass conservation equation (7.9a) shows that it is satisfied identically:

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial x}\left(\int^x i(x')\,\mathrm{d}x'-\frac{\partial V}{\partial t}\right)=i.$$

   Now we just take equation (7.11) for Q and to use the upstream volume identity (7.14) $Q=\int^x i(x')\,\mathrm{d}x'-\partial V/\partial t$ to give the equation in terms of V:

$$\frac{\partial V}{\partial t}+K\sqrt{\frac{\tilde{S}-\frac{1}{B}\frac{\partial^2 V}{\partial x^2}}{1-\frac{\beta}{\Lambda P}\frac{\partial^2 V}{\partial x^2}}}=\int^x i(x')\,dx'. \tag{7.15}$$

   This is a single equation in a single unknown, the upstream volume V. The only approximation that has been made is that variation with time (and space) is slow. It is most suited to flood routing problems, and not to problems of waves caused by fast irrigation gate movements.

There is an interesting approximation to equation (7.15) that we can obtain by linearising the equation for small disturbances about a steady uniform flow with no inflow $i=0$. Let

$$V=A_0x-Q_0t+v,$$

where v is a small quantity. Then we have

$$A=\frac{\partial V}{\partial x}=A_0+\frac{\partial v}{\partial x}$$

and as $K = K(A)$ we have

$$K(A) = K\left(A_0 + \frac{\partial v}{\partial x}\right) = K(A_0) + \left.\frac{dK}{dA}\right|_0 \frac{\partial v}{\partial x} + \ldots,$$

writing it as a Taylor series. The square root term in equation (7.15) can be evaluated using power series expansions $(1+\delta)^n = 1 + n\delta + \ldots$ and as $\tilde{S} = S_0$ for the underlying uniform flow

$$\sqrt{\frac{\tilde{S} - \frac{1}{B}\frac{\partial^2 V}{\partial x^2}}{1 - \frac{\beta}{\Lambda P}\frac{\partial^2 V}{\partial x^2}}} = \sqrt{S_0}\sqrt{\frac{1 - \frac{1}{BS_0}\frac{\partial^2 v}{\partial x^2}}{1 - \frac{\beta}{\Lambda P}\frac{\partial^2 v}{\partial x^2}}} \approx \sqrt{S_0}\left(1 + \left(\frac{\beta}{2\Lambda_0 P_0} - \frac{1}{2BS_0}\right)\frac{\partial^2 v}{\partial x^2} + \ldots\right),$$

and multiplying through by the Taylor expansion for K and substituting into equation (7.15) we get

$$-Q_0 + \frac{\partial v}{\partial t} + K(A_0)\sqrt{S_0} + \sqrt{S_0}\left.\frac{dK}{dA}\right|_0 \frac{\partial v}{\partial x} + K(A_0)\sqrt{S_0}\left(\frac{\beta}{2\Lambda_0 P_0} - \frac{1}{2B_0 S_0}\right)\frac{\partial^2 v}{\partial x^2} = 0,$$

and as the underlying flow is given by $Q_0 = K(A_0)\sqrt{S_0}$ the leading terms cancel and we get

$$\begin{aligned}\frac{\partial v}{\partial t} + \sqrt{S_0}\left.\frac{dK}{dA}\right|_0 \frac{\partial v}{\partial x} &= K(A_0)\sqrt{S_0}\left(\frac{1}{2BS_0} - \frac{\beta}{2\Lambda_0 P_0}\right)\frac{\partial^2 v}{\partial x^2} \\ &= \frac{Q_0}{2B_0 S_0}\left(1 - \frac{\beta B_0 S_0}{\Lambda_0 P_0}\right)\frac{\partial^2 v}{\partial x^2}. \qquad (7.16)\end{aligned}$$

The left side is an advective derivative, and the coefficient of the $\partial v/\partial x$ term plays the role of an advective velocity so that we write $c_{\mathrm{k}} = \sqrt{S_0}\, dK/dA|_0$, which is called the kinematic wave speed. Now, as $Q_0 = K_0\sqrt{S_0}$, this means that

$$c_{\mathrm{k}} = \frac{dQ_0}{dA_0},$$

which is called the Kleitz-Seddon equation.

From the uniform flow relation we have

$$Q_0 = A_0\sqrt{\frac{g}{\Lambda_0}\frac{A_0}{P_0}S_0},$$

from which it is possible to show that

$$c_{\mathrm{k}} = \frac{3}{2}U_0\left(1 - \frac{1}{3}\frac{A_0}{\Lambda_0 P_0}\left.\frac{d(\Lambda P)}{dA}\right|_0\right). \qquad (7.17)$$

For wide channels, the derivative of ΛP with area should be small, and so we have the useful formula that the speed of propagation of long disturbances in streams is $c_{\mathrm{k}} \approx 3U_0/2$.

From the uniform flow formula the term in the brackets in the diffusion coefficient of equation (7.16) can be simplified to give

$$\frac{\partial v}{\partial t} + c_{\mathrm{k}}\frac{\partial v}{\partial x} = \frac{Q_0}{2B_0 S_0}\left(1 - \beta F_0^2\right)\frac{\partial^2 v}{\partial x^2}, \qquad (7.18)$$

which is an advection-diffusion equation. We have shown that this describes disturbances in channels where variation is slow and deviations about uniform flow are relatively small, and have observed the nature of behaviour of solutions in §6.4. These are relatively minor limitations in many cases, and the equation is a good model of motion in a river or canal.

Now we non-dimensionalise the advection equation (7.18). We consider a time scale of variation T, and assume that to first order the length scale $L \approx c_{\mathrm{k}}T$. We introduce dimensionless variables $x_* = x/L$ and $t_* = t/T$, so that the scale of variation of both x_* and t_* is about 1. Substituting into the equation, and assuming that $\beta F_0^2 \ll 1$:

$$\frac{1}{T}\frac{\partial v}{\partial t_*} + \frac{c_{\mathrm{k}}}{L}\frac{\partial v}{\partial x_*} = \frac{Q_0}{2B_0 S_0}\frac{1}{L^2}\frac{\partial^2 v}{\partial x_*^2},$$

and multiplying through,

$$\frac{\partial v}{\partial t_*} + \frac{\partial v}{\partial x_*} = \frac{Q_0}{2B_0 S_0 c_{\mathrm{k}}^2 T} \frac{\partial^2 v}{\partial x_*^2}.$$

As the time and space differentiation are now of a scale of 1, the dimensionless diffusion coefficient on the right now shows the relative importance of diffusion. We substitute $c_{\mathrm{k}} = 3U_0/2$, $Q_0 = U_0 A_0$, and making the wide-channel approximation, such that $A_0 = B_0 h_0$ and $P_0 = B_0$ the dimensionless diffusion coefficient becomes

$$\frac{2}{9T}\sqrt{\frac{h_0 \Lambda_0}{g S_0^3}},$$

and we can see that diffusion is small for slowly-varying waves (T large) on steeper streams, S_0 large. In the case where it is negligible, the advection equation results, with solution $v = f(x - c_{\mathrm{k}} t)$, where $f(.)$ is given by the upstream boundary condition, which is a travelling wave with no diminution.

8. The analysis and use of stage and discharge measurements

Almost universally the routine measurement of the state of a river is that of the stage, the surface elevation at a gauging station, usually specified relative to an arbitrary local datum. While surface elevation is an important quantity in determining the danger of flooding, another important quantity is the actual flow rate past the gauging station. Accurate knowledge of this instantaneous discharge - and its time integral, the total volume of flow - is crucial to many hydrologic investigations and to practical operations of a river and its chief environmental and commercial resource, its water. Examples include decisions on the allocation of water resources, the design of reservoirs and their associated spillways, the calibration of models, and the interaction with other computational components of a network.

8.1 Stage discharge method

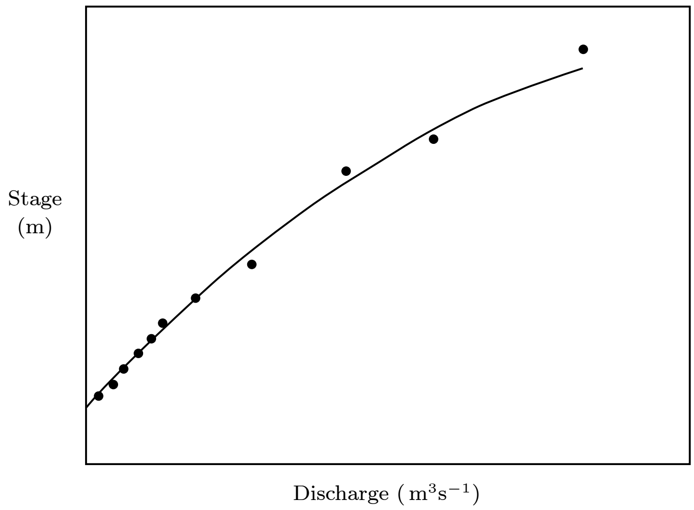

Figure 8-1. Rating curve (stage-discharge diagram) and data points to which it has been fitted

The traditional way in which volume flow is inferred is for a *Rating Curve* to be derived for a particular gauging station, which is a relationship between the stage measured and the actual flow passing that point. The measurement of flow is done at convenient times by traditional hydrologic means, with a current meter measuring the flow velocity at enough points over the river cross section so that the volume of flow can be obtained for that particular stage, measured at the same time. By taking such measurements for a number of different stages and corresponding discharges over a long period of time, a number of points can be plotted on a stage-discharge diagram, and a curve drawn through those points, giving what is hoped to be a unique relationship between stage and flow, the *Rating Curve*, as shown in Figure 8-1. If the supposedly unique relationship between the flow rate and the stage is written $Q_{\mathrm{r}}(\eta)$, subsequent measurements of the surface elevation at some time t, such as an hourly or daily measurement, are then used to give the discharge:

$$Q(t) = Q_\mathrm{r}(\eta(t)).$$

It is assumed that for any stage reading, which is the routine periodic measurement, the corresponding discharge can be calculated. This is what happens when the stage is read and telemetered to a central data management authority. From the rating curve for that stage, the corresponding discharge can be calculated. This is very widely used and is the routine method of flow measurement. It begs a number of questions.

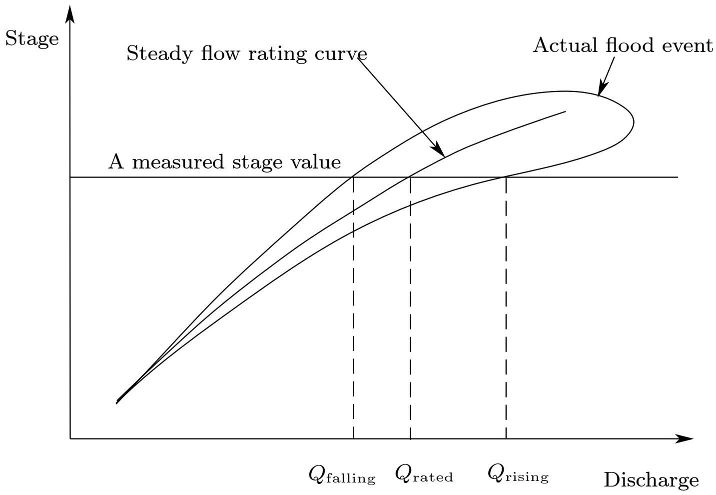

Figure 8-2. Stage-discharge diagram showing the steady-flow rating curve and an exaggerated looped trajectory of a particular flood event, which can be due to two effects: changing roughness and unsteady flow effects

There are several problems associated with the use of a Rating Curve:

- The assumption of a unique relationship between stage and discharge may not be justified.
- Discharge is rarely measured during a flood, and the quality of data at the high flow end of the curve might be quite poor.
- It is usually some sort of line of best fit through a sample made up of a number of points - sometimes extrapolated for higher stages.
- It has to describe a range of variation from no flow through small but typical flows to very large extreme flood events.
- There are a number of factors which might cause the rating curve not to give the actual discharge, some of which will vary with time. Factors affecting the rating curve include:
  - The channel changing as a result of modification due to dredging, bridge construction, or vegetation growth.
  - Sediment transport - where the bed is in motion, which can have an effect over a single flood event, because the effective bed roughness can change during the event. As a flood increases, any bed forms present will tend to become larger and increase the effective roughness, so that friction is greater after the flood peak than before, so that the corresponding discharge for a given stage height will be less after the peak. This will contribute to a flood event showing a looped curve on a stage-discharge diagram as is shown on Figure 8-2.
  - Backwater effects - changes in the conditions downstream such as the construction of a dam or flooding in the next waterway.
  - Unsteadiness - in general the discharge will change rapidly during a flood, and the slope of the water surface will be different from that for a constant stage, depending on whether the discharge is increasing or decreasing, also contributing to a flood event appearing as a loop on a stage-discharge diagram such as Figure 8-2.
  - Variable channel storage - where the stream overflows onto flood plains during high discharges, giving

rise to different slopes and to unsteadiness effects.

– Vegetation - changing the roughness and hence changing the stage-discharge relation.

– Ice - which we will ignore.

Some of these can be allowed for by procedures which we will describe later.

8.1.1 The hydraulics of a gauging station

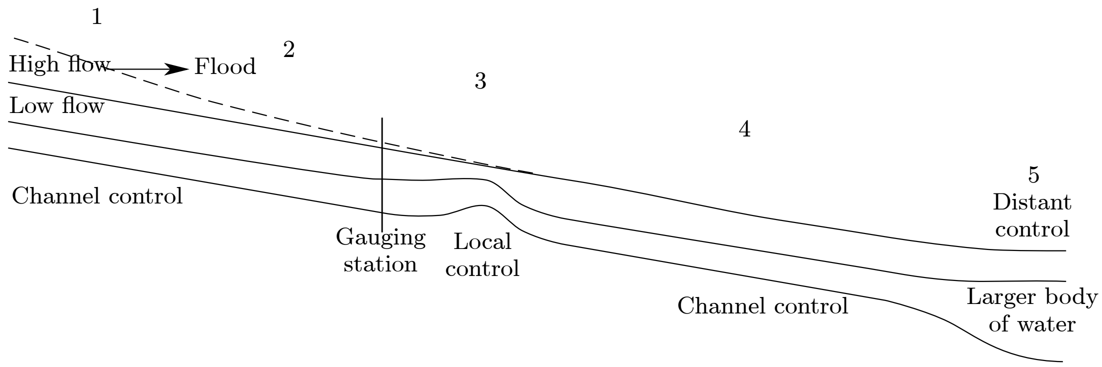

Figure 8-3. Section of river showing different controls at different water levels with implications for the stage discharge relationship at the gauging station shown

A typical set-up of a gauging station where the water level is regularly measured is given in Figure 8-3 which shows a longitudinal section of a stream. Downstream of the gauging station is usually some sort of fixed control which may be some local topography such as a rock ledge which means that for relatively small flows there is a relationship between the head over the control and the discharge which passes. This will control the flow for small flows. For larger flows the effect of the fixed control is to ”drown out”, to become unimportant, and for some other part of the stream to control the flow, such as the larger river downstream shown as a distant control in the figure, or even, if the downstream channel length is long enough before encountering another local control, the section of channel downstream will itself become the control, where the control is due to friction in the channel, giving a relationship between the slope in the channel, the channel geometry and roughness and the flow. There may be more controls too, but however many there are, if the channel were stable, and the flow steady (i.e. not changing with time anywhere in the system) there would be a unique relationship between stage and discharge, however complicated this might be due to various controls. In practice, the natures of the controls are usually unknown.

Something which the concept of a rating curve overlooks is the effect of unsteadiness, or variation with time. In a flood event the discharge will change with time as the flood wave passes, and the slope of the water surface will be different from that for a constant stage, depending on whether the discharge is increasing or decreasing. Figure 8-3 shows the increased surface slope as a flood approaches the gauging station. The effects of this are shown on Figure 8-2, in somewhat exaggerated form, where an actual flood event may not follow the rating curve but will in general follow the looped trajectory shown. As the flood increases, the surface slope in the river is greater than the slope for steady flow at the same stage, and hence, according to conventional simple hydraulic theory explained below, more water is flowing down the river than the rating curve would suggest. This is shown by the discharge marked Q_{rising} obtained from the horizontal line drawn for a particular value of stage. When the water level is falling the slope and hence the discharge inferred is less.

The effects of this might be important - the peak discharge could be significantly underestimated during highly dynamic floods, and also since the maximum discharge and maximum stage do not coincide, the arrival time of the peak discharge could be in error and may influence flood warning predictions. Similarly water-quality constituent loads could be underestimated if the dynamic characteristics of the flood are ignored, while the use of a discharge hydrograph derived inaccurately by using a single-valued rating relationship may distort estimates for resistance coefficients during calibration of an unsteady flow model.

8.1.2 Rating curves – representation, approximation and calculation

Figure 8-4 shows the current rating curve for the Ovens River at Wangaratta in Victoria, Australia, where flow

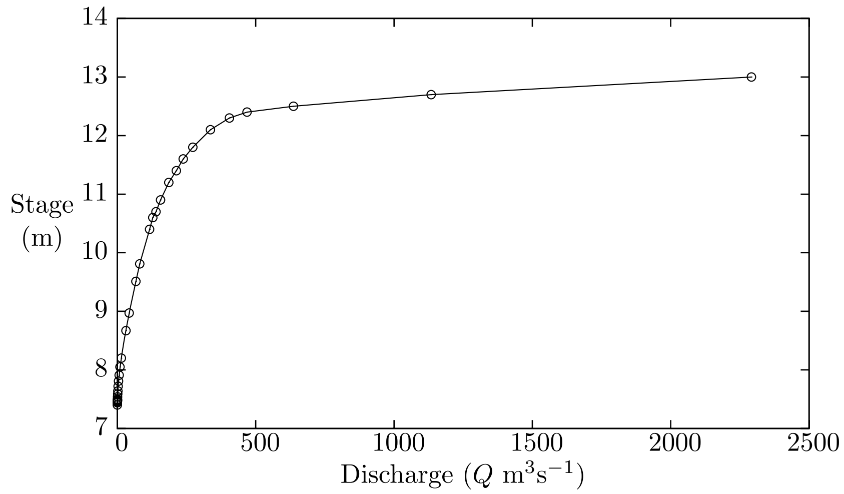

Figure 8-4. Rating curve using natural (Q, η) axes

measurements have been made since 1891. There are a couple of difficulties with such a curve, including reading results off for small flows, where the curve is locally vertical, and for high flows where it is almost horizontal. A traditional way of overcoming the difficulty of representing rating curves over a large range has been to use log-log axes. However, this has no physical basis and has a number of practical difficulties, although it has been recommended by International Standards.Hydraulic theory can help here, for it can be used to show that the stage-discharge relationship will tend to show stage varying approximately like $\eta \sim Q^{1/2}$, for both cases:

1. Flow across a U-shaped (parabolic) weir, the approximate situation for *low flow* at a gauging station, when a local control such as a rock ledge controls the flow, and
2. Uniform flow down a U-shaped (parabolic) waterway for *large flows*, when the local control is washed out and the waterway acts more like a uniform flow governed by Manning's law.

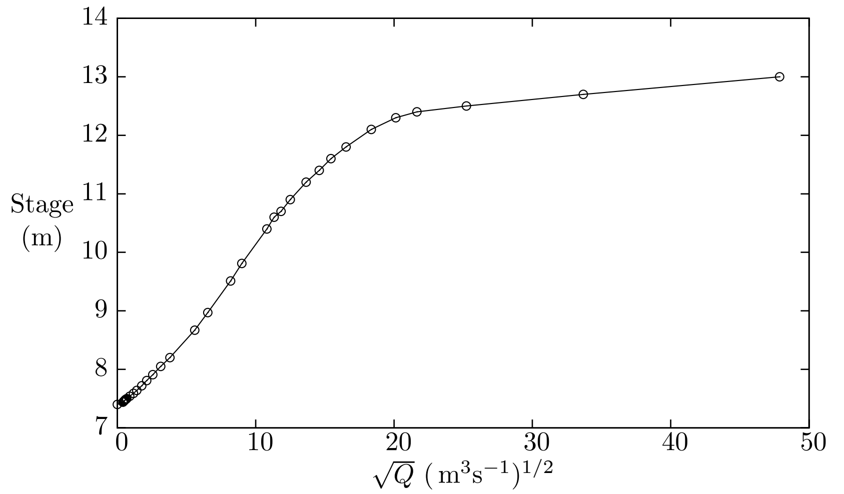

Figure 8-5. The same rating curve as in figure 8-4 but using $(\sqrt{Q}, \eta)$ axes

In these cases, both parts of the relationship would plot as (different) straight lines on $\left(\sqrt{Q},\eta\right)$ axes. In figure 8-5 we plot the results from the above figure on such a square root scale for the discharge, and we see that indeed at both small and large flows the rating curve is a straight line. This means that simpler procedures of numerical approximation and interpolation could be used. Sometimes results have to be taken by extrapolating the curve. If this has to be done, then linear extrapolation on the $\left(\sqrt{Q},\eta\right)$ axes might be reasonable, but it is still a procedure to be followed with great caution, as the actual geometry for above-bank flows can vary a lot.

A number of such practical considerations are given in Fenton (2001).

8.2 Stage-discharge-slope method

This is presented in some books and in International Standards, however, especially in the latter, the presentation is confusing and at a low level, where no reference is made to the fact that underlying it the *slope* is being measured. Instead, the *fall* is described, which is the change in surface elevation between two surface elevation gauges and is simply the slope multiplied by the distance between them. No theoretical justification is provided and it is presented in a phenomenological sense (see, for example, Herschy 1995).

Although the picture in Figure 8-3 of the factors affecting the stage and discharge at a gauging station seems complicated, the underlying processes are capable of quite simple description. In a typical stream, where all wave motion is of a relatively long time and space scale, the governing equations are the long wave equations, which are a pair of partial differential equations for the stage and the discharge at all points of the channel in terms of time and distance along the channel. One is a mass conservation equation, the other a momentum equation. Under the conditions typical of most flows and floods in natural waterways, however, the flow is sufficiently slow that the equations can be simplified considerably. Most terms in the momentum equation are of a relative magnitude given by the square of the Froude number, which is U^2/gD, where U is the fluid velocity, g is the gravitational acceleration, and D is the mean depth of the waterway. In most rivers, even in flood, this is small, and the approximation may be often used. For example, a flow of $1\,\mathrm{ms}^{-1}$ with a depth of $2\,\mathrm{m}$ has $F^2 \approx 0.05$.

In Appendix §A we show that under these circumstances, a surprisingly good approximation to the momentum equation of motion for flow in a waterway is the simple equation:

$$Q(t) = K(\eta(t))\sqrt{-\partial\eta/\partial x(t)}, \tag{8.1}$$

where we have shown that the slope of the free surface might be a function of time. This gives us a formula for calculating the discharge Q which is as accurate as is reasonable to be expected in river hydraulics, provided we know

1. the stage and the dependence of conveyance K on stage at a point from either measurement or the G-M-S or Chézy's formulae, and
2. the slope of the surface

Stage-conveyance curves

Equation (8.1) shows how the discharge actually depends on both the stage and the surface slope, whereas traditional hydrography assumes that it depends on stage alone. If the slope does vary under different backwater conditions or during a flood, then a better hydrographic procedure would be to gauge the flow when it is steady, and to measure the surface slope , thereby enabling a particular value of K to be calculated for that stage. If this were done over time for a number of different stages, then a stage-conveyance relationship could be developed which should then hold whether or not the stage is varying. Subsequently, in day-to-day operations, if the stage and the surface slope were measured, then the discharge calculated from equation (8.1) should be quite accurate, within the relatively mild assumptions made so far. All of this holds whether or not the gauging station is affected by a local or channel control, and whether or not the flow is changing with time.

This suggests that a better way of determining streamflows in general, but primarily where backwater and unsteady effects are likely to be important, is for the following procedure to be followed:

1. At a gauging station, two measuring devices for stage be installed, so as to be able to measure the slope of the water surface at the station. One of these could be at the section where detailed flow-gaugings are taken, and the other could be some distance upstream or downstream such that the stage difference between the two points is enough that the slope can be computed accurately enough. As a rough guide, this might be, say $10\,\mathrm{cm}$, so that if the water slope were typically 0.001, they should be at least $100\,\mathrm{m}$ apart.
2. Over time, for a number of different flow conditions the discharge Q would be measured using conventional methods such as by current meter. For each gauging, both surface elevations would be recorded, one becoming the stage η to be used in the subsequent relationship, the other so that the surface slope S_η can be calculated. Using equation (8.1), $Q = K(\eta)\sqrt{S_\eta}$, this would give the appropriate value of conveyance K for that stage,

automatically corrected for effects of unsteadiness and downstream conditions.

3. From all such data pairs (η_i, K_i) for $i = 1, 2, \ldots$, the conveyance curve (the functional dependence of K on η) would be found, possibly by piecewise-linear or by global approximation methods, in a similar way to the description of rating curves described below. Conveyance has units of discharge, and as the surface slope is unlikely to vary all that much, we note that there are certain advantages in representing rating curves on a plot using the square root of the discharge, and it my well be that the stage-conveyance curve would be displayed and approximated best using $(\sqrt{K}, \eta)$ axes.
4. Subsequent routine measurements would obtain both stages, including the stage to be used in the stage-conveyance relationship, and hence the water surface slope, which would then be substituted into equation (8.1) to give the discharge, corrected for effects of downstream changes and unsteadiness.

If hydrography had followed the path described above, of routinely measuring surface slope and using a stage-conveyance relationship, the ”science” would have been more satisfactory. Effects due to the changing of downstream controls with time, downstream tailwater conditions, and unsteadiness in floods would have been automatically incorporated, both at the time of determining the relationship and subsequently in daily operational practice.

However, for the most part slope has not been measured, and hydrographic practice has been to use rating curves instead. The assumption behind the concept of a discharge-stage relationship or rating curve is that the slope at a station is constant over all flows and events, so that the discharge is a unique function of stage $Q_{\mathrm{r}}(\eta)$ where we use the subscript r to indicate the rated discharge. Instead of the empirical/rational expression (8.1), traditional practice is to calculate discharge from the equation

$$Q(t) = Q_{\mathrm{r}}(\eta(t)), \tag{8.2}$$

thereby ignoring any effects that downstream backwater and unsteadiness might have, as well as the possible changing of a downstream control with time.

In comparison, equation (8.1), based on a convenient empirical approximation to the real hydraulics of the river, contains the essential nature of what is going on in the stream. It shows that, although the conveyance might be a unique function of stage which it is possible to determine by measurement, because the surface slope will in general vary throughout different flood events and downstream conditions, discharge in general does not depend on stage alone.

8.3 Looped rating curves – correcting for unsteady effects in obtaining discharge from stage

In conventional hydrography the stage is measured repeatedly at a single gauging station so that the time derivative of stage can easily be obtained from records but the surface slope along the channel is not measured at all. The methods of this section are all aimed at obtaining the slope in terms of the stage and its time derivatives at a single gauging station. The simplest and most traditional method of calculating the effects of unsteadiness has been the Jones formula, derived by B. E. Jones in 1916 (see for example Chow 1959, Henderson 1966). The principal assumption is that to obtain the slope, the x derivative of the free surface, we can use the time derivative of stage which we can get from a stage record, by assuming that the flood wave is moving without change as a kinematic wave (Lighthill and Whitham, 1955) such that it obeys the partial differential equation:

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0, \tag{8.3}$$

where h is the depth and c is the kinematic wave speed. Solutions of this equation are simply waves travelling at a velocity c without change. The equation was obtained as the last of a series of approximations in Section ??. The kinematic wave speed c is given by the derivative of flow with respect to cross-sectional area, the Kleitz-Seddon law

$$c = \frac{1}{B}\frac{dQ_{\mathrm{r}}}{d\eta} = \frac{1}{B}\frac{dK}{d\eta}\sqrt{\tilde{S}}, \tag{8.4}$$

where B is the width of the surface and Q_{r} is the steady rated discharge corresponding to stage η, and where we have expressed this also in terms of the conveyance K, where $Q_{\mathrm{r}} = K(\eta)\sqrt{\tilde{S}}$, and the slope $\sqrt{\tilde{S}}$ is the mean slope of the stream. A good approximation is $c \approx 5/3 \times U$, where U is the mean stream velocity.

The Jones method assumes that the surface slope S_η in equation (8.1) can be simply related to the rate of change of stage with time, assuming that the wave moves without change. Thus, equation (8.3) gives an approximation for the surface slope: $\partial h/\partial x \approx -1/c \times \partial h/\partial t$. We then have to use the simple geometric relation between surface gradient and depth gradient, that $\partial \eta/\partial x = \partial h/\partial x - \tilde{S}$, such that we have the approximation

$$S_\eta = -\frac{\partial \eta}{\partial x} = \tilde{S} - \frac{\partial h}{\partial x} \approx \tilde{S} + \frac{1}{c}\frac{\partial h}{\partial t}$$

and recognising that the time derivative of stage and depth are the same, $\partial h/\partial t = \partial \eta/\partial t$, equation (8.1) gives

$$Q = K\sqrt{\tilde{S} + \frac{1}{c}\frac{\partial \eta}{\partial t}} \tag{8.5}$$

If we divide by the steady discharge corresponding to the rating curve we obtain

$$\frac{Q}{Q_{\mathrm{r}}} = \sqrt{1 + \frac{1}{c\tilde{S}}\frac{\partial \eta}{\partial t}} \tag{Jones}$$

In situations where the flood wave does move as a kinematic wave, with friction and gravity in balance, this theory is accurate. In general, however, there will be a certain amount of diffusion observed, where the wave crest subsides and the effects of the wave are smeared out in time.

To allow for those effects Fenton (1999) provided the theoretical derivation of two methods for calculating the discharge. The derivation of both is rather lengthy. The first method used the full long wave equations and approximated the surface slope using a method based on a linearisation of those equations. The result was a differential equation for dQ/dt in terms of Q and stage and the derivatives of stage $d\eta/dt$ and $d^2\eta/dt^2$, which could be calculated from the record of stage with time and the equation solved numerically. The second method was rather simpler, and was based on the next best approximation to the full equations after equation (8.3). This gives the *advection-diffusion equation*

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = \nu\,\frac{\partial^2 h}{\partial x^2}, \tag{8.6}$$

where the difference between this and equation (8.3) is the diffusion term on the right, where ν is a diffusion coefficient (with units of $\mathrm{L}^2\mathrm{T}^{-1}$), given by

$$\nu = \frac{K}{2B\sqrt{\tilde{S}}}.$$

Equation (8.6), is a consistent low-inertia approximation to the long wave equations, where inertial terms, which are of the order of the square of the Froude number, which approximates motion in most waterways quite well. However, it is not yet suitable for the purposes of this section, for we want to express the x derivative at a point in terms of time derivatives. To do this, we use a small-diffusion approximation, we assume that the two x derivatives on the right of equation (8.6) can be replaced by the zero-diffusion or kinematic wave approximation as above, $\partial/\partial x \approx -1/c \times \partial/\partial t$, so that the surface slope is expressed in terms of the first two time derivatives of stage. The resulting expression is:

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = \frac{\nu}{c^2}\,\frac{\partial^2 h}{\partial t^2},$$

and solving for the x derivative, we have the approximation

$$S_\eta = -\frac{\partial \eta}{\partial x} = \tilde{S} - \frac{\partial h}{\partial x} \approx \tilde{S} + \frac{1}{c}\frac{\partial h}{\partial t} - \frac{\nu}{c^3}\frac{d^2 h}{dt^2},$$

and substituting into equation (8.1) gives

$$Q = Q_{\mathrm{r}}(\eta)\sqrt{\underbrace{\underbrace{1}_{\text{Rating curve}} + \frac{1}{c\tilde{S}}\frac{d\eta}{dt}}_{\text{Jones formula}} \underbrace{- \frac{\nu}{c^3\tilde{S}}\frac{d^2\eta}{dt^2}}_{\text{Diffusion term}}} \tag{8.7}$$

where Q is the discharge at the gauging station, $Q_{\mathrm{r}}(\eta)$ is the rated discharge for the station as a function of stage, $\tilde{S}$ is the bed slope, c is the kinematic wave speed given by equation (8.4):

$$c = \frac{\sqrt{\tilde{S}}}{B}\frac{dK}{d\eta} = \frac{1}{B}\frac{dQ_{\mathrm{r}}}{d\eta},$$

in terms of the gradient of the conveyance curve or the rating curve, B is the width of the water surface, and where the coefficient ν is the diffusion coefficient in advection-diffusion flood routing, given by:

$$\nu = \frac{K}{2B\sqrt{\tilde{S}}} = \frac{Q_{\mathrm{r}}}{2B\tilde{S}}. \tag{8.8}$$

In equation (8.7) it is clear that the extra diffusion term is a simple correction to the Jones formula, allowing for the subsidence of the wave crest as if the flood wave were following the advection-diffusion approximation, which is a good approximation to much flood propagation. Equation (8.7) provides a means of analysing stage records and correcting for the effects of unsteadiness and variable slope. It can be used in either direction:

- If a gauging exercise has been carried out while the stage has been varying (and been recorded), the value of Q obtained can be corrected for the effects of variable slope, giving the steady-state value of discharge for the stage-discharge relation,
- And, proceeding in the other direction, in operational practice, it can be used for the routine analysis of stage records to correct for any effects of unsteadiness.

The ideas set out here are described rather more fully in Fenton & Keller (2001).

An example

A numerical solution was obtained for the particular case of a fast-rising and falling flood in a stream of $10\,\mathrm{km}$ length, of slope 0.001, which had a trapezoidal section $10\,\mathrm{m}$ wide at the bottom with side slopes of 1:2, and a Manning's friction coefficient of 0.04. The downstream control was a weir. Initially the depth of flow was $2\,\mathrm{m}$, while carrying a flow of $10\,\mathrm{m}^3\mathrm{s}^{-1}$. The incoming flow upstream was linearly increased ten-fold to $100\,\mathrm{m}^3\mathrm{s}^{-1}$ over 60 mins and then reduced to the original flow over the same interval. The initial backwater curve problem was solved and then the long wave equations in the channel were solved over six hours to simulate the flood. At a station halfway along the waterway the computed stages were recorded (the data one would normally have), as well as the computed discharges so that some of the above-mentioned methods could be applied and the accuracy of this work tested.

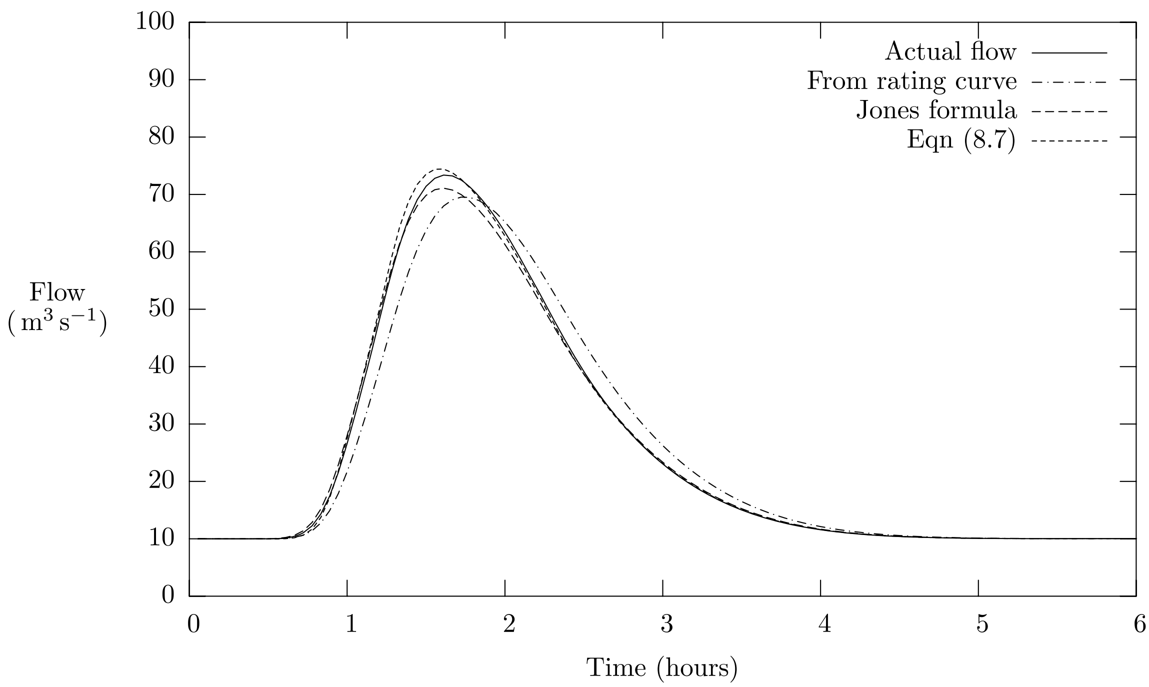

Figure 8-6. Simulated flood with hydrographs computed from stage record using three levels of approximation

Results are shown on Figure 8-6. It can be seen that the application of the diffusion level of approximation of equation (8.7) has succeeded well in obtaining the actual peak discharge. The results are not exact however, as the derivation depends on the diffusion being sufficiently small that the interchange between space and time differentiation will be accurate. In the case of a stream such as the example here, diffusion is relatively large, and our results are not exact, but they are better than the Jones method at predicting the peak flow. Nevertheless, the results from the Jones method are interesting. A widely-held opinion is that it is not accurate. Indeed, we see here that in predicting the peak flow it was not accurate in this problem. However, over almost all of the flood it was accurate, and predicted the *time* of the flood peak well, which is also an important result. It showed that both before and after the peak the ”discharge wave” led the ”stage wave”, which is of course in phase with the curve showing the flow computed from the stage graph and the rating curve. As there may be applications where it is enough to know the arrival time of the flood peak, this is a useful property of the Jones formula. Near the crest, however, the rate of rise became small and so did the Jones correction. Now, and only now, the inclusion of the extra diffusion term gave a significant correction to the maximum flow computed, and was quite accurate in its prediction that the real flow was some 10% greater than that which would have been calculated just from the rating curve. In this fast-rising example the application of the unsteady corrections seems to have worked well and to be justified. It is no more difficult to apply the diffusion correction than the Jones correction, both being given by derivatives of the stage record.

8.4 The effects of bed roughness on rating curves

8.4.1 Introduction

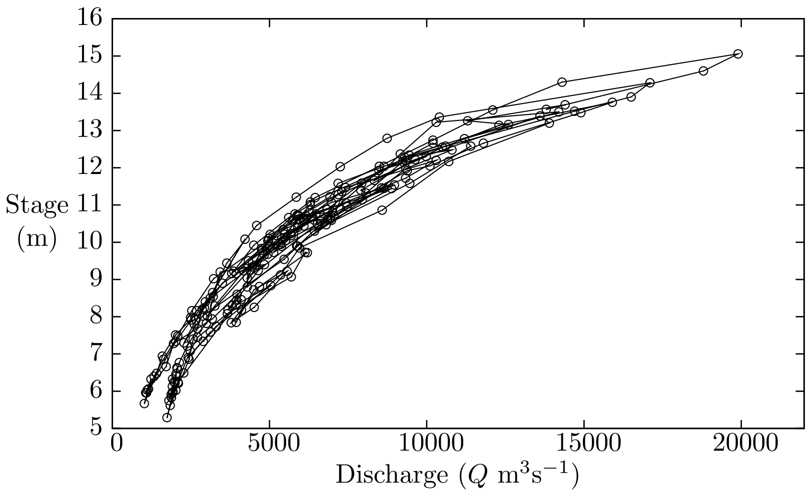

Figure 8-7. Flood trajectories for Station 41 on the Red River, showing raw data – corrected data was everywhere obscured

We have considered the detailed records for 1995 and 1996 for the Red River, Viet Nam. The trajectories, plotted on Stage/Discharge axes on Figure 8-7, show considerable loopiness. In the expectation that the cause was the unsteady wave effects as described above, we applied the theory described above, assuming for simple purposes, a slope $S_0 = 0.0015$, and using the recorded values of stage, depth, area, and breadth. Everywhere, no visible correction was made, and the methods of the section above have failed to describe the loopiness. It seems that the loopiness in this case must be due to effects of the bed topography and friction changing with flow. This is the more common situation, and it is bed roughness changes which will more often be the cause of loopiness.

Simons & Richardson (1962) have written on the nature of the relationship between stage and discharge when the river bed is mobile, when bed forms may change, depending on the flow, such that in a flood there may be different bed roughness before and after, and the stage-discharge trajectory may show loopiness, as shown in Figure 8-7 above. This suggests a different approach to the relationship between stage and discharge observed in a river, when the river itself is the control, rather than a hydraulic structure.

Let us consider the mechanism by which changes in roughness cause the flood trajectories to be looped, by considering a hypothetical and idealised situation. In Figure 8-8 is shown a plot of Stage versus Discharge. The rating curves which would apply if the bedforms were held constant are shown – for a flat bed and for various increasing

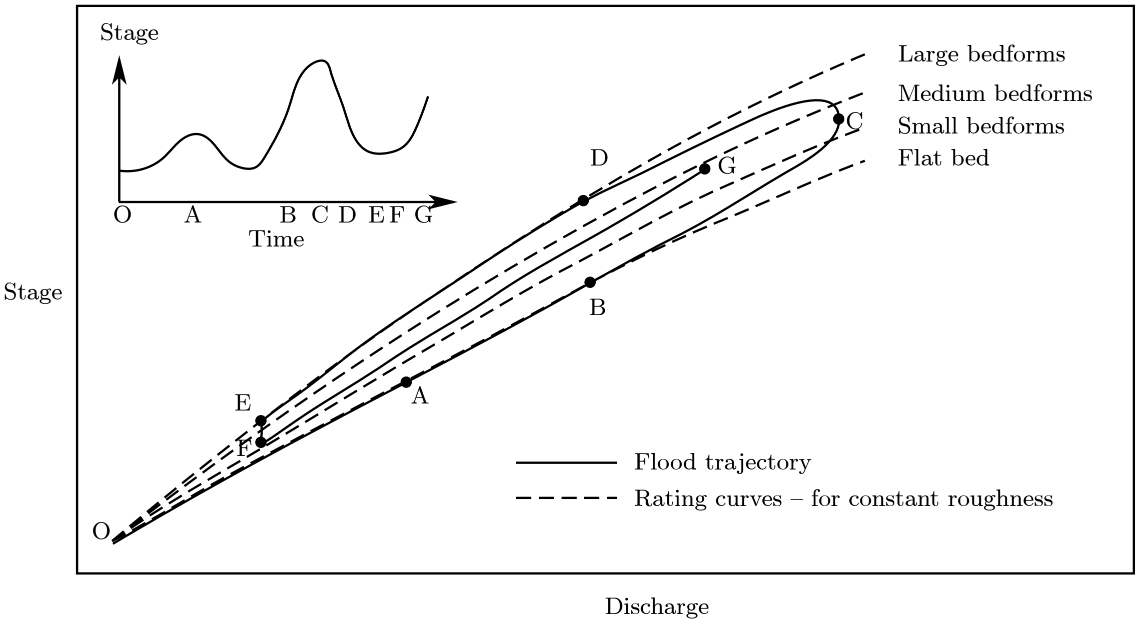

Figure 8-8. Rating curves for different bedforms and a looped flood trajectory

bedform roughness. In the left corner is a stage-time graph with three flood events, one small, one large, and one not yet completed. The points labelled O, A, ..., G are also shown on the *flood trajectory*, showing the actual relationship between stage and discharge at each time.

O: flow is small, over a flat bed.

A: a small flood peak has arrived. The flow is not enough to change the nature of the bed, and the flood trajectory follows the flat-bed rating curve back down to a smaller flow, and then back up to a larger flow.

B: the bed is no longer stable and bed forms, with increasing roughness, start to form. Accordingly, the carrying capacity of the channel is reduced and the stage increases.

C: the flood peak has arrived, and the bed forms continue to grow, so that a little time later the stage is a maximum.

D: the bedforms have continued to grow until here, although the flow is decreasing.

E: the flow has decreased much more quickly than the bed-forms can adjust, and the point is only a little below the rating curve corresponding to the largest bedforms.

F: over the intervening time, flow has been small and almost constant, however the time has been enough to reduce the bed-forms. Now another flood starts to arrive, and this time, instead of following the flat-bed curve, it already starts from a finite roughness, such as we seem to see in Figure 8-7.

G: after this, the history of the stage will still depend on the history of the flow and the characteristics of the bed-form rate of growth.

8.4.2 A theoretical approach

This process could be modelled mathematically. If we had an expression for the rate of growth of roughness as a function of discharge, depth, and roughness itself, say,

$$\frac{d\Omega}{dt} = f(Q, h, \Omega), \tag{8.9}$$

where Q is discharge, h is depth, and Ω is some measure of roughness, and if we had a channel friction law such as Gauckler-Manning-Strickler, which we write in a general way:

$$Q = F(h, \Omega, \bar{S}), \tag{8.10}$$

then if we assume that slope is constant, as it almost always is except when unsteady effects are important, then differentiating (8.10) with respect to time,

$$\begin{aligned}\frac{dQ}{dt} &= \frac{\partial F}{\partial h}\frac{dh}{dt} + \frac{\partial F}{\partial \Omega}\frac{d\Omega}{dt} \\ &= \frac{\partial F}{\partial h}\frac{dh}{dt} + \frac{\partial F}{\partial \Omega} f(Q, h, \Omega),\end{aligned}$$

having substituted equation (8.9), giving us an ordinary differential equation for Q as a function of t provided we know the stage hydrograph $h(t)$, which is what is usually measured.

In general the condition of the river bed, whether smooth, or with dunes, antidunes or standing waves, will depend on the time history of the flow. That is, the roughness now depends on the preceding conditions for a period of time.

8.4.3 An empirical approach

This immediately suggests the concept of using some form of convolution, where the effects of preceding events are incorporated in an integral sense. In hydrology, this is most familiar as the unit hydrograph, see for example, Chapter 7 of Chow et al. (1988). In the case of a river with data such as in Figure 8-7 we suggest the following nonlinear influence function, where the discharge Q at time t can be written as nonlinear functions of stage η at previous times $t - \Delta$, $t - 2\Delta$, *etc.*:

$$\begin{aligned}Q(t) = a_{00} \quad & + a_{01}\eta(t) + a_{02}\eta^2(t) + \ldots \\ & + a_{11}\eta(t-\Delta) + a_{12}\eta^2(t-\Delta) + \ldots \\ & + a_{21}\eta(t-2\Delta) + a_{22}\eta^2(t-2\Delta) + \ldots \\ & + \ldots .\end{aligned}$$

If only the first line of that equation had been taken, then that is a conventional rating curve, where the discharge at t is assumed to be a function of the stage at t.

Now, the procedure would be to calculate the coefficients a_{ij} from a long data sequence, using least squares methods. This could then be then applied to future events so that a better prediction of the actual discharge could be obtained.

Appendix A. The momentum equation simplified

Somehow a very important simplification of the momentum equation got lost in between parts of the lecture notes. Consider the momentum equation (3.8):

$$\frac{\partial Q}{\partial t}+2\beta\frac{Q}{A}\frac{\partial Q}{\partial x}+\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial\eta}{\partial x}=\beta\frac{Q^2B}{A^2}\tilde{S}-\Lambda P\frac{Q^2}{A^2}, \tag{3.8}$$

where we consider problems where the flow is only in one direction, $Q\left|Q\right|=Q^2$. If we consider problems where the boundary conditions change gradually in both time, such that $\partial Q/\partial t$ is small, and hence the spatial derivative $\partial Q/\partial x$ is small as well, such as when waves are very long, then if we neglect the terms with those two derivatives, both usually the case, the momentum equation becomes

$$\left(gA-\beta\frac{Q^2B}{A^2}\right)\frac{\partial\eta}{\partial x}=\beta\frac{Q^2B}{A^2}\tilde{S}-\Lambda P\frac{Q^2}{A^2},$$

which can simply be solved for Q as an algebraic equation, giving

$$Q=\sqrt{\frac{gA^3}{P\Lambda}}\sqrt{\frac{-\partial\eta/\partial x}{1-\frac{\beta B}{P\Lambda}\left(S+\partial\eta/\partial x\right)}}.$$

We call the quantity $\sqrt{gA^3/P\Lambda}$ the conveyance K, or using other forms of the resistance term, from equation (3.12)

$$K=\sqrt{\frac{g}{\Lambda}\frac{A^3}{P}}=C\sqrt{\frac{A^3}{P}}=\frac{1}{n}\frac{A^{5/3}}{P^{2/3}}=k_{\mathrm{St}}\frac{A^{5/3}}{P^{2/3}}. \tag{3.12}$$

The equation for Q becomes

$$Q=K\sqrt{\frac{-\partial\eta/\partial x}{1-\frac{\beta B}{P\Lambda}\left(S+\partial\eta/\partial x\right)}}, \tag{A-1}$$

and in many (most) cases of interest the terms involving β, which are of a magnitude that of the Froude number squared, can be ignored, giving the momentum equation in the form

$$Q=K\sqrt{-\frac{\partial\eta}{\partial x}}. \tag{A-2}$$

We have shown that in most practical situations that the momentum equation in the form of equation (3.8) can be approximated very simply by equation (A-1) or even more simply by equation (A-2). It does not look like a momentum equation, but it is – and one is free to use any of the common resistance formulae, equation (3.12)! It shows that in a stream the discharge is given simply by $K(\eta)$, a function of the local stage, and the slope of the free surface $\partial\eta/\partial x$, whether this is changing in time and space or not. This makes a number of deductions rather simpler. We used this in §7.7 to get simple (and accurate enough!) formulae for unsteady flood routing. Additionally, it provides the physical justification of the stage-discharge-slope method in §8.2, and the explanation for the looped rating curve in §8.3.