Institute of Hydraulic and Water Resources Engineering, Vienna University of Technology

Coastal & Ocean Engineering

Tutorial Sheet 3 – Irrotational incompressible flow theory

Revision of curl operator: The vorticity $\boldsymbol{\omega}$ of a fluid flow is given by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix},$$

where the general expression for the velocity vector $\mathbf{u}$ is $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = (u, v, w)$,and the gradient operator is $\nabla = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z = (\,\partial/\partial x,\, \partial/\partial y,\, \partial/\partial z)$.

1. Verify that $\nabla \times \nabla\phi = 0$ for all ϕ. (Hence if a flow is irrotational, $\nabla \times \mathbf{u} = 0$, there exists a velocity potential ϕ such that $\mathbf{u} = \nabla\phi = \mathbf{i}\,\partial\phi/\partial x + \mathbf{j}\,\partial\phi/\partial y + \mathbf{k}\,\partial\phi/\partial z = (\,\partial\phi/\partial x,\, \partial\phi/\partial y,\, \partial\phi/\partial z)$.)

2. The mass conservation equation for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

   Verify that it can be written in vector form $\nabla \cdot \mathbf{u} = 0$.

3. Show that the planar flow field given by $\mathbf{u} = \cos x\, \cosh z\, \mathbf{i} + \sin x\, \sinh z\, \mathbf{k}$ is that of an incompressible irrotational fluid. (This field occurs in the flow caused by steadily-progressing water waves).

4. A question with rotation: if a rigid body rotates about the z axis with angular speed Ω
   a. Show that the velocity components are: $u = -\Omega y$, $v = \Omega x$, $w = 0$.
   b. Obtain the vorticity, the curl of the velocity field. (*Ans.:* $2\Omega\mathbf{k}$, a constant, equal to twice the angular velocity!)

5. Given the solution for velocity potential

$$\phi = A(x^2 - z^2), \text{ where } A \text{ is a constant,}$$

   corresponding to a stagnation point flow, either the flow of a planar jet directed against a wall or the flow in a right-angled corner.
   a. Obtain expressions for the velocity components u and w.
   b. Show that the flow is irrotational and the fluid incompressible.
   c. Verify that the flow satisfies the kinematic boundary condition on the vertical wall $x = 0$.
   d. Verify that the flow satisfies the symmetry condition or the kinematic boundary condition $w = 0$ on $z = 0$.
   e. Sketch the flow.

6. Consider the two dimensional problem of the reflection of waves by a vertical wall such that a standing wave pattern is produced. The velocity potential ϕ is given by:

$$\phi(x, z, t) = B\, \cos kx\, \cosh kz\, \sin \sigma t,$$

   in which B is a constant, σ is the radian frequency of the wave motion, $k = 2\pi/L$ is the wavenumber, L is the wavelength, and d is the mean depth.

   Verify that ϕ satisfies Laplace's equation and the boundary conditions on (i) the bottom: $w = \partial\phi/\partial z = 0$ on $z = 0$, and (ii) the wall $u = \partial\phi/\partial x = 0$ on $x = 0$.