Institute of Hydraulic and Water Resources Engineering, Vienna University of Technology

Coastal & Ocean Engineering

Tutorial Sheet 2 – Kinematic boundary conditions

The scalar or dot product

The scalar product between two vectors gives the combined effect of the two in a scalar sense. Consider two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$:

- The scalar product is $\mathbf{a} \cdot \mathbf{b}$ and is equal to the component of $\mathbf{a}$ in the direction of $\mathbf{b}$ times the magnitude of $\mathbf{b}$, and *vice versa*.
- This is simply evaluated: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- An important special case is the scalar product of a vector with a unit vector; it is simply the component of that vector in the direction of the unit vector:

$$\mathbf{a} \cdot \hat{\mathbf{n}} = \text{The component of } \mathbf{a} \text{ in the direction of } \hat{\mathbf{n}}$$

- The length of any vector is given by the square root of the dot product of the vector with itself:

$$a = |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Finding the direction of the normal to a surface

One of the most useful applications of the gradient operator ∇ is finding the direction of the normal to a boundary, which can be used to give boundary conditions, even for unsteady flow problems of a complicated nature. One of the properties of the gradient operator is that it is perpendicular to a level surface, *i.e.* if there is a scalar field ϕ (temperature, for example), then $\nabla\phi$ at any point is perpendicular to the surface on which ϕ is a constant. This leads to a very nice way of determining the normal vector, and the unit normal vector, on any solid surface. All we have to do is to introduce some function ϕ which is constant on the surface we are considering, then $\nabla\phi$ gives the direction of the normal, and the unit normal $\hat{\mathbf{n}}$ is given by

$$\begin{aligned} \hat{\mathbf{n}} &= \frac{\text{Any vector normal to the surface}}{\text{The magnitude of that vector}} \\ &= \frac{\nabla\phi}{|\nabla\phi|}, \end{aligned}$$

where

$$|\nabla\phi| = \sqrt{\nabla\phi \cdot \nabla\phi} = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}.$$

The real trick is to determine the function which is constant on the surface we are considering – usually the easiest way is to have the constant be zero on that surface.

Example: *An underwater pipeline of radius a rests on a horizontal seabed. Find an expression for the unit vector normal to the pipe. Use co-ordinates at the centre of the pipe.*

The equation of the pipeline is

$$x^2 + z^2 = a^2.$$

(Note our use of z for the vertical co-ordinate. In general in coastal and ocean engineering problems are three-dimensional, so that we need all three co-ordinates, and the special one is vertical, the direction in which gravity acts.)

We introduce the general function of our two co-ordinates:

$$\phi(x,z) = x^2 + z^2 - a^2,$$

which is obviously constant (zero!) on the pipeline, and $\nabla\phi$ is normal to it. Now we evaluate $\nabla\phi$:

$$\nabla\phi = 2x\mathbf{i} + 2z\mathbf{k},$$

and dividing by $|\nabla\phi|$:

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\mathbf{i} + 2z\mathbf{k}}{\sqrt{4x^2 + 4z^2}},$$

but we have an additional simplification, for on the pipe $\sqrt{x^2 + z^2} = a^2$, so that

$$\hat{\mathbf{n}} = \frac{x}{a}\mathbf{i} + \frac{z}{a}\mathbf{k},$$

where x and z are not independent but are connected by $\sqrt{x^2 + z^2} = a^2$.

Check: Let's check by considering obvious points. Consider the point on the right side of the pipe, $x = a$, $z = 0$: $\hat{\mathbf{n}} = \mathbf{i}$, which is right. Also consider the point on the top of the pipe, $x = 0$, $z = a$: $\hat{\mathbf{n}} = \mathbf{k}$. Good. Finally let's just check on the other side of the pipe: $x = -a$, $z = 0$: $\hat{\mathbf{n}} = -\mathbf{i}$, which is also right, having picked up the fact that the normal is in the opposite direction to the first one we considered.

Boundary condition on stationary solid boundaries

We have several different types of solid boundaries to consider, such as retaining walls, steel structures, pipelines, and even the seabed itself (provided we do not have to allow for flow in and out of the bed). They all have the same boundary condition:

> On solid boundaries the condition that flow not cross the solid boundary is that the velocity component normal to the boundary is zero:
>
> $$\mathbf{u} \cdot \hat{\mathbf{n}} = 0,$$
>
> where $\mathbf{u}$ is the fluid velocity and $\hat{\mathbf{n}}$ is a unit normal to the boundary.

Note that if we had included viscosity, then the boundary condition on solid boundaries is the simpler condition $\mathbf{u} = \mathbf{0}$, namely that all velocity components are zero. What our inviscid condition says is that, while the component perpendicular to the body is zero, there is no restriction on the component tangential to that – water can rush past the body without any restriction. For the high Reynolds numbers we consider, this is quite reasonable.

Example (cont.): *Obtain the boundary condition on the pipe in the above example, where the velocity field is* $\mathbf{u} = (u, v, w) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

Obtaining the scalar product, we have

$$\mathbf{u} \cdot \hat{\mathbf{n}} = \left(\frac{x}{a}\mathbf{i} + \frac{z}{a}\mathbf{k}\right) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = \frac{ux}{a} + \frac{wz}{a} = 0,$$

and we can remove the non-zero common factor to give the condition on the circular cylinder,

$$ux + wz = 0.$$

Or, we can recognise that on the pipe, $x = a\cos\theta$, $z = a\sin\theta$, such that the condition can be written $u\cos\theta + w\sin\theta = 0$.Note that no condition on the velocity normal to the plane of flow is obtained.

Exercises

1. Repeat the above Example, using a different co-ordinate system, with origin on the bed (remember that the general equation for a circle with centre at (x_0, z_0) is $(x - x_0)^2 + (z - z_0)^2 = a^2$). Find an expression for a unit vector and then an expression connecting the velocity components such that water does not cross the solid boundary.

2. Ripples on a seabed are quite closely sinusoidal in nature. We will need to find a general expression for the boundary condition on the seabed. Consider a sea-bed (impervious to flow, for our purposes) given by: $z = A\ \cos x$.

   a. Obtain an expression for the unit normal.

   b. Obtain the kinematic boundary condition on the bed. (*Ans.:* $uA \sin x + w = 0$ on $z = a \cos x$).

   c. Check that your answer makes sense for $x = 0, \pi/2, \pi$.

3. An unsteady case! The kinematic boundary condition is

$$\mathbf{u} \cdot \hat{\mathbf{n}} = \mathbf{U} \cdot \hat{\mathbf{n}},$$

   where $\mathbf{U}$ is the velocity of the local boundary

   A circular cylinder of radius a moves along the x axis with constant velocity U. At time $t = 0$ it is at the origin.

   a. Verify to your own satisfaction that the equation of the cylinder is

$$(x - Ut)^2 + z^2 = a^2.$$

   b. Obtain the kinematic boundary condition on the cylinder:

$$(u - U)\,(x - Ut) + wz = 0,$$

   where u and w are the velocity components in the x and z directions respectively,