Institute of Hydraulic and Water Resources Engineering, Vienna University of Technology

Coastal & Ocean Engineering

Solution Sheet 2 – Kinematic boundary conditions

1. *Repeat the above Example of a pipe resting on the seabed, using a different co-ordinate system, with origin on the bed (remember that the general equation for a circle with centre at (x_0, z_0) is $(x-x_0)^2 + (z-z_0)^2 = a^2$). Find an expression for a unit vector and then an expression connecting the velocity components such that water does not cross the solid boundary.*

   The equation of the circle is
$$x^2 + (z-a)^2 = a^2.$$
   Introduce
$$\phi(x,z) = x^2 + (z-a)^2 - a^2,$$
   which is of course 0 on the circle.
$$\begin{aligned} \nabla\phi &= 2x\mathbf{i} + 2(z-a)\mathbf{k} \\ \hat{\mathbf{n}} &= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\mathbf{i} + 2(z-a)\mathbf{k}}{\sqrt{4x^2 + 4(z-a)^2}}, \end{aligned}$$
   but $x^2 + (z-a)^2 = a^2$, so that
$$\hat{\mathbf{n}} = \frac{x}{a}\mathbf{i} + \frac{z-a}{a}\mathbf{k},$$
   where x and z are related by the equation of the circle.

   Check: Left side of the pipe at $(-a, a)$: $\hat{\mathbf{n}} = -\mathbf{i}$; top of the pipe at $(0, 2a)$: $\hat{\mathbf{n}} = \mathbf{k}$, so correct.

2. *Ripples on a seabed are quite closely sinusoidal in nature. We will need to find a general expression for the boundary condition on the seabed. Consider a sea-bed (impervious to flow, for our purposes) given by: $z = A\cos x$.*

   a. *Obtain an expression for the unit normal.*

   Let $\phi = z - A\cos x$, which is 0 on the bed. Then
$$\begin{aligned} \nabla\phi &= +A\sin x\mathbf{i} + \mathbf{k}, \\ \hat{\mathbf{n}} &= \frac{\nabla\phi}{|\nabla\phi|} = \frac{A\sin x\mathbf{i} + \mathbf{k}}{\sqrt{A^2\sin^2 x + 1}} \end{aligned}$$

   b. *Obtain the kinematic boundary condition on the bed.*
$$\begin{aligned} &\mathbf{u}\cdot\hat{\mathbf{n}} = (u\mathbf{i} + w\mathbf{k})\cdot\frac{A\sin x\mathbf{i} + \mathbf{k}}{\sqrt{A^2\sin^2 x + 1}} = 0, \\ &uA\sin x + w = 0 \end{aligned}$$
   on $z = a\cos x$.

   c. *Check that your answer makes sense for $x = 0, \pi/2, \pi$.*

   When $x = 0$, $z = A$, the crest of the wave (ripple), and we get $w = 0$.

   When $x = \pi/2$, $z = 0$, $uA + w = 0$, and $w = -uA$ (draw a sketch).

   When $x = \pi$, $z = -A$, the trough of the wave (ripple), and we get $w = 0$.

3. *An unsteady case! The kinematic boundary condition is*

$$\mathbf{u} \cdot \hat{\mathbf{n}} = \mathbf{U} \cdot \hat{\mathbf{n}},$$

where U is the velocity of the local boundary
A circular cylinder of radius a moves along the x axis with constant velocity U. At time $t = 0$ it is at the origin.

a. *Verify to your own satisfaction that the equation of the cylinder is*

$$(x - Ut)^2 + z^2 = a^2.$$

Trivial from $(x - x_0)^2 + (z - z_0)^2 = a^2$!

b. *Obtain the kinematic boundary condition on the cylinder:*

$$(u - U)(x - Ut) + wz = 0,$$

where u and w are the velocity components in the x and z directions respectively.

$$\begin{aligned}
\phi &= (x - Ut)^2 + z^2 - a^2 \\
\nabla\phi &= 2(x - Ut)\mathbf{i} + 2z\mathbf{k} \\
\hat{\mathbf{n}} &= \frac{2(x - Ut)\mathbf{i} + 2z\mathbf{k}}{(\ldots)} \\
\mathbf{U} &= U\mathbf{i} \\
\mathbf{u} \cdot \hat{\mathbf{n}} &= (u\mathbf{i} + w\mathbf{k}) \cdot \frac{2(x - Ut)\mathbf{i} + 2z\mathbf{k}}{(\ldots)} \\
\mathbf{U} \cdot \hat{\mathbf{n}} &= U\mathbf{i} \cdot \frac{2(x - Ut)\mathbf{i} + 2z\mathbf{k}}{(\ldots)} \\
2u(x - Ut) + 2wz &= 2U(x - Ut),
\end{aligned}$$

and the result follows.